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Now You See It, Now You Don't:

The Goldstone-Higgs-Kibble Effect

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ABSTRACT

One of the mechanisms of field theory which is rather well known to the broader audience and is also rather perplexing and mysterious for the uninitiated is the phenomenon of spontaneous symmetry breaking. Although physical manifestations of this phenomenon pervade our daily life (magnets, ice...), the mechanism itself had remained an often misunderstood and misinterpreted apparently *l'art pour l'art*-esque arabesque of the particle theorist's whim. Herein we strive to allay these misconceptions and explain and illustrate this mechanism, progressing from trivia to

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1. Broken Symmetries are not Broke

The concept of symmetries is one of the most important ideas in modern Physics. Its ubiquity ranges from practical calculations, where symmetry considerations often simplify the problem significantly, to the attempts at fundamental and philosophical understanding of how Nature operates.

Most generally, and at the risk of irreverency, *symmetry* can be defined as *action without tangible effect*. Indeed, rotate an egg, by an *arbitrary* angle, around the axis that goes through the middle of the 'sharp' and the 'flat' end—and none can tell the difference (except around Easter time, when the egg is multi-colored). The rotation is therefore as action the effect of which in undiscernable; we say that the egg has an axial symmetry. Numerous other examples should spring to mind and we hope no further illustration is needed here.

Related to this concept is also the notion of 'broken symmetry' or 'approximate symmetry'. The real world is not ideal: geese are not spherical black bodies, although the heat absorption of a goose in the oven is rather well described in this way; the angular dependence of the shape of a goose does not affect its cooking significantly [1)]. The intricate symmetry of many a flower is breath-taking, yet the shapeless blotches of vivid color in the paintings of Monet evoke the same feelings of liveliness and loveliness.

We shall however be concerned with a somewhat different notion of broken symmetry. One which allows us to think about water and ice as one thing; one which blends magnets with molten iron; one which unites the charged with the chargeless, one which lets us talk about how 'up' is 'down' and 'charm' is 'strange' and 'truth' is 'beauty' and one which can dampen the range of the Forces of Nature from $\frac{1}{r}$ to $\frac{1}{r}e^{-r/R}$.

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A frequently heard misconception is that "If a symmetry is broken, it jus' ain't there." The adjective 'broken' has been quite aptly chosen, and is preferred here to 'approximate', for the former implies the existence of at least one *ordering parameter*, Φ, such that in a certain regime of this Φ the symmetry is exact, while in the complementary regime of Φ the symmetry is gone. On the other hand, we will understand 'approximate' to imply nothing of the kind, simply that the symmetry is imperfect, with perhaps a measurable discrepancy. The latter notion *is* useful—see the last three chapters of Wybourne's book [1]. In fact, practically all our knowledge of nuclear structure derives from using such approximate (also called dynamical) symmetries [2]. In this articlet, however, we will be concerned exclusively with *broken symmetries*.

To illustrate what is meant by 'broken symmetries' behold an ice cube. It contains monocrystalline domains. Within each such domain, the water molecules exhibit the orderly pattern of the crystal lattice and hence its symmetry: move along the axes of the crystal and in jumps that are multiples of the lattice spacing and no change can be detected! Soon, the ice cube will melt and the symmetry is gone: the water molecules move

1) …although the drumstick bare bones should be protected by foil from charring…

about randomly and hold no definite positions. Clearly, the temperature acts as the ordering parameter and the value $0°C$ is the critical temperature, the boundary between the liquid and the crystalline phase of H_2O..

Note that there is another symmetry breaking going on in this example! At room temperature, the water molecules move with random speeds and in random directions. At every instant, the velocity distribution for the individual water molecules has a complete spherical symmetry. Now put the glass of water in the freezer. The water becomes ice, where the individual molecules still vibrate, given the thermal energy, but the velocity distribution is no longer spherically symmetric! This latter type of symmetry and symmetry breaking is however statistical in nature and we shall hereafter not consider it any more.

Another example is found by melting a magnet (of course, one need not be *that* drastic; substantial heating will suffice). It is by now well known that a magnetized piece of iron owes its magnetic personality to domains within which the magnetic dipoles are perfectly aligned and so reinforce each other's field, giving rise to an appreciable net magnetization. Zooming in into one of these domains, one sees a crystalline structure with magnetic dipoles at each node and they all point in the same direction—hence a translational symmetry: you can move along the axes of the crystal and in jumps that are multiples of the lattice spacing and nothing changes; hence a (discrete, finite-step) translational symmetry.

Now place the magnet on the range-top and turn the heat on—the magnetization will soon be gone. Whatsmore, a hot piece of iron cannot be magnetized [2)], no matter how strong magnetic field is placed through it. A little thought leads to the conclusion that below a certain critical temperature, T_c, the forces holding together a magnetized domain are strong enough, while above T_c, thermal vibrations are stronger and destroy the domains. The key point to realize here is the existence of at least two competitive forces, the balance of which depends on the temperature.

2. Description of Symmetry Breaking

The major goal in describing symmetry breaking is therefore a model that aptly describes the transition from one phase into another, one which can interpolate between the two states of the observed system, and preferably in a way that lets us predict both qualitative and quantitative characteristics of the system.

2.1. Mean field models

One such description generally goes under the name of 'mean field model', advanced originally by Landau and Ginzburg, whence these models are sometimes also called Landau-Ginzburg models. The main idea is to introduce a 'mean field', a collective variable of sorts, which represents or determines somehow a phenomenological quantity of interest, such as pressure, magnetization, directions of the lattice axes, degree of symmetry...

[2)] ...it is however quite challenging a task for the experimenter to keep the iron hot and not melt the magnetizing coil at the same time...

In the case of the magnetization, introduce a mean-field variable $M(t,x,y,z)$, describing the magnetization at the point x,y,z at time t. Clearly, the magnetization of a site can affect the magnetization of the neighboring ones and so a variation of the magnetization localized at one place can propagate in a wave-like manner, so we expect that the equation of motion for M should be something like

$$\big[\vec{\nabla}^2 - \frac{1}{c^2}\big]M \quad = \quad \ldots \;, \tag{2.1}$$

where c is the speed of propagation of magnetization, to be obtained by direct measurements. This part of the equation of motion would arise from varying the following terms in the action

$$S_{\text{kin.}} \quad = \quad \frac{1}{2}\int \mathrm{d}^4x \sum_{\mu=0}^{3} \big(\partial_\mu M\big)^2 \;, \tag{2.2}$$

where $\partial_\mu \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\mu}$ are spacetime derivatives; $x^0 = ct$ and $x^\mu = x,y,z$ for $\mu = 1,2,3$. Since such terms describe propagation, they are called 'kinetic'. The relativistic-looking notation is used with the forethought of later application in relativistic field theories. Here, c is not the speed of light but the speed of propagation of magnetization in your favorite magnetizable substance.

Now we come to the ellipses on the right hand side of the equation of motion (2.1). Consider adding a potential energy term to the action

$$S_{\text{pot.}} \quad = \quad \int \mathrm{d}^4 V(M) \;, \tag{2.3}$$

where $V(M)$ is some polynomial in M. Of course, $-\frac{\partial V}{\partial M}$ will then be the 'force' acting on the magnetization mean field, and hence driving its propagation. The equation of motion becomes

$$\big[\vec{\nabla}^2 - \frac{1}{c^2}\big]M \quad = \quad -\frac{\partial V}{\partial M} \;. \tag{2.4}$$

Choosing, for example, $V(M) = \frac{1}{2}\mu^2 M^2$, would produce a linear harmonic oscillator type restoring force.

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2.2. Enter temperature

Consider the following potential energy (density)

$$V(M) \quad = \quad \frac{1}{2}\big(T^2 - \mu^2\big)M^2 + \frac{\lambda}{4}M^4 \;, \tag{2.5}$$

borrowed from the anharmonic extension of the linear harmonic oscillator, where T represents the temperature, $\lambda > 0$ and the sign of μ^2 has been flipped by hand, no excuses.

For $T > \mu$, the force is negative, i.e., restoring and drives M towards $M = 0$. That is, at $T > \mu$, M hovers about the value $M = 0$ to the extent allowed by the energy stored in the kinetic term. The mean (expected, average) value of M

$$\langle M\rangle \quad = \quad \int \mathrm{d}M \cdot M \cdot e^{i(S_{\text{kin.}}+S_{\text{pot.}})} \;, \tag{2.6}$$

is zero. This can be seen without any calculation, simply by observing that the minimum of the potential lies at $M = 0$.

Consider now lowering the temperature. Below $T = \mu$, the potential develops a hump at $M = 0$ and develops two minima, at

$$\text{for } T < \mu, \qquad M_{\text{min.}} = \pm\sqrt{(\mu^2 - T^2)/\lambda}\ . \tag{2.7}$$

The 'force' $-\frac{\partial V}{\partial M}$ is now repulsive around the origin, but attractive around $M = M_{\text{min.}}$. Hence, we expect the magnetization to hover around these *nonzero* values. Indeed, it may be calculated that $\langle M \rangle = M_{\text{min.}}$—the system undergoes a spontaneous magnetization[3]!

Qualitatively, this describes the magnetization rather well. Quantitative predictions are another matter. Note that there are three quantities which can be measured: (1) the critical temperature, $T = \mu$, (2) the average value of magnetization $\langle M \rangle = M_{\text{min.}}$ in the magnetized phase, when $T < \mu$, and (3) the difference in the potential energy of the system. That is, for $T > \mu$, $\langle M \rangle = 0$ and so $V(\langle M \rangle) = 0$. For $T < \mu$, $\langle M \rangle = \pm M_{\text{min.}}$ and $V(\langle M \rangle) = -(\mu^2 - T^2)^2/4\lambda$. The potential energy of the system is thus expected to slide from the former to the latter value just after the temperature falls below $T = \mu$. This energy difference is measurable. On the other hand, there are only two parameters in the model, μ and λ. In practice, the phenomenon of spontaneous magnetization is thus better described by a potential energy (density)

$$V(M) \;=\; \frac{1}{2}\big(T^2 - \mu^2\big)M^2 + \frac{\lambda}{4}M^4 + \frac{\kappa}{6}M^6\ , \tag{2.8}$$

where now $\kappa > 0$ and λ need not be positive.

✎ Find all the distinct types of shapes that the above potential function can take in terms of relative values of μ, λ, κ. That is, classify the possibilities according to whether the potential function has only a minimum, several minima, which minima are lower then others, etc.

➠ Assuming that the temperature is slowly decreasing and that initially $M = 0$, discuss what happens to the system for each of the types of potential that you found in part ✎. In your arguments, keep in mind that M should be treated as a quantum field.

2.3. The lesson

Notice that nothing in the problem determines whether M is positive or negative; the system possesses a finite symmetry, $M \to -M$. Once M settles around the positive or the negative minimum for $T < \mu$, this symmetry is broken. In other words, the breaking of the $M \to -M$ symmetry is correlated with the emergence of magnetizability.

3) Contrary to how it sounds, "spontaneous magnetization" does not mean that and object self-magnetizes. Rather, it means that the system spontaneously acquires the *ability* to be magnetized, to retain magnetization after the external magnetizing field has been removed. "Spontaneous magnetizability" might have been clearer, but...

Now ask yourself *just what* really breaks the symmetry $M \to -M$. With the potential as in Eq, (2.5) and for $T < \mu$, $M = 0$ is still an extremum, but an unstable one. Also, $M = 0$ does not break the symmetry. A kick (an external magnetic field) is needed to start the magnetization process; the magnetization then proceeds, represented by M sliding down the potential towards one of the two minima. Since the direction of the initial kick determines which of the two minima will be reached and so eventually which way the symmetry is broken, it is this initial datum that breaks the symmetry, not the dynamics of the mean field M!

Even if you consider $M(t, x, y, z)$ as a statistical ensemble of infinitely many independent degrees of freedom [4), one per point in spacetime, the same result follows. As quantum variables, the product of the uncertainty in M and its conjugate momentum (proportional to $\dot{M}$) must be bigger then $\hbar$. With the derivatives negligible, the value of M must fluctuate wildly and so one does not need an initial 'kick'. However, the fluctuations are random, and so the positive and the negative minimum is reached with the same probability and $\langle M \rangle$ nevertheless vanishes.

Thus, taking the above model as a spontaneous symmetry breaking paradigm, we conclude that the symmetry is broken by initial (boundary) conditions, rather than the dynamics. Also note that the symmetry in this model was discrete rather than continuous.

Finally, the choice of the potential energy function was *ad hoc*. Indeed, one calls such potentials "phenomenological", meaning that their shape is dictated by some general principles (see below), but typically contain several parameters. These parameters are then determined by comparison with the experiment or so that the potential reproduces a desired phenomenon as was the case here. In principle at least, the mean-field variable such as the magnetization here can be calculated from the 'micro-theory', that is from the underlying, fundamental physics of the system considered. For the magnetization, this is a textbook "calculation in principle" (see Ref. [3]); more generally, such a 'micro-theory' may or may not be available or even conceivable. Phenomenological potentials, Lagrangians and actions should therefore not be regarded as the 'last word', but rather as an intermediate tool capable of a coarse description. Once a phenomenological action has been well tested and adjusted to describe a particular phenomenon, the development of a 'micro-theory' is somewhat simplified, because we know what the result should look like, at least in a good approximation.

3. Continuous Symmetry Breaking

Before turning to a particle physics model, we wish to develop our intuition by considering another easily understood illustration. This time, however, we are looking for a paradigm in which the symmetry to be broken will be continuous.

[4)] ...which happens when the derivatives are negligible...

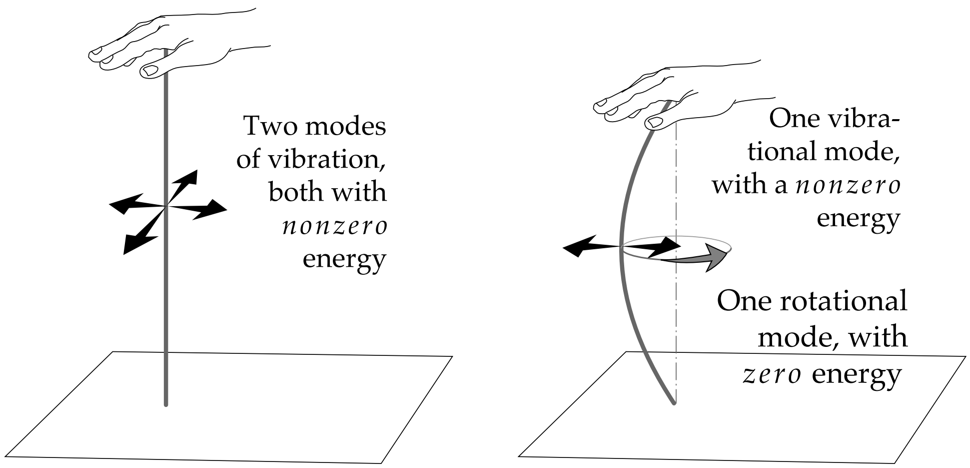

Figure 1: The bent rod example.

3.1. A simple illustration

The simplest continuous symmetry is the one-parameter group of rotations about a given axis in 3-dimensional space or about a point in a plane. Consider holding a thin elastic rod suspended between the tabletop and your palm as in Fig. 1. Initially, the rod has such an axial symmetry: the shape of the hand and the table top being immaterial, the rod can be freely rotated about its axis without changing the situation in the slightest. At the same time, the rod can be vibrated in two independent directions, as indicated on the left hand side of Fig. 1. It should be obvious that it takes work to excite these two vibrational modes; both of these modes store energy, i.e., have mass.

Now press the rod at the upper end until it bends as depicted in the right hand side of Fig. 1. It is obvious that the axial symmetry of the rod has been broken *by altering of the boundary condition*. The rod can still be vibrated in the radial direction as depicted on the right hand side of Fig. 1 by the solid black arrows. This is *one* vibrational mode for which it takes work to excite it and which therefore stores energy; this is the one 'massive' vibrational mode. Along with this mode, there is also a rotational mode, as depicted in the right hand side of Fig. 1 by the framed grey arrow. Most importantly, this mode takes no work to excite! That is, were it not for the friction between the rod and the tabletop on one end and the palm on the other, a small initial kick would send the rod revolving forever! This "vibrational" mode is therefore 'massless' in that it stores no energy and requires no work to be excited.

Note that the massless 'vibrational' mode that has been created is a rotation—precisely as was the symmetry which is broken by the bending of the rod. Furthermore, for small oscillations about the shapes depicted at the left and the right of Fig. 1, there is a linear harmonic restoring force, the strength of which is determined by the elasticity of the rod. These modes store energy just as is usual for linear harmonic motion; hence the mass for these modes. The upshot is that for the straight rod, the straight configuration is the equilibrium, i.e., the ground state and that is unique since rotations about its axis are unobservable.

For the bent rod, the bent shape is the ground state, but the ground state is degenerate: there are continuously many such states—one for each angle. So the massless mode transports the system among its continuously many degenerate ground states. The masslessness of this mode may also be seen as the fact that one may always chose a co-rotating coordinate frame in which the rod seems stationary.

This massless mode is called the Goldstone mode.

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The potential energy for the vibrations of the straight rod looks like a paraboloid with the rod at the central axis. The potential energy for the vibrations of the bent rod on the other hand looks like the bottom of a champaign bottle or a sombrero, with a hump around the origin, a circular valley around the origin the deepest part of which is at the distance of the considered part of the rod from the axis of rotation (the dot'n'dash line on the right in Fig. 1) and sweeping upward outside of that perimeter.

3.2. A complex field

Consider now a field theory involving a single complex scalar field Φ the dynamics of which is governed by the following Lagrangian density:

$$\mathcal{L}(\Phi) \;=\; \|\partial_\mu\Phi\|^2 \;-\; V(\Phi)\;, \tag{3.1}$$

where $\|\partial_\mu\Phi\|^2 = \sum_{\mu=0}^{3}|\frac{\partial\phi}{\partial x^\mu}|^2$ is the kinetic energy density and $V(\phi)$ is the potential energy (density). This we *choose* to be a polynomial of degree at most 4, to ensure renormalizability [5)].

Since the action must be real, so must the Lagrangian density be, and it follows that $V(\Phi)$ is in fact a function of $|\Phi|$. Thus, the redefinition $\Phi \to e^{i\varphi}\Phi$ is unobservable and therefore a symmetry! Write $\Phi(x) = \rho(x)e^{i\phi(x)}$, whereupon

$$\mathcal{L}(\Phi) \;=\; \|\partial_\mu\rho\|^2 + \rho^2\|\partial_\mu\phi\|^2 - \; V(\rho)\;, \tag{3.2}$$

and the $\Phi \to e^{i\varphi}\Phi$ symmetry is now seen as to act by leaving $\rho(x)$ as it is, while shifting $\phi(x) \to \phi(x) + \varphi$. note however that this 'decoupled' action of the symmetry is a consequence of the transcendental change of variables $\Phi(x) = \rho(x)e^{i\phi(x)}$.

Now consider the potential energy function

$$V(\Phi,\overline{\Phi}) \;=\; \frac{1}{2}(T^2-\mu^2)|\Phi|^2 + \frac{\lambda}{4}\big(|\Phi|^2\big)^2\;, \tag{3.3}$$

and note that this is the most general real polynomial of order 4 in the complex field Φ. The emphasis on maximum generality may appear trivial here, but it is one of the crucial

[5)] If you don't know what that is, never mind for now.

principles in particle theory modeling, since the missing terms will inevitably be introduced by quantum corrections unless there is a symmetry preventing them from doing so.

> This seems to be a good place to define renormalizability. An action is said to be renormalizable if quantum corrections do not add new terms to the action but only renormalize the parameters in it. Therein lies the utility of considering the most general renormalizable actions: all the parameters are accounted for and quantum effects will simply drive them to some particular values. Testing and proving renormalizability is discussed in every field theory textbook; suffice it here to note that in 4-dimensional spacetime, a renormalizable potential must be no higher than 4th order in bosonic fields and no higher than quadratic in fermionic fields. In 3-dimensional spacetime (suitable for describing "surface physics"), a renormalizable potential must be no higher than 6th order in bosonic and no higher than quadratic in fermionic fields. In 2-dimensional spacetime (suitable for "string physics"), a a renormalizable potential can be an arbitrary power series in the bosonic fields and no higher than 4th order in fermions.

The minima of the potential (3.3) are

$$\Phi_{\text{min.}} = 0 \ , \qquad T > \mu \ , \tag{3.4a}$$

$$\Phi_{\text{min.}} = \sqrt{\frac{(\mu^2 - T^2)}{\lambda}}\, e^{i\phi} \ , \qquad T < \mu \ . \tag{3.4b}$$

The rotational degree of freedom ϕ in (3.4b) represents the degeneracy of the ground states for $T < \mu$, when the $\Phi \to e^{i\varphi}\Phi$ symmetry is broken. That is, any one of the degenerate ground states will have a definite, fixed value of ϕ and the $\Phi \to e^{i\varphi}\Phi$ transformation is no longer a symmetry, but rotates one of the ground states into another.

Somewhat remarkably, the bent stick paradigm is applicable practically verbatim in this case. Indeed, once the field Φ settles about some particular ground state, small oscillations about this ground state may be described as follows. Since the initial value of ϕ in $\Phi = \rho e^{i\phi}$ is arbitrary, we may chose the ground state where $\phi = 0$ without loss of generality. Also, denoting $r = \sqrt{((\mu^2 - T^2)/\lambda)}$, we may write

$$\Phi = (r + \hat{\rho})e^{i\hat{\phi}} \ , \tag{3.5}$$

where $\hat{\rho}$ and $\hat{\phi}$ represent small oscillations about $\Phi = r$. Just as before, in (3.2), we obtain

$$\mathcal{L}(\hat{\rho}, \hat{\phi}) = \|\partial_\mu \hat{\rho}\|^2 + (r + \hat{\rho})^2 \|\partial_\mu \hat{\phi}\|^2 - \frac{1}{2}(T^2 - \mu^2)(r + \hat{\rho})^2 + \frac{\lambda}{4}(r + \hat{\rho})^4 \ . \tag{3.6}$$

Dropping all the terms higher than quadratic in the $\hat{\rho}$ and $\hat{\phi}$ (small oscillations, remember?), we obtain

$$\mathcal{L}(\hat{\rho}, \hat{\phi}) = \mathcal{L}(\hat{\rho}) + \mathcal{L}(\hat{\phi}) + \ \ldots \tag{3.7}$$

where the ellipses denote higher order terms and where

$$\mathcal{L}(\hat{\phi}) = \|\partial_\mu \tilde{\phi}\|^2 \ , \tag{3.8}$$

whence the rescaled field $\tilde{\phi}(x) = r\hat{\phi}(x)$ is the massless Goldstone boson, while

$$\mathcal{L}(\hat{\rho}, \hat{\phi}) = \|\partial_\mu \hat{\rho}\|^2 + \frac{1}{4\lambda}(\mu^2 - T^2)^2 - (\mu^2 - T^2)\hat{\rho}^2 , \tag{3.9}$$

and $\hat{\rho}$ has mass $\sqrt{\mu^2 - T^2}$ when $T < \mu$. Clearly, if the initial symmetry were larger and there were more scalar fields breaking the symmetry, we should expect one massless Goldstone boson for every broken symmetry degree of freedom. The number of massive remaining fields (small fluctuations about the chosen ground state) will depend on the total number of scalar fields introduced in the beginning.

Thus, we have covered the first half of the journey. We have concluded that for every broken symmetry there ought to be a massless Goldstone boson. In fact this is a rigorously proven result, known as the Goldstone theorem.

4. Gauge symmetry

Look back at the Lagrangian (density) in Eq. (3.1), which is invariant under the $\Phi \to e^{i\varphi}\Phi$ symmetry—provided ϕ is a constant. Also, this same symmetry occurs since the observable information in the field Φ is measured by the probability density $|\Phi(x)|^2$, which exhibits the same undefinedness.

The unifying *Leitmotiv* of modern particle theory is the simple observation that, if the phase of Φ is unobservable, it might as well be varying all over the place.

> Regarding nomenclature, the word 'gauge symmetry' has stuck from the translation of 'Eichen', used first by H. Weyl, since the early practitioners were more concerned with attempts at *fixing* a definite choice of such unobservable degrees of freedom. This procedure is known as "fixing the gauge" and which in electromagnetism implies, among other things, fixing a definite value for the 'zero-potential', hence fixing or adopting a gauge or standard of the potential.

However, replace $\Phi \to e^{i\varphi}\Phi$ in (3.1), and take into account that $\partial_\mu\varphi \neq 0$. The lagrangian density will change

$$\mathcal{L}(\Phi) \to \mathcal{L}(\Phi) + |\Phi|^2\|\partial_\mu\varphi\|^2 - 2(\partial_\mu\varphi)\,\Im m\left[(\Phi\partial_\mu\overline{\Phi})e^{-i\varphi}\right] . \tag{4.1}$$

To cancel this change of the lagrangian, the derivative ∂_μ is changed into a *covariant* derivative, ∇_μ, which is defined by demanding that under the gauge transformation $\Phi \to e^{i\varphi}\Phi$, it should transform as

$$\nabla_\mu \to e^{i\varphi}\nabla_\mu e^{-i\varphi} , \tag{4.2}$$

since then

$$\nabla_\mu\Phi \to e^{i\varphi}\nabla_\mu\Phi , \tag{4.3}$$

and $\|\nabla_\mu\Phi\|^2$ is invariant. Straightforward calculation produces

$$\nabla_\mu = \partial_\mu + A_\mu , \tag{4.4}$$

where A_μ is the gauge field and transforms simultaneously with $\Phi \to e^{i\varphi}\Phi$ as

$$A_\mu \quad \to \quad A_\mu \; - \; i(\partial_\mu \varphi) \; . \tag{4.5}$$

This is very much like we have found for the Fermionic electron field, in class.

The general principle of making the dynamics invariant under gauge transformations is as follows.

1. Specify the action of the gauge symmetry on the 'matter' fields, such as $\Phi \to e^{i\varphi}\Phi$.
2. In the action that governs the dynamics of Φ, replace partial derivatives by covariant ones, defined by Eq. (4.2), and given explicitly in (4.4).
3. Add a kinetic term for the gauge fields A_μ

$$\mathcal{L}_{\text{kin.,gauge}}(A_\mu) \quad \stackrel{\text{def}}{=} \quad -\frac{1}{4}\|F_{\mu\nu}\|^2 \; = \; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \; , \tag{4.6}$$

where

$$F_{\mu\nu} \quad \stackrel{\text{def}}{=} \quad \big[\, \nabla_\mu \, , \, \nabla_\nu \,\big] \tag{4.7}$$

is the field strength, a.k.a. the stress tensor.

The Lagrangian density $\mathcal{L}(\Phi, A_\mu)$ obtained in step **3.** of the above algorithm then describes a propagating matter field Φ interacting with the (also propagating) gauge field A_μ. To such a model we shall now add a potential for Φ and adjust the parameters (μ, λ) so that the symmetry $\Phi \to e^{i\varphi}\Phi$ be broken as studied above. This time, however, our aim is to see what happens to the gauge field A_μ.

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Consider the effect of the change of variables (3.5) in the term $\|(\partial_\mu + A_\mu)\Phi\|^2$. The derivative clearly annihilates the constant r, but a term of a new kind, $r^2\|A_\mu\|^2$, emerges. The combined lagrangian becomes

$$\begin{aligned} \mathcal{L} \; &= -\tfrac{1}{4}\|\partial_\mu A_\nu - \partial_\nu A_\mu\|^2 + \|(\partial_\mu + A_m)(r+\rho)\|^2 \\ &- \; \tfrac{1}{2}(T^2 - \mu^2)(r-\rho)^2 - \frac{\lambda}{4}(r+\rho)^4 \; . \end{aligned} \tag{4.8}$$

✎ Verify the formula Eq. (4.8). That is, find out how to obtain this form and then also find all the mistakes (signs, factors of 2, etc.).

Most importantly, the phase degree of freedom in $\Phi = (r+\rho)e^{i\phi}$ has vanished! The GOldstone boson has vanished! To achieve this vanishing act, a suitable gauge transformation was performed (hint for the exercise!), adding a multiple of $(\partial_\mu\varphi)$ to A_μ; call this shifted gauge field $\tilde{A}_\mu$. Recall now that the longitudinal polarization of A_μ was unphysical. However, the longitudinal mode of $\varphi_\mu\varphi$ is simply proportional to φ itself (go to the Fourier transforms to see this). Thus, unlike the original gauge field A_μ, $\tilde{A}_\mu$ does have a non-zero longitudinal component, which in fact is the (derivative of the) 'vanished' Goldstone boson φ.

Thus, before symmetry breaking, there is a massless gauge field A_μ and a massive complex scalar field Φ (fluctuating around $\Phi = 0$). The total number of degrees of freedom is 4: two physical polarizations of A_μ and two real component fields of Φ.

After symmetry breaking, there is a massive gauge field (the mass term being $r^2\|A_m\|^2$) and a single massive scalar field, ρ. The total number of degrees of freedom is again 4: three physical polarizations of the vector field $\tilde{A}_\mu$ and one real scalar field $\rho(x)$.

$$\boxed{\text{This switcheroo is called the Higgs mechanism.}}$$

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5. Nonabelian Gauge Fields and the Higgs Mechanism

'The time has come,' the Walrus said, 'to speak of many things: of shoes and ships and sealing-wax, of cabbages and kings, and why the sea is boiling hot and whether pigs have wings.'

The generalization of the Higgs mechanism to nonabelian symmetries [6)] is found in any respectable field theory textbook; try for example p.612–616 of Ref. [4]. Their conventions are notably somewhat outdated, but the algebra involved in translating their formulae into our above notation and so as to compare with other texts. A telegraphic account of the Higgs phenomenon, followed by a description of experimental tests of the Glashow-Weinberg-Salam model for weak interactions is found in Ref. [5], starting at page 332.

References

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[6)] While groups of commuting symmetries are called Abelian, in honor of Henrik Abel, those which do not commute are not called Cainian, but nonabelian. So, the group of 2×2 unimodular and unitary matrices, $SU(2)$ is nonabelian in virtue of the well known fact that $n \times n$ matrices do not commute unless diagonal.