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Elusive Conifold Compactifications

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ABSTRACT

Compactification on conifolds, which have only conical singularities, interfaces topologically distinct (super)string models. The part of the connecting path, recently argued to be of infinite length in Zamolodchikov's metric, is shown to be a closed loop (at best). The essential problem in such models is identified and we discuss possible remedies.

Introduction. Smooth Calabi-Yau spaces are known to describe superstring vacuua [1]. They form many topological (indeed, homotopy) classes, for each of which the choice of the complex structure and the Kähler class can be varied through a (typically many-dimensional) moduli space. Points in the boundary of these moduli space correspond to singular Calabi-Yau spaces, amongst which orbifolds are the best known examples. The rather simple geometry of the latter allows an exact stringy analysis, and shows that cyclic quotient singularities are harmless for string propagation. One then expects other singularities to be innocuous too (see Ref. [2] for preliminary results).

So-called double points (nodes) are the next simplest type of singularity. They occur on hypersurfaces, at points where the gradient of the defining constraint vanishes but the matrix of second derivatives is regular. In the more familiar case of Riemann surfaces, this is the only "bad" thing possible and happens when a cycle is pinched to a point. In complex 3-folds, the situation is much more complicated [3]; the regularity of the matrix of second derivatives however is a hallmark of nodes and does not happen for any higher case. While bounds on the severity of physically admissible singularities are not known (see however [2]), I believe there is no doubt in the reader's mind that conifolds, possessing only such simple singularities,

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should be included in the arena of (super)string compactification. In addition, preliminary analysis indicates that conifolds can also occur in cosmologically interesting situations such as the stringy cosmic strings of Ref. [4][♯2].

Consider the topology-changing process [5,6,7]

$$\begin{array}{ccc} & & \mathcal{M} \\ & & \uparrow \wp^{-1} \\ \mathcal{M}^\flat & \xrightarrow{\varepsilon\to 0} & \mathcal{M}^\sharp \end{array} , \tag{1}$$

where

$$\mathcal{M}^\flat \quad : \qquad S(x)P(x) - Q(x)R(x) \quad = \quad \varepsilon T(x) \ , \tag{2}$$

$$\mathcal{M} \quad : \quad \begin{cases} p \;=\; P(x)y_0 + Q(x)y_1 \;=\; 0 \ , \\ q \;=\; R(x)y_0 + S(x)y_1 \;=\; 0 \ , \end{cases} \tag{3}$$

are two Calabi-Yau manifolds with Euler characteristics $\chi^\flat_{_E} = -200$ and $\chi_{_E} = -168$, respectively. $x = (x_0{:}x_1{:}x_2{:}x_3{:}x_4)$ are homogeneous coordinates on $\mathbb{CP}^4$, $y = (y_0{:}y_1)$ on $\mathbb{CP}^1$, $P(x)$ and $Q(x)$ are quartic polynomials, $R(x)$ and $S(x)$ are linear in x, $T(x)$ is a non-singular quintic and ε is a complex parameter.

The interfacing Calabi-Yau conifold,

$$\mathcal{M}^\sharp \quad : \quad C^\sharp \;\stackrel{\rm def}{=}\; S(x)P(x) - Q(x)R(x) \quad = \quad 0 \ , \tag{4}$$

has 16 nodes, where $\nabla C^\sharp{=}0$, and $\chi^\sharp_{_E} = -184$. Clearly, $\mathcal{M}^\sharp$ is obtained from $\mathcal{M}^\flat$ in the limit $\varepsilon \to 0$. From the other side, the projection of $\mathcal{M}$ along $\mathbb{CP}^1$ (elimination of the coordinates of $\mathbb{CP}^1$) yields $\mathcal{M}^\sharp$. In the opposite process, each node of $\mathcal{M}^\sharp$ gets replaced by a copy of the $\mathbb{CP}^1$ spanned by y; this is a “small resolution”. Note that *this is not blowing up* : if it were, each node would be replaced by $\mathbb{CP}^1 \times \mathbb{CP}^1$. Since either one of the two factors may occur in a small resolution, there is a total of 2^{16} choices. Most of these do not result in a Kähler manifold, but the choice taken automatically by the process $\mathcal{M}^\sharp \stackrel{\wp^{-1}}{\longrightarrow} \mathcal{M}$—does [6,7].

Towards the RG-Fixed Point. The superpotential of a Landau-Ginzburg model [8] pertaining to $\mathcal{M}^\flat$ is :

$$W^\flat \;\stackrel{\rm def}{=}\; S(\boldsymbol{X})\,P(\boldsymbol{X}) - Q(\boldsymbol{X})\,R(\boldsymbol{X}) - \varepsilon T(\boldsymbol{X}) \ , \tag{5}$$

where $\boldsymbol{X}^\mu$ are superfields in which the scalar component field corresponds to x_μ. Clearly, the scaling weights are $w(\boldsymbol{X}^\mu) = 1/5$, whence the central charge is $c = 9$. For any given S, P, Q, R and T, the chiral ring [9] can be computed and is indeed finite dimensional. Moreover,

[♯2] Important discussions on this point with C. Vafa and S.-T. Yau are gratefully acknowledged.

variations of $W^\flat$ are truly marginal deformations, correspond to deformations of the complex structure on $\mathcal{M}^\flat$ and give rise to the (2,1)-moduli.

In the limit when $\varepsilon \to 0$, the superpotential becomes

$$W^\sharp \stackrel{\text{def}}{=} S(\boldsymbol{X})P(\boldsymbol{X}) - Q(\boldsymbol{X})R(\boldsymbol{X}) \ . \tag{6}$$

It exhibits the following problems (not having satisfied the requirements of Ref. [9]). Rather than only at the origin of the field space, $\boldsymbol{X}^\mu$=0, the conditions $\nabla W^\sharp = 0$ are satisfied at a bouquet of 16 $\mathbb{C}$-like rays which all meet at the origin. These are the vacuua since $V = \|\nabla W^\sharp\|^2$. Such flat directions create two related problems.

(a) The numerical characteristics of the model are ill-defined; e.g., the Witten index is a difference of two divergent integrals rather then two finite integers.

(b) The chiral ring [9] is infinite dimensional. This is easy to see on recalling that the chiral ring is obtained by taking the quotient of the ring of all polynomials by the Jacobian ideal $\Im(\nabla W^\sharp)$, which is generated by the gradients of the superpotential,

$$R^\sharp[\boldsymbol{X}] \ \stackrel{\text{def}}{=} \ \Big\{P[\boldsymbol{X}]\Big/\Im(\nabla W^\sharp)\Big\} \ . \tag{7}$$

Most notably, however, the 16 $\mathbb{C}$-like rays intersect the projectivised field space precisely at the 16 singular points of $\mathcal{M}^\sharp$; the vacuua of $W^\sharp$ match the singularities of $\mathcal{M}^\sharp$. The above mentioned problems with flat directions are thus seen to parallel the failure of conventional differential geometry at these singular points.

To approach the limit $W^\sharp$ from the other side, in Ref. [10], the superpotential

$$W_{\text{naïve}} \stackrel{\text{def}}{=} P(\boldsymbol{X})\,\boldsymbol{Y}^0 + Q(\boldsymbol{X})\,\boldsymbol{Y}^1 + R(\boldsymbol{X})\,\boldsymbol{Y}^0 + S(\boldsymbol{X})\,\boldsymbol{Y}^1 \tag{8}$$

was considered. However, $W_{\text{naïve}}$ is unsatisfactory—it fails to meet several of the necessary requirements [9], even though it (supposedly) corresponds to a *smooth* space, $\mathcal{M}$.

1. To make $W_{\text{naïve}}$ quasihomogeneous, one has to require $w(\boldsymbol{X}^\mu) = 0$ and $w(\boldsymbol{Y}^\alpha) = 1$. While this produces $c = 9$ as desired, it also means that the superfields $\boldsymbol{X}^\mu$ must not scale at all. For describing a renormalization flow fixed point which is rescaling invariant, this is unnatural, to say the least.
2. $W_{\text{naïve}}$ is very badly degenerate, *much* worse than $W^\sharp$. Rather than only at the origin of the field space, $\boldsymbol{X}^\mu$=0, or even at the 16 $\mathbb{C}$-like rays of $W^\sharp$, $W_{\text{naïve}}$ is flat along a mash of 16 $\mathbb{C}^3$-bodies which intersect each other. Moreover, these 16 $\mathbb{C}^3$-bodies have nothing in common with the 16 $\mathbb{C}$-like rays of $W^\sharp$ vacuua—except the origin. Thus, the vacuua of $W_{\text{naïve}}$ have nothing to with the singularity structure of $\mathcal{M}^\sharp$ and even less with that of $\mathcal{M}$, which in fact is smooth. As with $W^\sharp$, this implies the same two problems, (a) and (b) as above, except that they are much worse than in the case of $W^\sharp$, whereas $\mathcal{M}$ is a smooth Calabi-Yau space and should have led to no problems whatsoever.

The first problem with $W_{\text{naïve}}$ can be resolved in a very simple way. (It is gratifying to see that the second problem also gets resolved to a great extent.) Consider

$$\begin{array}{ccc} & & \mathcal{M}' \in \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}\!\!\begin{Vmatrix} 4&1&0 \\ 1&0&1 \\ 0&1&1 \end{Vmatrix} \\ & & \uparrow \wp'^{-1} \\ & & \mathcal{M} \in \begin{bmatrix} 4 \\ 1 \end{bmatrix}\!\!\begin{Vmatrix} 4&1 \\ 1&1 \end{Vmatrix} \\ & & \uparrow \wp^{-1} \\ [4\|5] \ni \mathcal{M}^\flat & \xrightarrow{\varepsilon\to 0} & \mathcal{M}^\sharp \end{array} \tag{9}$$

i.e., we 'split' once more, obtaining $\mathcal{M}'$ from (3) :

$$\mathcal{M}' \ : \quad \begin{cases} p = P(x)y_0 + Q(x)y_1 = 0 \ , \\ r = R(x)z_0 + S(x)z_1 = 0 \ , \\ s = y_1z_0 - y_0z_1 = 0 \ , \end{cases} \tag{10}$$

It is easy to prove that in fact $\mathcal{M}$ and $\mathcal{M}'$ are identically the same manifold. It suffices to show that $\wp'$ creates no singularities. If it did, the singularities would have to be described by one of the two embeddings

$$\text{Sing}(\wp') = \begin{bmatrix} 4 \\ 1 \end{bmatrix}\!\!\begin{Vmatrix} 4&1&1&0&0 \\ 1&0&0&1&1 \end{Vmatrix} \quad \text{or} \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix}\!\!\begin{Vmatrix} 4&4&1&0&0 \\ 0&0&1&1&1 \end{Vmatrix} \ , \tag{11}$$

but both of these are empty sets. More explicitly, since $z = (z_0{:}z_1)$ are homogeneous coordinates of a $\mathbb{CP}^1_Z$, they cannot vanish simultaneously and, when projecting along $\mathbb{CP}^1_Z$ to obtain $\mathcal{M}$, the determinant of the system $\{r = 0 = s\}$ is required to vanish. This is precisely the equation $q = 0$, which defines a smooth hypersurface.

For $\mathcal{M}'$, however, a Landau-Ginzburg superpotential is readily written as

$$W' \stackrel{\text{def}}{=} P(\boldsymbol{X})\boldsymbol{Y}^0 + Q(\boldsymbol{X})\boldsymbol{Y}^1 + R(\boldsymbol{X})\boldsymbol{Z}^0 + S(\boldsymbol{X})\boldsymbol{Z}^1 + \boldsymbol{Y}^1\boldsymbol{Z}^0 - \boldsymbol{Y}^0\boldsymbol{Z}^1 \ . \tag{12}$$

Now $w(\boldsymbol{X}^\mu) = w(\boldsymbol{Y}^\alpha) = 1/5$ and $w(\boldsymbol{Z}^\beta) = 4/5$ are all positive and $c = 9$, as required. Consider next the partition functional

$$\int D[\boldsymbol{X}]\, D[\boldsymbol{Y}]\, D[\boldsymbol{Z}]\, \mathrm{e}^{-\int d^2\sigma(\int d^2\theta\, W' \ + \ \text{h.c.})} \ , \tag{13}$$

where we have dropped the kinetic terms [9,8]. Integration over the superfields $\boldsymbol{Z}^\beta$ yields δ-functionals :

$$\int D[\boldsymbol{Z}^0] \implies \delta[R(\boldsymbol{X}) + \boldsymbol{Y}^1] \ , \qquad \int D[\boldsymbol{Z}^1] \implies \delta[S(\boldsymbol{X}) - \boldsymbol{Y}^0] \tag{14}$$

in the path integral. These allow us to drop the $\int D[\boldsymbol{Y}]$ path integrations and substitute

$$\boldsymbol{Y}^1 \to -R(\boldsymbol{X}) \ , \quad \boldsymbol{Y}^0 \to S(\boldsymbol{X}) \ , \qquad \implies \ W' \to W^\sharp \ . \tag{15}$$

We have just derived that path integration over the $\boldsymbol{Z}$'s implies $W' = W^\sharp$. It is easy to see that integrating over $\boldsymbol{Y}$'s first has the same effect. Also, since $\mathcal{M} \equiv \mathcal{M}'$, the fixed point of the renormalization flow characterized by the superpotential W' describes superstring compactification on both $\mathcal{M}'$ and $\mathcal{M}$ and we may drop the prime from W'. This proves :

The superpotential $W^\sharp$ (6), corresponding to $\mathcal{M}^\sharp$, and the superpotential W (12), corresponding to $\mathcal{M}$, describe one and the same Landau-Ginzburg model.

With this reinterpretation, W does not suffer from the second problem of $W_{\text{naïve}}$ either : W is flat precisely where it must be to match the singular points of $\mathcal{M}^\sharp$.

This should have been expected. The superpotential is constructed from the defining polynomials, the variations of which are tangential to (a subspace of) the space of complex structures. The resolution $\mathcal{M}^\sharp \to \mathcal{M}$ occurs by a variation in the Kähler cone, keeping the complex structure fixed. The Landau-Ginzburg analysis, as presented in Ref. [8,9], carries no information about the Kähler cone and it had to be the case that $W^\sharp = W$.

In Ref. [10], the transition from $W_{\text{naïve}}$ to $W^\sharp$ was parametrized by a path which was found infinitely long in Zamolodchikov's metric. In the interpolating theories, the superpotential is not quasihomogeneous, the two superfields $\boldsymbol{Y}^\alpha$ and also the linear combinations $R(\boldsymbol{X})$ and $S(\boldsymbol{X})$ acquire masses and the dominant term in the superpotential is $m_0S^2{+}m_1R^2$, which does not scale at all; this is unlike any Calabi-Yau compactification. Assuming, as in Ref. [10], that $W_{\text{naïve}}$ describes compactification on $\mathcal{M}$ and since W (which we have proven to describe it) equals $W^\sharp$, it follows that the path of Ref. [10] corresponds to an infinitely long round-trip from $W^\sharp$, through the wilderness of 2-dimensional field theories, back to itself.

Summary and Discussion. As the first step beyond Ref. [8], we have shown : 1. Landau-Ginzburg treatment of Calabi-Yau conifolds leads to degenerate models, in which—at best—the degeneracy of the vacuua copies the singularity structure of the conifold[♯3]. 2. For smooth Calabi-Yau manifolds such as $\mathcal{M}$ in Eq. (3), hence unfortunately most members of the unifying web [6,7], Landau-Ginzburg models are inapproపariate in their present form.

Recall that in a constrained $\mathbb{CP}^4_X \times \mathbb{CP}^1_Y$ model [11,12],

$$L \;=\; \int \mathrm{d}^4\theta \Big(\boldsymbol{X}^\dagger_\mu \mathrm{e}^{-\boldsymbol{V}_X} \boldsymbol{X}^\mu + f_X \boldsymbol{V}_X \;+\; \boldsymbol{Y}^\dagger_\alpha \mathrm{e}^{-\boldsymbol{V}_Y} \boldsymbol{Y}^\alpha + f_Y \boldsymbol{V}_Y \Big) \;+\; \Big(\int \mathrm{d}^2\theta\, \boldsymbol{\Lambda}_a\, P^a(\boldsymbol{X},\boldsymbol{Y}) + \text{h.c.} \Big)\,, \tag{16}$$

the choice of the $\boldsymbol{V}_i{=}0$ and $\boldsymbol{\Lambda}_a{=}\lambda_a$ (=*const.*) gauge yields the Landau-Ginzburg Lagrangian. But, it is precisely $\boldsymbol{V}_i$ which projectivise[♯4] the $\boldsymbol{X}$ and $\boldsymbol{Y}$ field space into $\mathbb{CP}^4_X \times \mathbb{CP}^1_Y$ and carry the information about the Kähler class on $\mathcal{M}$. It is usually assumed that D-terms

[♯3] As a few more examples would demonstrate, it is in general very hard to find such "well aligned" models.

[♯4] Each of these non-propagating gauge superfields is the connection for a complexified $U(1)$, denoted $\mathbb{C}^*$. Passing to usual gauge-slices indeed yields $\mathbb{CP}^n = (\mathbb{C}^{n+1}{-}0)/\mathbb{C}^*$.

become irrelevant operators and thus can be ignored. However, without the constraint terms, integrating out the $\boldsymbol{X}$'s and $\boldsymbol{Y}$'s yields bona-fide marginal operators [13] :

$$L_{\text{eff.}} \;=\; -\sum_{i=X,Y}\frac{n_i{+}1}{4\pi}\Big(\int \mathrm{d}\theta^1\mathrm{d}\bar\theta^2\,\boldsymbol{S}_i[1-\log(\boldsymbol{S}_i/\mu)] \;+\; \text{h.c.}\Big) \;+\ldots\;, \tag{17}$$

where $\boldsymbol{S}_i \stackrel{\text{def}}{=} D_2\bar D_1\boldsymbol{V}_i$, $i=X,Y$. In the presence of the constraint terms, we cannot integrate out explicitly the $\boldsymbol{X}$'s and $\boldsymbol{Y}$'s but one should expect a more complicated variant of Eq. (17) retaining however marginality. Since the Calabi-Yau manifold $\mathcal{M}$ does define a consistent superstring vacuum, we suspect that the problems of the superpotential W stem from the above choice of gauge. Regretably, we have not been able to find another gauge in which computations could be done to verify more explicitly the consistency of the model.

Finally, a remark is in order regarding general arguments about the decoupling of "extra" fields when discussing the transition among compactifications on $\mathcal{M}^\flat$ and $\mathcal{M}$. It is often ignored that some of the *exact* $\mathbf{27}^3$ Yukawa couplings in the effective 4-dimensional field theory (even when properly normalized) diverge in the limit $\mathcal{M}^\flat \xrightarrow{\varepsilon\to0} \mathcal{M}^\sharp$ [7]. Thus, certain fields enter a strong Yukawa coupling regime as $\varepsilon\to0$ and are effectively constrained in the limit. Given such a violent behaviour of the effective 4-dimensional theory, it is expected that the underlying 2-dimensional theory should exhibit some form of strong coupling too. This unfortunately seems to put our understanding of conifold compactifications in the same situation as confinement in QCD.

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