UTTG-30-90
Mod. Phys. Lett. **A6** (1991) 207–216
29. VIII 1990.

How Singular a Space Can Superstrings Thread ?

Tristan Hübsch†
Theory Group, Department of Physics
University of Texas, Austin, TX 78712, U.S.A.

ABSTRACT

Many superstring models with N=1 supergravity in 4-dimensional Minkowski spacetime involve σ-models with complex three dimensional, Ricci-flat target manifolds. In general, inclusion of singular target spaces probes the boundary of the moduli space and completes it. Studing suitably singular σ-models, I find certain criteria for the severity of admissible singularizations.

†Supported by the Robert A. Welch Foundation and the NSF Grant PHY8605978. Address after Aug. 1990 : Harvard University Mathematics Department, Cambridge, MA 02138. On leave of absence from the „Ruđer Bošković" Institute, Bijenička 54, 41 000 Zagreb, Croatia, Yugoslavia.

1 Calabi-Yau Manifolds and Beyond

Since Ref. [1], Calabi-Yau manifolds received perpetually returning interest of the physics community. These (Ricci-flat and Kähler) manifolds have been shown, to a constantly improving accuracy [2,3], to describe superstring vacuua.

1.1 The Pointillist Picture

The sector of the world sheet action which pertains to the Calabi-Yau compactification features rigid (2,2)-supersymmetry in the standard ansatz and takes the form

$$S^{(2,2)}_{CY} \;=\; \int_\Sigma \mathrm{d}^2\sigma\, \mathrm{d}^2\varsigma\, \mathrm{d}^2\bar\varsigma\;\; \mathbf{K}^{(CY)}(\boldsymbol{Z},\bar{\boldsymbol{Z}})\ , \tag{1.1}$$

where

$$\boldsymbol{Z}^\mu \;\stackrel{\rm def}{=}\; Z^\mu \;+\; \varsigma^+\zeta_+{}^\mu \;+\; \varsigma^-\zeta_-{}^\mu \;+\; \varsigma^+\varsigma^-\mathsf{Z}^\mu \tag{1.2}$$

is the string supercoordinate. Upon the $\mathrm{d}^2\varsigma\,\mathrm{d}^2\bar\varsigma$ -integration and the elimination of the auxiliary component fields Z^μ, $S^{(2,2)}_{CY}$ describes the dynamics of the component fields Z^μ, $\zeta^\mu_\pm$ and their conjugates. For (1.1) to describe (super)string vacua, $\mathbf{K}^{(CY)}$ is chosen so that $S^{(2,2)}_{CY}$ is superconformally invariant.

The partition function is

$$\mathcal{Z}^{(2,2)}_{CY} \stackrel{\rm def}{=} \int D[\boldsymbol{Z}]\, D[\bar{\boldsymbol{Z}}]\;\; \exp\{\, i\, S^{(2,2)}_{CY}\,\} \tag{1.3}$$

where the integration measure $D[\boldsymbol{Z}]\,D[\bar{\boldsymbol{Z}}]$ can be organized hierarchically, starting with the classical component fields in $\boldsymbol{Z}^\mu$,

$$\int_{\mathcal{M}} \mathrm{d}^6 z \sqrt{\det(g)}\;\; \int_{T_z(\mathcal{M})} \mathrm{d}^6\zeta_\pm\;\; \int_{\bar T_z(\mathcal{M})} \mathrm{d}^6\bar\zeta_\pm\ , \tag{1.4}$$

where $\det(g)$ is the determinant of $g_{\mu\bar\nu} = \mathbf{K}^{(CY)}_{,\mu\bar\nu}$. We expect $\mathcal{Z}^{(2,2)}_{CY}$ to be dominated by quantum fields (not necessarily smooth or continuous) that are 'near' the classical ones so that the latter represent fairly well the average of the dominant contributions.

On the other hand, the component fields of $\boldsymbol{Z}$ include the coordinates and the (co)tangent vectors on the internal space $\mathcal{M}$. All information that these can yield we can also obtain by various techniques of standard algebraic geometry, which is what one conventionally does in the so-called 'point-limit' of (super)strings. The observed stoutness of the 'point-limit' results (with world sheet instanton effects included) [2,3] can thus be taken as circumstantial evidence for our expectations regarding the dominant contributions to the path-integral in (1.3).

1.2 The Instanton Retouch

In the σ-model with the action (1.1), Z^μ maps the world sheet Σ into the target space $\mathcal{M}$, while the $\zeta_\pm{}^\mu$ map Σ into the (holomorphic) tangent space of $\mathcal{M}$ at Z^μ. The holomorphic instantons [2] are classical configurations of Z^μ, inequivalent to the 'tree-level' Z^μ which is constant on Σ. In other words, they arise as homotopically non-trivial maps $\Sigma \to \mathcal{M}$.

Now, the world sheet being a Riemann surface, only holomorphic maps (in)to even-dimensional subspaces of the target manifold $\mathcal{M}$ can agree with its complex structure, which in turn is crucial for $(2,2)$-supersymmetry. Since the harmonic (1,1)-forms are dual to 2-cycles (which may be thought of as formal sums of certain 2-dimensional subspaces) in $\mathcal{M}$, holomorphic instantons are easily seen to interfere with any result related to (1,1)-forms. Odd-dimensional subspaces, on the other hand, can only be interfered with through their boundaries or the spaces that they themselves bound. Thus, results that depend only on odd-dimensional cycles in $\mathcal{M}$ cannot be affected by holomorphic instantons. Such results are then exact, moreover, to any finite order in string loops [2].

In particular, *triple-(2,1) Yukawa couplings cannot be renormalized* (cf. Ref. [3]).

Moreover, the world sheet is naturally mapped to 2-cycles, which are dual to (1,1)-cohomology. To any basis for the latter, there corresponds a basis for the former, hence a parametrization for the instantons. Many features of their effects can therefore be obtained with no actual evaluation [4], aside from the fact that the evaluation also can be performed, at least in principle.

1.3 Who Ordered Singularities ?

Perhaps the sincerest argument for considering singular target spaces for (super)string propagation is that there is no compelling reason against them. Now, parameter spaces of most known (smooth) Calabi-Yau manifolds connect when certain conical singularizations are included [5,6]. Also, spaces with cyclic quotient singularities can be smoothed into Calabi-Yau manifolds [7]. However, the general theory of singularities and their smoothings [8,9] is fairly complicated and it is unclear at present which are the singularities that can be resolved to give a Calabi-Yau manifold and precisely which are innocuous for the (super)string.

In the present paper, certain criteria are found for admissible singularities. These are satisfied, for example, by all simple (A, D, E), modality-1 and -2 singularities as listed in the first book of Ref. [8](p.158–160). This includes the conifolds of Ref. [5,6] as the simplest case [10].

An important obstacle in finding a *complete* answer to the question raised in the title of this paper is the awesome number and complexity of ways that a complex 3-fold can singularize. In fact, only the simplest singularities are classified and their smoothings well understood [8,9]. We must therefore refrain from complete generality and content ourselves presently to study singularities which typically arise.

In all constructions that I am aware of, singularities arise as follows. (1) When embedding via simultaneous constraints, these may fail to be transverse. (2) When passing to the quotient by the action of a finite group, its fixed points yield singularities. (3) Subspaces in a given smooth Calabi-Yau manifold may be crushed into smaller and singular subspaces; e.g., the reverse of blowing up a cyclic quotient singularity of an orbifold. It is fairly obvious that type (3) brings nothing new; for it is the reverse of smoothing the singularities of type (1) or (2). Cyclic quotient singularities are innocuous for (super)strings [11] (see also Ref. [12,13,14]). I therefore focus on type (1) and note that these are described completely by a non-transverse system of algebraic equations in *local* coordinates. Thus, Lagrangian analysis of constrained (2,2)-supersymmetric σ-models [15,16] is well suited for the task and is presented in section 2. The existence of the nowhere vanishing holomorphic volume-form is studied in section 3. Some brief remarks and discussion are left for section 4.

2 Constrained Supersymmetric σ-Models

The main disadvantage of the world sheet action (1.1) is that it requires an explicit Kähler potential $\mathbf{K}^{(CY)}$ with which $S_{CY}^{(2,2)}$ exhibits superconformal invariance; none are known for any Calabi-Yau manifold. Also, the functional dependence of $\mathbf{K}^{(CY)}$ on $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ is often irrelevant; many physically observable quantities depend only on some rather coarse ('topological') properties of $\mathbf{K}^{(CY)}$. On the other hand, many of the Calabi-Yau manifolds are constructed as embedded in a bigger and better understood space and the induced Kähler potential suffices for many of the physically motivated computations [4,16]. We therefore consider a Calabi-Yau manifold defined as the solution of a system of K homogeneous, holomorphic polynomial constraints

$$P^a(x) = 0 \ , \qquad \forall x \in \mathcal{M} \hookrightarrow \mathcal{X} \ , \qquad a = 1, \ldots, K \tag{2.1}$$

in some complex, compact $(K+3)$-dimensional *ambient* space $\mathcal{X}$ (such as $\mathbb{P}^4$).

The world sheet action of a corresponding σ-model is then [15,16]

$$S \ = \ \int_\Sigma \mathrm{d}^2\sigma \left(\mathcal{L}_{\text{kin.}} \ + \ \mathcal{L}_{\text{con.}} \right) \ , \tag{2.2}$$

where

$$\mathcal{L}_{\text{kin.}} = \int \mathrm{d}^2\varsigma \, \mathrm{d}^2\bar{\varsigma} \ \ \mathbf{K}(\boldsymbol{X}, \overline{\boldsymbol{X}}) \ , \qquad \mathcal{L}_{\text{con.}} = \int \mathrm{d}^2\varsigma \ \boldsymbol{\Lambda}_a \, P^a(\boldsymbol{X}) \ \ + \ \ \text{h.c.} \tag{2.3}$$

and $\mathbf{K}(\boldsymbol{X}, \overline{\boldsymbol{X}})$ is a suitably chosen Kähler potential on $\mathscr{X}$. Here

$$\boldsymbol{X}^\mu \stackrel{\text{def}}{=} X^\mu + \varsigma^+ \xi_+{}^\mu + \varsigma^- \xi_-{}^\mu + \varsigma^+\varsigma^- \mathsf{X}^\mu \tag{2.4}$$

are $K+3$ local coordinate superfields for $\mathscr{X}$ and

$$\boldsymbol{\Lambda}_a \stackrel{\text{def}}{=} \Lambda_a + \varsigma^+ \lambda_{+a} + \varsigma^- \lambda_{-a} + \varsigma^+\varsigma^- \mathsf{L}_a \tag{2.5}$$

are Lagrange superfields.

Path-integration over $\Lambda_a, \lambda_{\pm a}, \mathsf{L}_a$ and X^μ yield delta-functionals which enforce algebraic field equations. Upon the fermionic integration, these are :

$$\int D[\Lambda_a] \;\Rightarrow\; 0 = P^a_{,\mu\nu}\, \xi_+{}^\mu \xi_-{}^\nu - P^a_{,\mu}\, \mathsf{X}^\mu , \tag{2.6}$$

$$\int D[\lambda_{\pm a}] \;\Rightarrow\; 0 = P^a_{,\mu}\, \xi_\pm{}^\mu , \tag{2.7}$$

$$\int D[\mathsf{L}_a] \;\Rightarrow\; 0 = P^a , \tag{2.8}$$

$$\int D[\mathsf{X}^\mu] \;\Rightarrow\; 0 = G_{\mu\overline{\nu}}\, \mathsf{X}^{\overline{\nu}} - \Gamma_{\mu\overline{\nu}\,\overline{\rho}}\, \xi_+{}^{\overline{\nu}} \xi_-{}^{\overline{\rho}} - \Lambda_a\, P^a_{,\mu} , \tag{2.9}$$

with $G_{\mu\overline{\nu}} = \mathbf{K}_{,\mu\overline{\nu}}$ and $\Gamma_{\mu\overline{\nu}\,\overline{\rho}} = \mathbf{K}_{,\mu\overline{\nu}\,\overline{\rho}}$. Contracting Eq. (2.9) with $G^{\mu\overline{\rho}}\, \overline{P}^{\overline{a}}_{,\overline{\rho}}$, we obtain

$$\Lambda_a\, M^{a\overline{a}} = \overline{P}^{\overline{a}}_{,\overline{\mu}}\, \mathsf{X}^{\overline{\mu}} - \Gamma^{\overline{\mu}}{}_{\overline{\rho}\,\overline{\sigma}}\, \overline{P}^{\overline{a}}_{,\overline{\mu}}\, \xi_+{}^{\overline{\rho}} \xi_-{}^{\overline{\sigma}} , \tag{2.10}$$

$$M^{a\overline{a}} \stackrel{\text{def}}{=} (\, P^a_{,\mu}\, G^{\mu\overline{\mu}}\, \overline{P}^{\overline{a}}_{,\overline{\mu}}\,) , \tag{2.11}$$

here and hereafter, $G^{\mu\overline{\mu}}$ is used to raise indices.

2.1 Smooth target space

$M^{a\overline{a}}$ is easily seen to be invertible and actually positive definite on the constrained subspace, provided the matrix of gradients $[P^a_{,\mu}(X)]$ is of rank K where $P^a(X) = 0$, i.e., provided the $\{P^a(X) = 0\}$ subspace is smooth. Then,

$$\Lambda_a = M_{a\overline{a}}\, \overline{P}^{\overline{a}}_{,\overline{\mu}}\, \mathsf{X}^{\overline{\mu}} - M_{a\overline{a}}\, \overline{P}^{\overline{a}}_{,\overline{\mu}}\, \Gamma^{\overline{\mu}}{}_{\overline{\rho}\,\overline{\sigma}}\, \xi_+{}^{\overline{\rho}} \xi_-{}^{\overline{\sigma}} . \tag{2.12}$$

Path-integration over X^μ results in a delta-function that enforces Eq. (2.12).

Note that the matrix of gradients $[P^a_{,\mu}]$ is a proper tensor and independent of any connection Γ_μ on the constrained subspace where $P^a = 0$ and so $P^a_{;\mu} \stackrel{\text{def}}{=} P^a_{,\mu} + \Gamma_\mu \cdot P^a = P^a_{,\mu}$. Furthermore, since the polynomials $P^a(\boldsymbol{X})$ are locally transverse to the subspace $\mathcal{M} \hookrightarrow \mathscr{X}$, the matrix $[P^a_{,\mu}(X)]$ projects onto the normal space to $\mathcal{M}$ at Z^μ.

Upon elimination of the X's and $\overline{\mathsf{X}}$'s by means of their new equations of motion, the Lagrangian in (2.3) contains the four-fermion term

$$\Big[R_{\mu\overline{\nu}\rho\overline{\sigma}} \;+\; P^a_{,\mu;\rho}\, M^{a\overline{a}}\, \overline{P}^{\overline{a}}_{,\overline{\nu};\overline{\sigma}}\Big]\, \zeta_+{}^{\mu}\, \zeta_-{}^{\rho}\, \zeta_+{}^{\overline{\nu}}\, \zeta_-{}^{\overline{\sigma}} \quad , \tag{2.13}$$

where

$$R_{\mu\overline{\nu}\rho\overline{\sigma}} \;\stackrel{\text{def}}{=}\; \mathbf{K}_{,\mu\rho\overline{\nu}\overline{\sigma}} \;-\; \Gamma_{\mu\rho}{}^{\kappa}\, \Gamma_{\kappa\overline{\nu}\overline{\sigma}} \;, \qquad P^a_{,\mu;\nu} \;\stackrel{\text{def}}{=}\; P^a_{,\mu\nu} \;+\; \Gamma_{\mu\nu}{}^{\rho}\, P^a_{,\rho} \tag{2.14}$$

are the standard Riemann tensor on $\mathcal{X}$ and the extrinsic curvature of the constrained subspace, respecively. The sum of the two contributions in Eq. (2.13) is indeed the induced Riemann tensor on the constrained subspace, to which the four-fermion term should couple.

2.2 Singular target space

We now prove the following.

> *For a constrained σ-model to inherit the* (2,2)*-supersymmetry of the ambient σ-model and the possible singularities to be innocuous, it suffices that the constraints have nonzero Taylor series up to and including second order.*

Consider the case where $\text{rank}[P^a_{,\mu}] < K$ at a finite number of points of the $\{P^a = 0\}$ subspace $\mathcal{M}$, which is therefore singular at those points. Also, for ease of notation, we take $K = 1$. At a singular point, both P^1 and $P^1_{,\mu}$ vanish and there Eqs. (2.6) becomes

$$\int D[\Lambda_1] \;\Rightarrow\; 0 = P^1_{,\mu\nu}\, \xi_+{}^{\mu}\, \xi_-{}^{\nu} \;, \tag{2.15}$$

Eq. (2.8) is unchanged and restricts the string coordinates X^{μ} from $\mathcal{X}$ to the 3-dimensional constrained hypersurface $\mathcal{M}$ while Eq. (2.7) becomes trivial, $0 = 0$. However, the $\xi_{\pm}{}^{\mu}$ are now constrained by Eq. (2.15)—as long as the matrix $[P^1_{,\mu\nu}]$ does not vanish. It is easy to see that the $\xi_{\pm}{}^{\mu}$ now span a 3-dimensional cone with the tip at the singularity, as they should, tangential to the constrained hypersurface spanned by the X^{μ} subject to $P^1(X) = 0$.

Note that, for each fixed a, the matrix of second derivatives $[P^a_{,\mu\nu}]$, the so-called Hessian, is independent of any connection at a singularity since both the polynomials P^a and the gradients $P^a_{,\mu}$ vanish there and so

$$P^a_{;\mu\nu} \;\stackrel{\text{def}}{=}\; P^a_{,\mu\nu} \;+\; \Gamma_{\mu,\nu} \cdot P^a \;+\; \Gamma_{\mu\nu}{}^{\rho} P^a_{,\rho} \;=\; P^a_{,\mu\nu} \;.$$

The rank of $[P^a_{,\mu\nu}]$ has therefore an invariant meaning. In case of more constraints, the matrices of second derivatives $P^a_{,\mu\nu}$ form a direct sum and the analogous conclusion follows immediately.

Note also that, at a singularity, Eq. (2.9) becomes

$$\int D[\mathsf{X}^\mu] \quad \Rightarrow \quad 0 = G_{\mu\overline{\nu}}\, \mathsf{X}^{\overline{\nu}} - \Gamma_{\mu\overline{\nu}\,\overline{\sigma}}\, \xi_+^{\overline{\nu}}\, \xi_-^{\overline{\sigma}} \tag{2.16}$$

and decouples from Eqs. (2.7), (2.8) and (2.15). Using this to eliminate all the X^μ's at the singularity, the four-fermion term there becomes $R_{\mu\overline{\nu}\rho\overline{\sigma}}\, \zeta_+{}^\mu\, \zeta_-{}^\rho\, \zeta_+{}^{\overline{\nu}}\, \zeta_-{}^{\overline{\sigma}}$, with the ambient space Riemann tensor defined in (2.14) and the fermions subject to Eq. (2.15). At the singularity, the σ-model decouples from the extrinsic curvature term which occurred in Eq. (2.13) and would have been divergent since $M_{a\overline{a}}$ is the inverse of $M^{a\overline{a}}$ and the latter vanishes at any singularity. Writing $P^a = \phi^a + t\varphi^a$ with ϕ^a singular and φ^a smooth, divergent terms include t^{-1} and are seen to decouple in view of Eq. (2.15).

We remark that all simple (A, D, E), modality-1 and modality-2 singular polynomials (see Ref. [8], p.158–160 of the first book) satisfy this criterion. To see this, take for example the A_k polynomial germ, $f(x) = x^{k+1}$. A germ represents all constraints in a $\mathbb{C}^n$-like neighbourhood which can be brought, through a holomorphic change of local coordinates, to the form

$$f(x_1) \ + \ x_2{}^2 \ + \ \ldots \ + \ x_n{}^2 \ = \ 0 \ . \tag{2.17}$$

They all exhibit the same type of singularity and can be smoothed in the same way. The inclusion of squares of all the local coordinates not involved in the germ is called Morsification. Since the germs for all simple, modality-1 and modality-2 singularities involve less then four local coordinates, their Morsification always contains at least one coordinate occurring as a square and the matrix of second derivatives has at least rank one. It is important to note that the omission of even one local coordinate in Eq. (2.17) represents a *much worse* singularity than is indicated by the germ.

3 The Holomorphic Volume-Form

We have thus shown that constrained supersymmetric $\mathbb{CP}^n$ σ-models with isolated singular points in the target space stemming from non-transversality of their defining equations do inherit the supersymmetry of the $\mathbb{CP}^n$ models. For the resulting constrained σ-model to lead to a 2-dimensional field theory valid for (super)string compactification, we must prove the existence of the nowhere vanishing holomorphic (3,0)-form Ω (which can be suitably generalized for singular spaces [9]).

In the case when the constrained subspace is smooth, Ref. [17] defines Ω explicitly and its existence is equivalent to the vanishing of the first Chern class. For K constraints in a product of m (weighted) complex projective spaces,

$$\Omega \stackrel{\rm def}{=} \oint_{\gamma_{P^1}} \cdots \oint_{\gamma_{P^K}} \frac{\prod_{i=1}^{m} (x^0_{(i)} {\rm d}x^1_{(i)} \ldots {\rm d}x^{n_i}_{(i)})}{P^1(x) \cdots P^K(x)} \quad . \tag{3.1}$$

Here n_i is the dimension of the i^{th} $\mathbb{P}^n$ factor and γ_{P^1} is a contour in $\prod_{i=1}^{m} \mathbb{P}^{n_i}_i$ around the hypersurface defined by $P^1(x) = 0$. The contour integrals are evaluated by means of residues since division by each $P^a(x)$ yields a simple pole at $\mathcal{M}$, where each $P^a(x)$ vanishes but not all its gradients.

Now, when the matrix of gradients $[P^a_{,\mu}]$ is not of maximum rank, division by the $P^a(x)$'s will create singularities worse than a product of K simple poles and to evaluate the expression (3.1), we need to go beyond simple residues. Instead of pursuing this line here, we shall follow the closely related analysis of Ref. [18].

We begin by noting that the holomorphic factor of the volume form of the ambient space can be written locally

$${\rm d}\,Vol(\mathcal{X}) \;=\; \Omega \;\wedge\; {\rm d}P^1 \wedge \ldots \wedge {\rm d}P^K \ . \tag{3.2}$$

Appropriately chosen contour integration will then yield Eq. (3.1). Suppose, for simplicity, $\dim \mathcal{X} = 4$ and $K = 1$. Choose local coordinates $x, y, u, v \in \mathcal{X}$ such that, in a small neighborhood, the origin is the only singularity, $\mathcal{M}$ is given by $\phi(x, y, u, v) = 0$ and a, b, c, d are the respective (integral, positive) scaling weights of x, y, u, v.

It will unfortunately not be possible to discuss all such singularity types as there simply are too many of them [8]. However, the analysis can be repeated for any other $\phi(x, y, u, v)$ of particular interest. Consider for example

$$\phi(x, y, u, v) \;=\; x^{f/a} \;+\; y^{f/b} \;+\; u^{f/c} \;+\; v^{f/d} \ , \tag{3.3}$$

where $f = w(\phi)$, the weight of $f(x, y, u, v)$, is divisible by all four of a, b, c, d. Since locally ${\rm d}\,Vol(\mathcal{X}) = {\rm d}x \wedge {\rm d}y \wedge {\rm d}u \wedge {\rm d}v$, relation (3.2) implies that $w(\Omega) = a{+}b{+}c{+}d{-}f$.

Now, if $w(\Omega) > 0$, i.e., if

$$a \;+\; b \;+\; c \;+\; d \;\;>\;\; f \ , \tag{3.4}$$

then [18] Ω in Eq. (3.1) equals the limit $\lim_{t\to 0} \Omega_t$, obtained from the holomorphic (3,0)-form on the smoothed model with the defining polynomial locally

$$\phi_t(x, y, u, v) \;\stackrel{\rm def}{=}\; \phi(x, y, u, v) \;+\; t\, \varphi(x, y, u, v) \ , \tag{3.5}$$

where $\varphi(x,y,u,v)$ is a smoothing perturbation of $\phi(x,y,u,v)$. Thus, for singularities of the type (3.3), the condition (3.4) ensures the existence of Ω.

Again, all simple (A,D,E), modality-1 and modality-2 singularities satisfy this relation. Take for example the A_{2k} type

$$ {x_1}^{2k+1} \ + \ {x_2}^2 \ + \ {x_3}^2 \ + \ {x_4}^2 \quad = \ 0 \ . $$

Here $w(x_1)=2$, $w(x_2)=w(x_3)=w(x_4)=2k{+}1$ and so $w(\Omega)=2k{+}3>0$.

Note that the value of $w(\Omega)$ of course depends on the choice of coordinates, but its positivity does not. Consider the determinantal conifold of Ref. [6], defined by

$$ X(z)\,Y(z) \ - \ U(z)\,V(z) \quad = \ 0 \ , $$

where X,U are quartic and Y,V are linear in the homogeneous $z^i \in \mathbb{P}^4$. Choosing X,Y,U,V for local coordinates, their weights are $4,1,4,1$ respectively and the defining equation has weight 5; then $w(\Omega)=5$. By a complex linear transformation, the defining equation can be written as $\sum_{i=1}^4 W_i^{\,2}(z)=0$, in which case the weights of all W_i must be the same and can be normalized to 1 so that the defining equation has weight 2; now $w(\Omega)=2$.

4 Remarks and Discussion

4.1 Relation to Landau-Ginzburg models

Since the simple (A,D,E) singularities have found their way into 2-dimensional field theory, it is worth pointing out that here we encounter them in a different way. In Ref. [12,13,14], certain Landau-Ginzburg models have been studied. These may be considered as gauge-fixed version of (2.3), where $\mathbf{\Lambda}_a$ have all been set equal to 1 and some generic D-term has been chosen for $\mathbf{K}(\boldsymbol{X},\overline{\boldsymbol{X}})$. Since the anomalous dimension of D-terms is typically positive, $\mathcal{L}_{\text{kin.}}$ in (2.3) will, typically, supply irrelevant operators to the action. The terms in $\mathcal{L}_{\text{con.}}$ do not get renormalized and determine the renormalization flow fixed point. However, having ignored the D-terms, the field space of the σ-model is not projectivised, and in particular, the origin is not excluded.

As a consequence, polynomials which are smooth in projective spaces become singular. For example, the Fermat quintic $\sum_{i=1}^5 {x_i}^5=0$ is non-singular in $\mathbb{P}^4$ but is singular in $\mathbb{C}^5$. In fact, it decouples into five copies of A_4 singularity in $\mathbb{C}^1$. Most importantly [13], the singular point is isolated and defines the ground state of the Landau-Ginzburg theory. Upon a suitable $\mathbb{Z}_5$ projection, renormalization flow leads to an exactly soluble model, physically equivalent to compactification on the quintic in $\mathbb{P}^4$ [19,13].

The polynomials I discussed here are singular already on the projective spaces. Ignoring projectivity, they have (at least) bouquets of $\mathbb{C}^1$-like singularities meeting at the origin. Naïve Landau-Ginzburg models inherit these bouquets of $\mathbb{C}^1$-like rays as ground states, exhibit continuous degeneracy, have exactly flat directions in the field space and infinte-dimensional chiral ring [20]. If, nevertheless, we take these models seriously, the flat directions would indicate de-compactification. *Per se*, there is nothing wrong with this; cosmologically, compactification may have parallelled the smoothing process. However, this property may simply be the artifact of the particular gauge-fixing by which the Landau-Ginzburg models were obtained from constrained $\mathbb{CP}^n$.

4.2 A peek into the moduli space

As with all Calabi-Yau compactifications, there are two types of moduli fields, corresponding to two types of marginal perturbations of (2.3). One type is often exhausted by listing deformations of the defining polynomials $P^a(\boldsymbol{X})$. These correspond to deformations of the complex structure on $\mathcal{M}$. The other type corresponds to variations of the Kähler class on $\mathcal{M}$ and may be realized as variations of the Kähler potential in (2.3)[♯1].

In general, when the system $P^a(\boldsymbol{X})$ is not transversal, the deformations of the polynomials split into those that preserve the character of the singularity and those that don't, the latter most often producing a smoothing of the singularities. The former span the space of complex structures for the singular space.

Consider for a moment cyclic quotient singularities. Points in the moduli space which correspond to manifolds with a $\mathbb{Z}_n$ symmetry are themselves $\mathbb{Z}_n$-orbifold points, i.e., cyclic quotients of the same order, n. The moduli subspace of $\mathbb{Z}_n$-symmetric manifolds is then the moduli space for the $\mathcal{M}/\mathbb{Z}_n$ quotients. The volume of the singular cone in the moduli space is $\frac{1}{n}$ of that of a flat ball, as it takes n iterations of the symmetry action (a $e^{2i\pi/n}$ rotation) to obtain $\mathbb{1}$.

Consider now a 1-parameter smoothing deformation of a non-transversal system. Let the complex parameter t be chosen so that $t = 0$ corresponds to the singular Calabi-Yau space, while $t \neq 0$ label smooth models. It is easy to check that the aperture of a singularity in the space defined by a non-transversal system $P^a(x) = 0$ is not rational. For the simplest, A_1, singularity in $\mathbb{C}^2$,

$$(x_1 + iy_1)^2 \; + \; (x_2 + iy_2)^2 \;\; = \; 0 \; , \tag{4.1}$$

the cone is two-sheeted and with 90° opening. The volume, i.e., area of the cone is $1/\sqrt{2}$ times that of a flat disk. In analogy with the orbifold example, it would take ∞ iterations

[♯1] After all, we know that all marginal operators can be found as variations of the Kähler potential in the 'intrinsic' formulation with $S^{(2,2)}_{CY}$ (1.1).

of the $e^{2i\pi/\sqrt{2}}$ 'rotations' to obtain $\mathbb{1}$ and we therefore expect that the $t = 0$ point behaves like a cusp in the t-disk (with the volume of the singular cone being $0 = \frac{1}{\infty}$). Indeed, this is verified by explicit calculation[♯2] in the case of the quintic

$$\sum_{i=1}^{5}(z_i)^5 \quad - \quad 5\psi \prod_{i=1}^{5} z_i \quad = \quad 0 \ ,$$

which develops nodes (A_1-singular points) at $\psi^5 = 1$.

Interestingly, the well known moduli space of torus exhibits points with the same qualitative features. The point $\tau = i$ corresponds to a $\mathbb{Z}_2$-symmetric torus and the area of the singular cone (the boundary arcs extending on the two sides of $\tau = i$ are identified) is $\frac{1}{2} \cdot \pi r^2$. The point $\tau = e^{2i\pi/3}$ labels a $\mathbb{Z}_3$-symmetric torus, with the area of the singular cone being $\frac{1}{3} \cdot \pi r^2$. Finally, at $\tau = i\infty$, the torus develops a node (4.1) and the area of the singular cone is 0. This is perhaps easier seen after $\tau \to -\frac{1}{\tau}$, whence the area in question lies between two overlapping tangents of the cusp at $\tau = 0$.

ACKNOWLEDGEMENT. It is a pleasure to thank Philip Candelas and Paul Green for many collaborative efforts which have unquestionably inspired the work presented here.

REFERENCES

[1] P.CANDELAS, G.HOROWITZ, A.STROMINGER and E.WITTEN: *Nucl. Phys.* **B258** (1985)46.

[2] E. WITTEN: *Nucl. Phys.* **B268** (1986)79;
M. DINE, N. SEIBERG, X.-G. WEN and E. WITTEN: *Nucl. Phys.* **B278** (1986)769, *ibid.* **B289** (1987)319;
E. MARTINEC: *Phys. Lett.* **171B** (1986)189;
M. DINE and N. SEIBERG: *Phys. Rev. Lett.* **57** (1986)2625;
M. DINE, N. SEIBERG and E. WITTEN: *Nucl. Phys.* **B289** (1987)589.

[3] J.DISTLER and B.R.GREENE: *Nucl. Phys.* **B309** (1988)295.

[4] J. DISTLER, B. GREENE, K. KIRKLIN and P. MIRON: *Phys. Lett.* **195B** (1987)41;
P. GREEN and T. HÜBSCH: *Class. Quant. Grav.* **6** (1989)311.

[5] P.S. GREEN and T. HÜBSCH: *Phys. Rev. Lett.* **61** (1988)1163, *Commun. Math. Phys.* **119** (1989)431.

[6] P. CANDELAS, P.S. GREEN and T. HÜBSCH: in "Strings '88", p.155, eds. S.J. Gates Jr., C.R. Preitschopf and W. Siegel (World Scientific, Singapore, 1989); *Nucl. Phys.* **B330** (1990)49.

[♯2] I wish to thank P. Candelas and P. Green for discussions on this point and the authors of Ref. [21] for sharing their results with me work prior to publication.

[7] S.-S. Roan and S.-T. Yau: *Acta Math. Sin.* **3**(1987)256;
A. Font, L.E. Ibañez, H.P. Nilles and F. Quevedo: *Nucl. Phys.* **B278** (1986)577;
M. Cvetič: in *Proceedings of the Eight Workshop on Grand Unification*, Syracuse, New York, April 1987; in *Superstrings, Unified Theories and Cosmology 1987*, p.138 (World Scientific, Singapore, 1988), *Phys. Rev. Lett.* **59** (1987)1795, *ibid.* **59** (1987)2989;
C.A. Lütken: *J. Phys.* **A21** (1988)1889.

[8] V.I. Arnold: *London Math. Soc. Lect. Note Series 53 (Cambridge University Press, Cambridge, 1981)*;
V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko: *Singularities of Differentiable Maps* (Birkhäuser, Boston, 1985).

[9] M. Reid: *Proc. Symp. in Pure Math.* **46**(1987)345.

[10] T. Hübsch: *Phys. Lett.* **B** (in press).

[11] L. Dixon, J.A. Harvey, C. Vafa and E. Witten: *Nucl. Phys.* **B261** (1985)678, *ibid.* **B274** (1986)285;
M. Mueller and E. Witten: *Phys. Lett.* **182B** (1986)28;
K.S. Narain, M.H. Sarmadi and C.Vafa: *Nucl. Phys.* **279** (1987)369;
L.E. Ibañez, J. Mas, H.P. Nilles and F. Quevedo: *Nucl. Phys.* **B301** (1988)157.

[12] E.J. Martinec: *Phys. Lett.* **217B** (1989)431, *University of Chicago report* April 1989, to appear in the V.G. Knizhnik memorial volume ed. L. Brink et. al.

[13] C. Vafa and N. Warner: *Phys. Lett.* **218B** (1989)51;
B.R. Greene, C. Vafa and N.P. Warner: *Nucl. Phys.* **B324** (1989)371.

[14] B.R. Greene: *Commun. Math. Phys.* **130** (1990)335.

[15] J.I. Latorre and C.A. Lütken: *Phys. Lett.* **222B** (1989)55.

[16] S.J. Gates Jr. and T. Hübsch: *Phys. Lett.* **226** (1989)100; *Nucl. Phys.* **B** (in press).

[17] P. Candelas: *Nucl. Phys.* **B298** (1988)458.

[18] P.S. Green: Singular Degenerations of Calabi-Yau Manifolds and the Weil-Petersson Metric. *University of Maryland report* May 1989.

[19] D. Gepner: *Phys. Lett.* **199B** (1987)380.

[20] W. Lerche, C. Vafa and N. Warner: *Nucl. Phys.* **B324** (1989)427.

[21] P. Candelas, X. de la Ossa, P. Green and L. Parkes: *University of Texas report* UTTG-25-90.