Quantum Mechanics II

Scattering (3): Formalities

The Lippmann-Schwinger Equation; The S-Matrix & the T-Matrix

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Formal Scattering

The Lippmann-Schwinger Equation

- Start with $\hat{H} = \hat{H}_0 + \hat{W}$ "Resolvents"
- then define $\hat{G}(z) := [z - \hat{H}]^{-1}$ $\hat{G}_0(z) := [z - \hat{H}_0]^{-1}$ $z \in \mathbb{C}$
- and $\hat{T}(z): \quad \hat{G}(z) =: \hat{G}_0(z) + \hat{G}_0(z)\hat{T}(z)\hat{G}_0(z)$

$$\hat{G}_0(z)\hat{T}(z)\hat{G}_0(z) = \hat{G}(z) - \hat{G}_0(z)$$

- From this,

$$\hat{T}\hat{G}_0 = \hat{G}_0^{-1}\hat{G} - \mathbb{1}$$

QM II

Formal Scattering

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$$\hat{G}_0(z)\hat{T}(z)\hat{G}_0(z) = \hat{G}(z) - \hat{G}_0(z)$$

- From this,

$$\hat{T}\hat{G}_0 = \hat{G}_0^{-1}\hat{G} - \mathbb{1} = \hat{G}_0^{-1}\hat{G} - \hat{G}^{-1}\hat{G} = (\hat{G}_0^{-1} - \hat{G}^{-1})\hat{G}$$

$$= (z - \hat{H}_0 - (z - \hat{H}))\hat{G} = \hat{W}\hat{G} \qquad \boxed{\hat{T}\hat{G}_0 = \hat{W}\hat{G}}$$

$$\hat{W}\hat{G} = \hat{G}_0^{-1}\hat{G} - \mathbb{1} \qquad \hat{G}_0\hat{W} = \mathbb{1} - \hat{G}_0\hat{G}^{-1} \qquad \hat{G}_0\hat{G}^{-1} = \mathbb{1} - \hat{G}_0\hat{W}$$

$$\hat{G}_0\hat{T} = \hat{G}\hat{G}_0^{-1} - \mathbb{1} = \cdots = \hat{G}\hat{W}$$

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Formal Scattering

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$$= \left(z - \hat{H}_0 - (z - \hat{H})\right)\hat{G} = \hat{W}\hat{G} \qquad \boxed{\hat{T}\hat{G}_0 = \hat{W}\hat{G}}$$

$$\hat{W}\hat{G} = \hat{G}_0^{-1}\hat{G} - \mathbb{1} \qquad \hat{G}_0\hat{W} = \mathbb{1} - \hat{G}_0\hat{G}^{-1} \qquad \hat{G}_0\hat{G}^{-1} = \mathbb{1} - \hat{G}_0\hat{W}$$

$$\hat{G}_0\hat{T} = \hat{G}\hat{G}_0^{-1} - \mathbb{1} = \cdots = \hat{G}\hat{W} \qquad \hat{G}\hat{G}_0^{-1} = \mathbb{1} + \hat{G}\hat{W}$$

$$\mathbb{1} = \hat{G}_0\hat{G}^{-1}\hat{G}\hat{G}_0^{-1} = [\mathbb{1} - \hat{G}_0\hat{W}][\mathbb{1} + \hat{G}\hat{W}]$$

Formal Scattering

The Lippmann-Schwinger Equation

- Start with $\hat{H} = \hat{H}_0 + \hat{W}$

"Resolvents"

- then define $\hat{G}(z) := [z - \hat{H}]^{-1}$ $\quad \hat{G}_0(z) := [z - \hat{H}_0]^{-1}$ $\quad z \in \mathbb{C}$

- and $\hat{T}(z): \quad \hat{G}(z) =: \hat{G}_0(z) + \hat{G}_0(z)\hat{T}(z)\hat{G}_0(z)$

$$\hat{G}_0(z)\hat{T}(z)\hat{G}_0(z) = \hat{G}(z) - \hat{G}_0(z)$$

- From this,

$$\hat{T}\hat{G}_0 = \hat{G}_0^{-1}\hat{G} - \mathbb{1} = \hat{G}_0^{-1}\hat{G} - \hat{G}^{-1}\hat{G} = (\hat{G}_0^{-1} - \hat{G}^{-1})\hat{G}$$

$$= \left(z - \hat{H}_0 - (z - \hat{H})\right)\hat{G} = \hat{W}\hat{G} \qquad \boxed{\hat{T}\hat{G}_0 = \hat{W}\hat{G}}$$

$$\hat{W}\hat{G} = \hat{G}_0^{-1}\hat{G} - \mathbb{1} \qquad \hat{G}_0\hat{W} = \mathbb{1} - \hat{G}_0\hat{G}^{-1} \qquad \hat{G}_0\hat{G}^{-1} = \mathbb{1} - \hat{G}_0\hat{W}$$

$$\hat{G}_0\hat{T} = \hat{G}\hat{G}_0^{-1} - \mathbb{1} = \cdots = \hat{G}\hat{W} \qquad \hat{G}\hat{G}_0^{-1} = \mathbb{1} + \hat{G}\hat{W}$$

$$\mathbb{1} = \hat{G}_0\hat{G}^{-1}\hat{G}\hat{G}_0^{-1} = [\mathbb{1} - \hat{G}_0\hat{W}][\mathbb{1} + \hat{G}\hat{W}] \qquad \boxed{[\mathbb{1} + \hat{G}\hat{W}] = [\mathbb{1} - \hat{G}_0\hat{W}]^{-1}}$$

Formal Scattering

The Lippmann-Schwinger Equation

$$[\mathbb{1} + \widehat{G}\widehat{W}] = [\mathbb{1} - \widehat{G}_0\widehat{W}]^{-1}$$

$$\widehat{T}\widehat{G}_0 = \widehat{W}\widehat{G}$$

- Now use this:

$$\widehat{G}(z) = [\widehat{G}_0(z) + \widehat{G}_0(z)\widehat{T}(z)\widehat{G}_0(z)]$$
$$= \widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\widehat{G}(z)$$

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Formal Scattering

The Lippmann-Schwinger Equation

$$[\mathbb{1} + \widehat{G}\widehat{W}] = [\mathbb{1} - \widehat{G}_0\widehat{W}]^{-1}$$

$$\widehat{T}\widehat{G}_0 = \widehat{W}\widehat{G}$$

- Now use this:

$$\widehat{G}(z) = \left[\widehat{G}_0(z) + \widehat{G}_0(z)\widehat{T}(z)\widehat{G}_0(z)\right]$$

$$= \widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\widehat{G}(z)$$

$$= \widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\left[\widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\widehat{G}(z)\right]$$

Formal Scattering

$$[\mathbb{1} + \hat{G}\hat{W}] = [\mathbb{1} - \hat{G}_0\hat{W}]^{-1}$$

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The Lippmann-Schwinger Equation

- Now use this:

$$\hat{G}(z) = [\hat{G}_0(z) + \hat{G}_0(z)\hat{T}(z)\hat{G}_0(z)]$$
$$= \hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}(z)$$
$$= \hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\left[\hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}(z)\right]$$
$$= \hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}_0(z)\hat{W}(z)\hat{G}(z)$$

$$\hat{G}(z) = \hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0 + \ldots$$

  - This computes the resolvent of the full Hamiltonian iteratively.
- The T-operator is then expressed similarly:

$$\hat{T}(z) = \hat{W}\hat{G}\hat{G}_0^{-1}$$

Formal Scattering

$$[\mathbb{1} + \widehat{G}\widehat{W}] = [\mathbb{1} - \widehat{G}_0\widehat{W}]^{-1}$$

$$\widehat{T}\widehat{G}_0 = \widehat{W}\widehat{G}$$

The Lippmann-Schwinger Equation

- Now use this:

$$\widehat{G}(z) = [\widehat{G}_0(z) + \widehat{G}_0(z)\widehat{T}(z)\widehat{G}_0(z)]$$

$$= \widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\widehat{G}(z)$$

$$= \widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)[\widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\widehat{G}(z)]$$

$$= \widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\widehat{G}_0(z) + \widehat{G}_0(z)\widehat{W}(z)\widehat{G}_0(z)\widehat{W}(z)\widehat{G}(z)$$

$$\widehat{G}(z) = \widehat{G}_0 + \widehat{G}_0\widehat{W}\widehat{G}_0 + \widehat{G}_0\widehat{W}\widehat{G}_0\widehat{W}\widehat{G}_0 + \widehat{G}_0\widehat{W}\widehat{G}_0\widehat{W}\widehat{G}_0\widehat{W}\widehat{G}_0 + \dots$$

  - This computes the resolvent of the full Hamiltonian iteratively.

- The T-operator is then expressed similarly:

$$\widehat{T}(z) = \widehat{W}\widehat{G}\widehat{G}_0^{-1} = \widehat{W}[\widehat{G}_0 + \widehat{G}_0\widehat{W}\widehat{G}_0 + \widehat{G}_0\widehat{W}\widehat{G}_0\widehat{W}\widehat{G}_0 + \dots]\widehat{G}_0^{-1}$$

Formal Scattering

$$[\mathbb{1} + \hat{G}\hat{W}] = [\mathbb{1} - \hat{G}_0\hat{W}]^{-1}$$

$$\hat{T}\hat{G}_0 = \hat{W}\hat{G}$$

The Lippmann-Schwinger Equation

- Now use this:

$$\begin{aligned}\hat{G}(z) &= [\hat{G}_0(z) + \hat{G}_0(z)\hat{T}(z)\hat{G}_0(z)] \\ &= \hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}(z) \\ &= \hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)[\hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}(z)] \\ &= \hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}_0(z) + \hat{G}_0(z)\hat{W}(z)\hat{G}_0(z)\hat{W}(z)\hat{G}(z)\end{aligned}$$

$$\hat{G}(z) = \hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0 + \ldots$$

  - This computes the resolvent of the full Hamiltonian iteratively.
- The T-operator is then expressed similarly:

$$\begin{aligned}\hat{T}(z) = \hat{W}\hat{G}\hat{G}_0^{-1} &= \hat{W}[\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0 + \ldots]\hat{G}_0^{-1} \\ &= \hat{W}[\mathbb{1} + \hat{G}_0\hat{W} + \hat{G}_0\hat{W}\hat{G}_0\hat{W} + \hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0\hat{W} + \ldots] \\ &= \hat{W} + \hat{W}[\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0 + \hat{G}_0\hat{W}\hat{G}_0\hat{W}\hat{G}_0 + \ldots]\hat{W}\end{aligned}$$

$$\hat{T}(z) = \hat{W} + \hat{W}\hat{G}\hat{W}$$

Formal Scattering

The Lippmann-Schwinger Equation

$$\widehat{G}_0(z) := [z - \widehat{H}_0]^{-1}$$
$$[\mathbb{1} + \widehat{G}\widehat{W}] = [\mathbb{1} - \widehat{G}_0\widehat{W}]^{-1}$$
$$\widehat{T}(z) = \widehat{W} + \widehat{W}\widehat{G}\widehat{W}$$

- So… What does this have to do with scattering?
- Well, consider $\widehat{H}\,|\Psi^{(+)}_{\mathbf{a}}\rangle = [\widehat{H}_0 + \widehat{W}]\,|\Psi^{(+)}_{\mathbf{a}}\rangle = E\,|\Psi^{(+)}_{\mathbf{a}}\rangle$

rearrange: $[E - \widehat{H}_0]\,|\Psi^{(+)}_{\mathbf{a}}\rangle = \widehat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle$ $\qquad [E - \widehat{H}_0]^{-1} = G_0(E^+)$

solve: $|\Psi^{(+)}_{\mathbf{a}}\rangle = |\Phi_{\mathbf{a}}\rangle + \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle$

Formal Scattering

The Lippmann-Schwinger Equation

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rearrange: $[E - \widehat{H}_0]\,|\Psi^{(+)}_{\mathbf{a}}\rangle = \widehat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle$

(a plane wave) the homogeneous solution

solve: $$|\Psi^{(+)}_{\mathbf{a}}\rangle = \underbrace{|\Phi_{\mathbf{a}}\rangle}_{\text{incident}} + \underbrace{\widehat{G}_0(E^+)\,\widehat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle}_{\text{scattering out-going}} \qquad [E - \widehat{H}_0]\,|\Phi_{\mathbf{a}}\rangle = 0$$

$$E^+ := \lim_{\epsilon \to 0}(E + i\epsilon)_{\epsilon \geqslant 0}$$

- …rearrange

$$|\Phi_{\mathbf{a}}\rangle = |\Psi^{(+)}_{\mathbf{a}}\rangle - \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle = [\mathbb{1} - \widehat{G}_0(E^+)\,\widehat{W}]\,|\Psi^{(+)}_{\mathbf{a}}\rangle$$

- …and solve

$$|\Psi^{(+)}_{\mathbf{a}}\rangle = [\mathbb{1} - \widehat{G}_0(E^+)\,\widehat{W}]^{-1}\,|\Phi_{\mathbf{a}}\rangle = [\mathbb{1} + \widehat{G}(E^+)\,\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

Formal Scattering

The Lippmann-Schwinger Equation

$$\hat{G}_0(z) := [z-\hat{H}_0]^{-1}$$
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rearrange: $[E-\hat{H}_0]\,|\Psi^{(+)}_{\mathbf{a}}\rangle = \hat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle$

solve: $|\Psi^{(+)}_{\mathbf{a}}\rangle = \underbrace{|\Phi_{\mathbf{a}}\rangle}_{\text{incident}} + \underbrace{\hat{G}_0(E^+)\,\hat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle}_{\text{scattering out-going}}$

the homogeneous solution (a plane wave): $[E-\hat{H}_0]\,|\Phi_{\mathbf{a}}\rangle = 0$

$$E^+ := \lim_{\epsilon\to 0}(E+i\epsilon)_{\epsilon\geqslant 0}$$

- …rearrange

$$|\Phi_{\mathbf{a}}\rangle = |\Psi^{(+)}_{\mathbf{a}}\rangle - \hat{G}_0(E^+)\,\hat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle = [\mathbb{1}-\hat{G}_0(E^+)\,\hat{W}]\,|\Psi^{(+)}_{\mathbf{a}}\rangle$$

- …and solve

$$|\Psi^{(+)}_{\mathbf{a}}\rangle = [\mathbb{1}-\hat{G}_0(E^+)\,\hat{W}]^{-1}\,|\Phi_{\mathbf{a}}\rangle = [\mathbb{1}+\hat{G}(E^+)\,\hat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

- Now,

$$\hat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle = [\hat{W}+\hat{W}\hat{G}(E^+)\,\hat{W}]\,|\Phi_{\mathbf{a}}\rangle = \hat{T}(E^+)\,|\Phi_{\mathbf{a}}\rangle$$

Formal Scattering

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(a plane wave) the homogeneous solution $[E - \hat{H}_0]\,|\Phi_{\mathbf{a}}\rangle = 0$

solve: $|\Psi^{(+)}_{\mathbf{a}}\rangle = \underbrace{|\Phi_{\mathbf{a}}\rangle}_{\text{incident}} + \underbrace{\hat{G}_0(E^+)\,\hat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle}_{\text{scattering out-going}}$

$E^+ := \lim_{\epsilon \to 0} (E + i\epsilon)_{\epsilon \geqslant 0}$

- …rearrange

$$|\Phi_{\mathbf{a}}\rangle = |\Psi^{(+)}_{\mathbf{a}}\rangle - \hat{G}_0(E^+)\,\hat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle = [\mathbb{1} - \hat{G}_0(E^+)\,\hat{W}]\,|\Psi^{(+)}_{\mathbf{a}}\rangle$$

- …and solve

$$|\Psi^{(+)}_{\mathbf{a}}\rangle = [\mathbb{1} - \hat{G}_0(E^+)\,\hat{W}]^{-1}\,|\Phi_{\mathbf{a}}\rangle = [\mathbb{1} + \hat{G}(E^+)\,\hat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

- Now,

$$\hat{W}\,|\Psi^{(+)}_{\mathbf{a}}\rangle = [\hat{W} + \hat{W}\hat{G}(E^+)\,\hat{W}]\,|\Phi_{\mathbf{a}}\rangle = \hat{T}(E^+)\,|\Phi_{\mathbf{a}}\rangle$$

- …project:

$$\langle\Phi_{\mathbf{b}}|\hat{T}(E^+)|\Phi_{\mathbf{a}}\rangle = \langle\Phi_{\mathbf{b}}|\hat{W}|\Psi^{(+)}_{\mathbf{a}}\rangle$$
$$= -\frac{2\pi\hbar^2|A|^2}{\mu}\,f^{(+)}_{\mathbf{a},b}(\Omega_{kb})$$

Q M II

Formal Scattering

The Lippmann-Schwinger Equation $|\Psi_{\mathbf{a}}^{(+)}\rangle = [\mathbb{1} + \widehat{G}(E^+)\,\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle$

- This results in

$$f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) = -\frac{\mu}{2\pi\hbar^2|A|^2}\langle\Phi_{\mathbf{b}}|\widehat{T}(E^+)|\Phi_{\mathbf{a}}\rangle$$

  - so that the scattering amplitude is (up to the numerical coefficients) the matrix element of the T-(transition)-operator.
  - Given the iterative computation of the T-operator, this provides an iterative computation for the scattering amplitude:

$$\begin{aligned} f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) = -\frac{\mu}{2\pi\hbar^2|A|^2}\langle\Phi_{\mathbf{b}}|\big[\widehat{W} &+ \widehat{W}\widehat{G}_0(E^+)\widehat{W} \\ &+ \widehat{W}\widehat{G}_0(E^+)\widehat{W}\widehat{G}_0(E^+)\widehat{W} \\ &+ \widehat{W}\widehat{G}_0(E^+)\widehat{W}\widehat{G}_0(E^+)\widehat{W}\widehat{G}_0(E^+)\widehat{W} + \ldots\big]\,| \end{aligned}$$

  - …and so on.
- This provides an iterative solution to the Lippmann-Schwinger equation and the scattering amplitude
  - formally analogous to the stationary state perturbation expansion.

Formal Scattering

The T- and the S-Matrix

$$|\Psi_{\mathbf{a}}^{(+)}\rangle = |\Phi_{\mathbf{a}}\rangle + \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi_{\mathbf{a}}^{(+)}\rangle$$
$$|\Psi_{\mathbf{a}}^{(+)}\rangle = [\mathbb{1} + \widehat{G}(E^+)\,\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

- The in- and the out-states are neither complete nor orthogonal
- Given $\widehat{H}\,|\Psi_{\mathbf{a}}^{(+)}\rangle = E_a\,|\Psi_{\mathbf{a}}^{(+)}\rangle \qquad \widehat{H}\,|\Psi_{\mathbf{a}}^{(-)}\rangle = E_a\,|\Psi_{\mathbf{a}}^{(-)}\rangle$
- compute

$$\langle\Psi_{\mathbf{a}}^{(+)}| = |\Psi_{\mathbf{a}}^{(+)}\rangle^\dagger = \Big([\mathbb{1} + \widehat{G}(E_a+i\epsilon)\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle\Big)^\dagger = \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{G}(E_a+i\epsilon)\widehat{W}]^\dagger$$
$$= \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{W}\widehat{G}(E_a-i\epsilon)]$$

$$\langle\Psi_{\mathbf{a}}^{(+)}|\Psi_{\mathbf{b}}^{(+)}\rangle = \langle\Phi_{\mathbf{a}}|[\mathbb{1} + \widehat{W}\widehat{G}(E_a-i\epsilon)]|\Psi_{\mathbf{b}}^{(+)}\rangle$$

Formal Scattering

The T- and the S-Matrix

$$|\Psi_{\mathbf{a}}^{(+)}\rangle = |\Phi_{\mathbf{a}}\rangle + \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi_{\mathbf{a}}^{(+)}\rangle$$
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- compute

$$\langle\Psi_{\mathbf{a}}^{(+)}| = |\Psi_{\mathbf{a}}^{(+)}\rangle^\dagger = \Big([\mathbb{1} + \widehat{G}(E_a{+}i\epsilon)\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle\Big)^\dagger = \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{G}(E_a{+}i\epsilon)\widehat{W}]^\dagger$$
$$= \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{W}\widehat{G}(E_a{-}i\epsilon)]$$

$$\begin{aligned}\langle\Psi_{\mathbf{a}}^{(+)}|\Psi_{\mathbf{b}}^{(+)}\rangle &= \langle\Phi_{\mathbf{a}}|[\mathbb{1} + \widehat{W}\widehat{G}(E_a{-}i\epsilon)]|\Psi_{\mathbf{b}}^{(+)}\rangle \\ &= \langle\Phi_{\mathbf{a}}|\Psi_{\mathbf{b}}^{(+)}\rangle + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a{-}i\epsilon - \widehat{H}]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle \\ &= \langle\Phi_{\mathbf{a}}|\Big(|\Phi_{\mathbf{b}}\rangle + [E_b{+}i\epsilon - \widehat{H}_0]^{-1}\widehat{W}]|\,|\Psi_{\mathbf{b}}^{(+)}\rangle\Big) \\ &\quad + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a{-}i\epsilon - E_b]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle\end{aligned}$$

Formal Scattering

The T- and the S-Matrix

$$|\Psi_{\mathbf{a}}^{(+)}\rangle = |\Phi_{\mathbf{a}}\rangle + \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi_{\mathbf{a}}^{(+)}\rangle$$
$$|\Psi_{\mathbf{a}}^{(+)}\rangle = [\mathbb{1} + \widehat{G}(E^+)\,\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

- The in- and the out-states are neither complete nor orthogonal
- Given $\widehat{H}\,|\Psi_{\mathbf{a}}^{(+)}\rangle = E_a\,|\Psi_{\mathbf{a}}^{(+)}\rangle \qquad \widehat{H}\,|\Psi_{\mathbf{a}}^{(-)}\rangle = E_a\,|\Psi_{\mathbf{a}}^{(-)}\rangle$
- compute

$$\langle\Psi_{\mathbf{a}}^{(+)}| = |\Psi_{\mathbf{a}}^{(+)}\rangle^\dagger = \Big([\mathbb{1} + \widehat{G}(E_a{+}i\epsilon)\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle\Big)^\dagger = \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{G}(E_a{+}i\epsilon)\widehat{W}]^\dagger$$
$$= \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{W}\widehat{G}(E_a{-}i\epsilon)]$$

$$\begin{aligned}
\langle\Psi_{\mathbf{a}}^{(+)}|\Psi_{\mathbf{b}}^{(+)}\rangle &= \langle\Phi_{\mathbf{a}}|[\mathbb{1} + \widehat{W}\widehat{G}(E_a{-}i\epsilon)]|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&= \langle\Phi_{\mathbf{a}}|\Psi_{\mathbf{b}}^{(+)}\rangle + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a{-}i\epsilon - \widehat{H}]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&= \langle\Phi_{\mathbf{a}}|\Big(|\Phi_{\mathbf{b}}\rangle + [E_b{+}i\epsilon - \widehat{H}_0]^{-1}\widehat{W}]|\,|\Psi_{\mathbf{b}}^{(+)}\rangle\Big) \\
&\qquad + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a{-}i\epsilon - E_b]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&= \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle + \langle\Phi_{\mathbf{a}}|[E_b{+}i\epsilon - E_a]^{-1}\widehat{W}]|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&\qquad + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a{-}i\epsilon - E_b]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle
\end{aligned}$$

Formal Scattering

The T- and the S-Matrix

$$|\Psi_{\mathbf{a}}^{(+)}\rangle = |\Phi_{\mathbf{a}}\rangle + \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi_{\mathbf{a}}^{(+)}\rangle$$
$$|\Psi_{\mathbf{a}}^{(+)}\rangle = [\mathbb{1} + \widehat{G}(E^+)\,\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

- The in- and the out-states are neither complete nor orthogonal
- Given $\widehat{H}\,|\Psi_{\mathbf{a}}^{(+)}\rangle = E_a\,|\Psi_{\mathbf{a}}^{(+)}\rangle \qquad \widehat{H}\,|\Psi_{\mathbf{a}}^{(-)}\rangle = E_a\,|\Psi_{\mathbf{a}}^{(-)}\rangle$
- compute

$$\langle\Psi_{\mathbf{a}}^{(+)}| = |\Psi_{\mathbf{a}}^{(+)}\rangle^\dagger = \Big([\mathbb{1} + \widehat{G}(E_a+i\epsilon)\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle\Big)^\dagger = \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{G}(E_a+i\epsilon)\widehat{W}]^\dagger$$
$$= \langle\Phi_{\mathbf{a}}|\,[\mathbb{1} + \widehat{W}\widehat{G}(E_a-i\epsilon)]$$

$$\langle\Psi_{\mathbf{a}}^{(+)}|\Psi_{\mathbf{b}}^{(+)}\rangle = \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle + \frac{\langle\Phi_{\mathbf{a}}|\widehat{W}|\Psi_{\mathbf{b}}^{(+)}\rangle}{E_b+i\epsilon - E_a} + \frac{\langle\Phi_{\mathbf{a}}|\widehat{W}|\Psi_{\mathbf{b}}^{(+)}\rangle}{E_a-i\epsilon - E_b}$$

$$= \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_a-\vec{k}_b)\delta_{a,b} \qquad \text{orthogonal}$$

- Similarly

$$\langle\Psi_{\mathbf{a}}^{(-)}|\Psi_{\mathbf{b}}^{(-)}\rangle = \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_a-\vec{k}_b)\delta_{a,b} \qquad \text{orthogonal}$$

Formal Scattering

The T- and the S-Matrix

$$|\Psi_{\mathbf{a}}^{(+)}\rangle = |\Phi_{\mathbf{a}}\rangle + \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi_{\mathbf{a}}^{(+)}\rangle$$
$$|\Psi_{\mathbf{a}}^{(+)}\rangle = [\mathbb{1} + \widehat{G}(E^+)\,\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

- To prove the $^+/_-$ non-orthogonality, calculate:

$$\begin{aligned}
\langle\Psi_{\mathbf{a}}^{(-)}|\Psi_{\mathbf{b}}^{(+)}\rangle &= \langle\Phi_{\mathbf{a}}|[\mathbb{1} + \widehat{G}(E_a - i\epsilon)\widehat{W}]^\dagger\,|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&= \langle\Phi_{\mathbf{a}}|\Psi_{\mathbf{b}}^{(+)}\rangle + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a + i\epsilon - \widehat{H}]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&= \langle\Phi_{\mathbf{a}}|\Big(|\Phi_{\mathbf{b}}\rangle + [E_b + i\epsilon - \widehat{H}_0]^{-1}\widehat{W}]|\,|\Psi_{\mathbf{b}}^{(+)}\rangle\Big) \\
&\qquad + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a + i\epsilon - E_b]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&= \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle + \langle\Phi_{\mathbf{a}}|[E_b + i\epsilon - E_a]^{-1}\widehat{W}]|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&\qquad + \langle\Phi_{\mathbf{a}}|\widehat{W}[E_a + i\epsilon - E_b]^{-1}|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&= \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle + \frac{\langle\Phi_{\mathbf{a}}|\widehat{W}|\Psi_{\mathbf{b}}^{(+)}\rangle}{E_b + i\epsilon - E_a} + \frac{\langle\Phi_{\mathbf{a}}|\widehat{W}|\Psi_{\mathbf{b}}^{(+)}\rangle}{E_a + i\epsilon - E_b} \\
&= \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle - \frac{2i\epsilon}{(E_a - E_b)^2 + \epsilon^2}\langle\Phi_{\mathbf{a}}|\widehat{W}|\Psi_{\mathbf{b}}^{(+)}\rangle \\
&\xrightarrow{\epsilon\to 0} \langle\Phi_{\mathbf{a}}|\Phi_{\mathbf{b}}\rangle - 2\pi i\,\delta(E_a - E_b)\langle\Phi_{\mathbf{a}}|\widehat{W}|\Psi_{\mathbf{b}}^{(+)}\rangle
\end{aligned}$$

Formal Scattering

The T- and the S-Matrix

$$|\Psi_{\mathbf{a}}^{(+)}\rangle = |\Phi_{\mathbf{a}}\rangle + \widehat{G}_0(E^+)\,\widehat{W}\,|\Psi_{\mathbf{a}}^{(+)}\rangle$$
$$|\Psi_{\mathbf{a}}^{(+)}\rangle = [\mathbb{1} + \widehat{G}(E^+)\,\widehat{W}]\,|\Phi_{\mathbf{a}}\rangle$$

- Finally,

$$\langle\Psi_{\mathbf{a}}^{(-)}|\Psi_{\mathbf{b}}^{(+)}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_a-\vec{k}_b)\delta_{a,b} - 2\pi i\,\delta(E_a-E_b)\langle\Phi_{\mathbf{a}}|\widehat{T}(E_a^+)|\Phi_{\mathbf{b}}\rangle$$

- Define:

$$\begin{aligned} S_{\mathbf{a},\mathbf{b}} &:= \langle\Psi_{\mathbf{a}}^{(-)}|\Psi_{\mathbf{b}}^{(+)}\rangle \\ &= (2\pi)^3\delta^{(3)}(\vec{k}_a-\vec{k}_b)\delta_{a,b} - 2\pi i\,\delta(E_a-E_b)\widehat{T}_{\mathbf{a},\mathbf{b}}(E_a^+) \end{aligned}$$

- This S-matrix “translates” in-states into out-states:

$$\underbrace{|\Psi_{\mathbf{a}}^{(+)}\rangle}_{\text{out}} = \sum\!\!\!\!\!\!\int_{\mathbf{b}} \underbrace{|\Psi_{\mathbf{b}}^{(-)}\rangle}_{\text{in}}\underbrace{\langle\Psi_{\mathbf{b}}^{(-)}|\Psi_{\mathbf{a}}^{(+)}\rangle}_{S_{\mathbf{b},\mathbf{a}}} = \frac{1}{(2\pi)^3}\sum_b\int \mathrm{d}^3\vec{k}_b\, S_{\mathbf{b},\mathbf{a}}\,\underbrace{|\Psi_{\mathbf{b}}^{(-)}\rangle}_{\text{in}}$$

- The δ-functions in the S-matrix insure that only states of the same energy are mixed (conservation of energy)

Formal Scattering

The T- and the S-Matrix

- The S-matrix is a unitary matrix:

$$|\Psi_{\mathbf{a}}^{(+)}\rangle = \sum\!\!\!\!\!\!\int_{\mathbf{b}} |\Psi_{\mathbf{b}}^{(-)}\rangle\, S_{\mathbf{b},\mathbf{a}} = \sum\!\!\!\!\!\!\int_{\mathbf{b}} |\Psi_{\mathbf{b}}^{(-)}\rangle \langle \Psi_{\mathbf{b}}^{(-)}|\Psi_{\mathbf{a}}^{(+)}\rangle$$

$$|\Psi_{\mathbf{b}}^{(-)}\rangle = \sum\!\!\!\!\!\!\int_{\mathbf{a}} |\Psi_{\mathbf{a}}^{(+)}\rangle \langle \Psi_{\mathbf{a}}^{(+)}|\Psi_{\mathbf{b}}^{(-)}\rangle = \sum\!\!\!\!\!\!\int_{\mathbf{a}} |\Psi_{\mathbf{a}}^{(+)}\rangle\, (\mathbb{S}^{-1})_{\mathbf{a},\mathbf{b}}$$

$$(\mathbb{S}^{-1})_{\mathbf{a},\mathbf{b}} = \langle \Psi_{\mathbf{a}}^{(+)}|\Psi_{\mathbf{b}}^{(-)}\rangle = (S_{\mathbf{b},\mathbf{a}})^* = (\mathbb{S}^{\dagger})_{\mathbf{a},\mathbf{b}}$$

  - ...but it is not the matrix representation of a unitary operator!
  - ...since scattering states do not form a complete basis.
- The S-matrix is only defined between states of the same energy:

$$S_{\mathbf{a},\mathbf{b}} = \delta_{\mathbf{a},\mathbf{b}} - 2\pi i\, \delta(E_a - E_b)\, T_{\mathbf{a},\mathbf{b}}(E_a^+) \qquad \delta_{\mathbf{a},\mathbf{b}} = (2\pi)^3 \delta^3(\vec{k}_a - \vec{k}_b)\delta_{a,b}$$

  - ...while the T-matrix is defined for all states:

$$T_{\mathbf{a},\mathbf{b}}(E) = \langle \Phi_{\mathbf{a}}|\widehat{T}(E)|\Phi_{\mathbf{b}}\rangle$$

$$= \langle \Phi_{\mathbf{a}}|\widehat{W} + \widehat{G}_0\widehat{W}\widehat{G}_0 + \widehat{G}_0\widehat{W}\widehat{G}_0\widehat{W}\widehat{G}_0 + \ldots |\Phi_{\mathbf{b}}\rangle$$

Formal Scattering

The T- and the S-Matrix

- Unitarity of the S-matrix implies a condition for the T-matrix

$$\begin{aligned}\delta_{\mathbf{a},\mathbf{c}} &= (\mathbb{S}^\dagger)_{\mathbf{a},\mathbf{b}}\, S_{\mathbf{b},\mathbf{c}} = S^*_{\mathbf{b},\mathbf{a}}\, S_{\mathbf{b},\mathbf{c}} \\ &= [\delta_{\mathbf{b},\mathbf{a}} + 2\pi i\, \delta(E_b - E_a) T^*_{\mathbf{b},\mathbf{a}}][\delta_{\mathbf{b},\mathbf{c}} - 2\pi i\, \delta(E_b - E_c) T_{\mathbf{b},\mathbf{c}}] \\ &= \delta_{\mathbf{a},\mathbf{c}} + 2\pi i\, \delta(E_c - E_a)[T^*_{\mathbf{c},\mathbf{a}} - T_{\mathbf{a},\mathbf{c}}] \\ &\qquad + 4\pi^2 \sum\!\!\!\!\!\!\int_{\mathbf{b}} \delta(E_b - E_a)\delta(E_b - E_c)\, T^*_{\mathbf{b},\mathbf{a}} T_{\mathbf{b},\mathbf{c}}\end{aligned}$$

- Now,

$$\delta(E_b - E_a)\delta(E_b - E_c) = \delta(E_b - E_a)\delta(E_a - E_c)$$

Formal Scattering

The T- and the S-Matrix

- Unitarity of the S-matrix implies a condition for the T-matrix

$$
\begin{aligned}
\cancel{\delta_{\mathbf{a,c}}} &= (\mathbb{S}^\dagger)_{\mathbf{a,b}}\, S_{\mathbf{b,c}} = S^*_{\mathbf{b,a}}\, S_{\mathbf{b,c}} \\
&= [\delta_{\mathbf{b,a}} + 2\pi i\, \delta(E_b - E_a) T^*_{\mathbf{b,a}}][\delta_{\mathbf{b,c}} - 2\pi i\, \delta(E_b - E_c) T_{\mathbf{b,c}}] \\
&= \cancel{\delta_{\mathbf{a,c}}} + 2\pi i\, \delta(E_c - E_a)[T^*_{\mathbf{c,a}} - T_{\mathbf{a,c}}] \\
&\qquad + 4\pi^2 \sum\!\!\!\!\!\!\int_{\mathbf{b}} \delta(E_b - E_a)\delta(E_b - E_c)\, T^*_{\mathbf{b,a}} T_{\mathbf{b,c}}
\end{aligned}
$$

- Now,

$$
\delta(E_b - E_a)\delta(E_b - E_c) = \delta(E_b - E_a)\delta(E_a - E_c) \qquad \delta(E_c - E_a) = \delta(E_a - E_c)
$$

- so

$$
0 = \frac{i}{2\pi}\cancel{\delta(E_a - E_c)}[T^*_{\mathbf{c,a}} - T_{\mathbf{a,c}}] + \sum\!\!\!\!\!\!\int_{\mathbf{b}} \delta(E_b - E_a)\cancel{\delta(E_a - E_c)}\, T^*_{\mathbf{b,a}} T_{\mathbf{b,c}}
$$

- *i.e.*,

$$
\frac{1}{2\pi i}[T^*_{\mathbf{c,a}} - T_{\mathbf{a,c}}] = \sum\!\!\!\!\!\!\int_{\mathbf{b}} \delta(E_b - E_a)\, T^*_{\mathbf{b,a}} T_{\mathbf{b,c}}
$$

QM II

Formal Scattering

The T- and the S-Matrix

$$T_{\mathbf{c},\mathbf{a}}(E_a^+) = -\frac{2\pi\hbar^2}{\mu}\, f^{(+)}_{\mathbf{a},c}(\Omega_{kc})$$

$$\frac{1}{2\pi i}\big[T^*_{\mathbf{c},\mathbf{a}} - T_{\mathbf{a},\mathbf{c}}\big] = \sum\!\!\!\!\!\!\!\int_{\mathbf{b}} \delta(E_b - E_a)\, T^*_{\mathbf{b},\mathbf{a}} T_{\mathbf{b},\mathbf{c}}$$

- For $\mathbf{c} = \mathbf{a}$, this implies

$$\frac{1}{2\pi i}\big[T^*_{\mathbf{a},\mathbf{a}} - T_{\mathbf{a},\mathbf{a}}\big] = \sum\!\!\!\!\!\!\!\int_{\mathbf{b}} \delta(E_b - E_a)\, T^*_{\mathbf{b},\mathbf{a}} T_{\mathbf{b},\mathbf{a}}$$

$$-\frac{1}{\pi}\,\Im m[T_{\mathbf{a},\mathbf{a}}] = \frac{1}{(2\pi)^3}\sum_b \int \mathrm{d}^3\vec{k}_b\, \delta(E_b - E_a)\, |T_{\mathbf{b},\mathbf{a}}|^2$$

$$\frac{2\hbar^2}{\mu}\,\Im m[f_{\mathbf{a},a}(\theta=0)] = \frac{1}{(2\pi)^3}\sum_b \int k_b^2 \mathrm{d}k_b \int \mathrm{d}^2\Omega_{kb}\, \delta\Big(\frac{\hbar^2 k_b^2}{2\mu} - \frac{\hbar^2 k_a^2}{2\mu}\Big)\, |T_{\mathbf{b},\mathbf{a}}|^2$$

$$= \frac{1}{(2\pi)^3}\sum_b \int k_b^2 \mathrm{d}k_b \int \mathrm{d}^2\Omega_{kb}\, \frac{\mu}{\hbar^2 k_b}\delta(k_b - k_a)\, |T_{\mathbf{b},\mathbf{a}}|^2$$

$$= \frac{1}{(2\pi)^3}\frac{\mu}{\hbar^2}\sum_b k_b \int \mathrm{d}^2\Omega_{kb}\, |T_{\mathbf{b},\mathbf{a}}|^2$$

forward scattering *amplitude*

!!!

scattering *probability* summed over all out-going states

Formal Scattering

The T- and the S-Matrix

- So, the unitarity of the S-matrix implies

$$\frac{2\hbar^2}{\mu}\Im m[f_{\mathbf{a},a}(\theta=0)] = \frac{1}{(2\pi)^3}\frac{\mu}{\hbar^2}\sum_b k_b \int d^2\Omega_{kb}\,|T_{\mathbf{b},\mathbf{a}}|^2$$

- Now,

$$\underset{\text{all-inclusive}}{\sigma_T} := \sigma_{\mathbf{a}\to\text{all}} = \sum_b \int d^2\Omega_{kb}\,\sigma_{\mathbf{a}\to\mathbf{b}}(\Omega_{kb}) = \sum_b \int d^2\Omega_{kb}\,\frac{k_b}{k_a}|f_{\mathbf{a}b}(\Omega_{kb})|^2$$

$$= \sum_b \int d^2\Omega_{kb}\,\frac{k_b}{k_a}\left|\tfrac{\mu}{2\pi\hbar^2}T_{\mathbf{b},\mathbf{a}}\right|^2 = \frac{\mu^2}{4\pi^2\hbar^4 k_a}\sum_b k_b \int d^2\Omega_{kb}\,|T_{\mathbf{b},\mathbf{a}}|^2$$

Formal Scattering

The T- and the S-Matrix

- So, the unitarity of the S-matrix implies

$$\frac{2\hbar^2}{\mu}\Im m[f_{\mathbf{a},a}(\theta=0)] = \frac{1}{(2\pi)^3}\frac{\mu}{\hbar^2}\sum_b k_b \int d^2\Omega_{kb}\,|T_{\mathbf{b},\mathbf{a}}|^2$$

- Now,

$$\underset{\text{all-inclusive}}{\sigma_T} := \sigma_{\mathbf{a}\to\text{all}} = \sum_b \int d^2\Omega_{kb}\,\sigma_{\mathbf{a}\to\mathbf{b}}(\Omega_{kb}) = \sum_b \int d^2\Omega_{kb}\,\frac{k_b}{k_a}|f_{\mathbf{ab}}(\Omega_{kb})|^2$$

$$= \sum_b \int d^2\Omega_{kb}\,\frac{k_b}{k_a}\left|\frac{\mu}{2\pi\hbar^2}T_{\mathbf{b},\mathbf{a}}\right|^2 = \frac{\mu^2}{4\pi^2\hbar^4 k_a}\sum_b k_b \int d^2\Omega_{kb}\,|T_{\mathbf{b},\mathbf{a}}|^2$$

$$= \frac{2\pi\mu}{\hbar^2 k_a}\boxed{\frac{1}{(2\pi)^3}\frac{\mu}{\hbar^2}\sum_b k_b \int d^2\Omega_{kb}\,|T_{\mathbf{b},\mathbf{a}}|^2} = \frac{2\pi\mu}{\hbar^2 k_a}\boxed{\frac{2\hbar^2}{\mu}\Im m[f_{\mathbf{a},a}(\theta=0)]}$$

- Finally, we have:

$$\boxed{\sigma_T = \frac{4\pi}{k_a}\Im m[f_{\mathbf{a},a}(\theta=0]}$$

optical theorem

Formal Scattering

The T- and the S-Matrix

- Using $S_{\mathbf{a},\mathbf{b}} := \langle \Psi_{\mathbf{a}}^{(-)} | \Psi_{\mathbf{b}}^{(+)} \rangle$ derive a few symmetry properties
- Time-reversal: $\widehat{T}\,|\Psi_{\mathbf{a}}^{(+)}\rangle = |\Psi_{T\mathbf{a}}^{(-)}\rangle \quad T\mathbf{a} = \widehat{T}(\vec{k}_a, a) = (-\vec{k}_a, Ta)$

$$\widehat{T}(S_{\mathbf{b},\mathbf{a}}) = \widehat{T}\langle \Psi_{\mathbf{b}}^{(-)} | \Psi_{\mathbf{a}}^{(+)} \rangle = \langle \Psi_{T\mathbf{b}}^{(+)} | \Psi_{T\mathbf{a}}^{(-)} \rangle = \langle \Psi_{T\mathbf{a}}^{(-)} | \Psi_{T\mathbf{b}}^{(+)} \rangle^* = S^*_{T\mathbf{a},T\mathbf{b}}$$

- On the other hand, time-reversal is antilinear $\widehat{T}(S_{\mathbf{b},\mathbf{a}}) = S^*_{\mathbf{b},\mathbf{a}}$
- This produces

equalities among scattering amplitudes

conjugation →

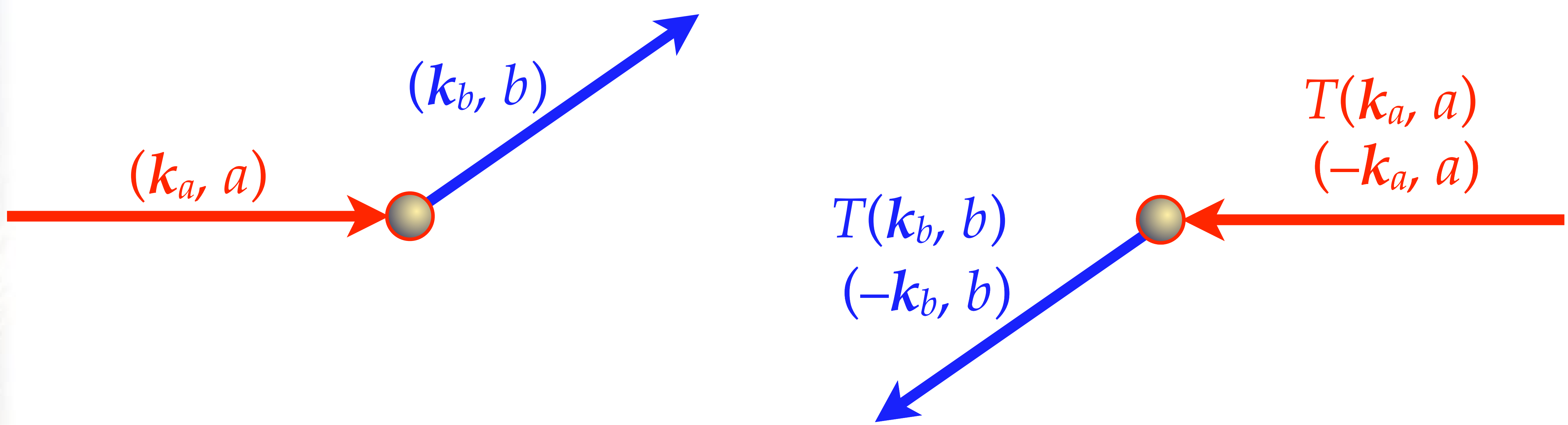
$$\begin{array}{ccccc} S^*_{T\mathbf{a},T\mathbf{b}} & = & \widehat{T}(S_{\mathbf{b},\mathbf{a}}) & = & S^*_{\mathbf{b},\mathbf{a}} \\ \downarrow * & & \downarrow * & & \downarrow * \\ S_{T\mathbf{a},T\mathbf{b}} & = & \widehat{T}(S^*_{\mathbf{b},\mathbf{a}}) & = & S_{\mathbf{b},\mathbf{a}} \end{array}$$

- For cross-sections, $k_a{}^2\,\sigma_{T\mathbf{a}\to T\mathbf{b}} = k_b{}^2\,\sigma_{\mathbf{b}\to\mathbf{a}}$, from the bottom line
- If $(T\mathbf{a}, T\mathbf{b}) = (\mathbf{a},\mathbf{b})$ up to 180°-rot (T-inv. int. states; e.g., no spin)

Formal Scattering

The T- and the S-Matrix

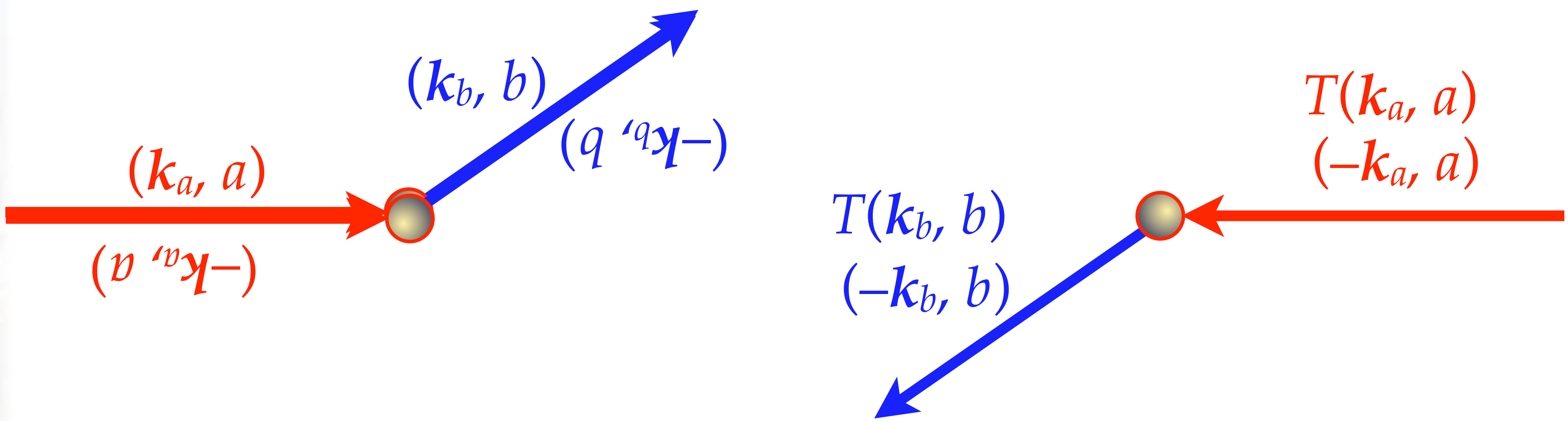
- If $(T\mathbf{a},T\mathbf{b}) = (\mathbf{a},\mathbf{b})$ up to 180°-rot (T-inv. int. states; e.g., no spin)



Formal Scattering

The T- and the S-Matrix

- If $(T\mathbf{a}, T\mathbf{b}) = (\mathbf{a}, \mathbf{b})$ up to 180°-rot (T-inv. int. states; e.g., no spin)



QM II

Formal Scattering

The T- and the S-Matrix

- Using $S_{\mathbf{a},\mathbf{b}} := \langle \Psi^{(-)}_{\mathbf{a}} | \Psi^{(+)}_{\mathbf{b}} \rangle$ derive a few symmetry properties
- Time-reversal: $\widehat{T}\,|\Psi^{(+)}_{\mathbf{a}}\rangle = |\Psi^{(-)}_{T\mathbf{a}}\rangle \quad T\mathbf{a} = \widehat{T}(\vec{k}_a, a) = (-\vec{k}_a, Ta)$

$$\widehat{T}(S_{\mathbf{b},\mathbf{a}}) = \widehat{T}\langle \Psi^{(-)}_{\mathbf{b}} | \Psi^{(+)}_{\mathbf{a}} \rangle = \langle \Psi^{(+)}_{T\mathbf{b}} | \Psi^{(-)}_{T\mathbf{a}} \rangle = \langle \Psi^{(-)}_{T\mathbf{a}} | \Psi^{(+)}_{T\mathbf{b}} \rangle^* = S^*_{T\mathbf{a},T\mathbf{b}}$$

- On the other hand, time-reversal is antilinear $\widehat{T}(S_{\mathbf{b},\mathbf{a}}) = S^*_{\mathbf{b},\mathbf{a}}$
- This produces

equalities among scattering amplitudes

conjugation →

$$\begin{array}{ccccc} S^*_{T\mathbf{a},T\mathbf{b}} & = & \widehat{T}(S_{\mathbf{b},\mathbf{a}}) & = & S^*_{\mathbf{b},\mathbf{a}} \\ \downarrow * & & \downarrow * & & \downarrow * \\ S_{T\mathbf{a},T\mathbf{b}} & = & \widehat{T}(S^*_{\mathbf{b},\mathbf{a}}) & = & S_{\mathbf{b},\mathbf{a}} \end{array}$$

- For cross-sections, $k_a{}^2\,\sigma_{T\mathbf{a}\to T\mathbf{b}} = k_b{}^2\,\sigma_{\mathbf{b}\to\mathbf{a}}$, from the bottom line
- If $(T\mathbf{a}, T\mathbf{b}) = (\mathbf{a},\mathbf{b})$ up to 180°-rot (T-inv. int. states; e.g., no spin)
  - $(S_{T\mathbf{a},T\mathbf{b}} = S_{\mathbf{a},\mathbf{b}}) = S_{\mathbf{b},\mathbf{a}}$ the S-matrix is symmetric; detailed balance holds and reverse (inverse) scattering has the same scattering amplitude

Quantum Mechanics II

Now, go forth and calculate!!

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http://physics1.howard.edu/~thubsch/