Quantum Mechanics II

Scattering (2): The Quantum Theory

The General Theory; The Scattering Amplitude Theorem; Applications & the Born Approximation

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Quantum Scattering

General Theory

- Scattering types:
  - Elastic scattering: $A + X \rightarrow A + X$
    - Kinetic energy is conserved; linear & angular momenta are conserved
    - Ex.: billiard balls, marbles,
  - Inelastic scattering: $A + X \rightarrow A^* + X^*$
    - Kinetic energy is *not* conserved, *total* energy is; must include internal energy

Quantum Scattering

General Theory

- Scattering types:
  - Elastic scattering: $A + X \to A + X$
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  - Inelastic scattering: $A + X \to A^* + X^*$
    - Kinetic energy is *not* conserved, *total* energy is; must include internal energy
    - Ex.: traffic collisions; vehicles and people absorb some of the energy
  - Rearrangement collision: $A + X \to B + Y$
    - Parts of the subsystems A and X get rearranged/exchanged
    - Ex.: nuclear and chemical reactions
  - Particle production: $A + X \to B + C + D + \ldots + Y + Z$
    - The final state consists of more than two (n) separate particles
    - ($n - 2$) of which are therefore *produced* by the collision
    - Total energy-momentum being conserved, some of the kinetic energy of the colliding objects A and X is transformed into the masses of the created particles

In fact, total energy-momentum and angular momentum are always conserved

Quantum Scattering

General Theory

- Focus on elastic and inelastic scattering / collisions
  - elastic scattering involves no exchange with internal energy
  - inelastic scattering involves some exchange with internal energy
- In- and out-states both involve two particles **work in CM frame**
  - For each of the different possible outcomes
  - …calculate a separate scattering amplitude and cross-section
  - The <u>*total*</u> cross-section is the the sum of these
  - The relative ratios are called branching ratios
- Set-up: $\widehat{H} = \underbrace{-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 + \widehat{h}_1 + \widehat{h}_2}_{=\widehat{H}_0} + \widehat{W}$ $\qquad \underbrace{[\widehat{h}_1 + \widehat{h}_2]\chi_a = \epsilon_a \chi_a}_{\text{"internal"}}$
  - separation of variables:

$$\widehat{H}\,\Psi_{\mathbf{a}}^{(+)} = E_{\mathbf{a}}\,\Psi_{\mathbf{a}}^{(+)} \qquad \Psi_{\mathbf{a}}^{(+)} = \sum_{a'} \psi_{\mathbf{a},a'}^{(+)}(\vec{r})\,\chi_{a'} \qquad \mathbf{a} := (\vec{k}_a, a)$$

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General Theory

$$\Psi_{\mathbf{a}}^{(+)} = \sum_{a'} \psi_{\mathbf{a},a'}^{(+)}(\vec{r})\,\chi_{a'}$$

- The general solution is expected in the form

the incident state: *same* state

$$\psi_{\mathbf{a}}^{(+)}(\vec{r}) \sim A\Big\{ \underbrace{e^{i\vec{k}_a\cdot\vec{r}}}_{\text{incident}} + \underbrace{f_{\mathbf{a}\,a}^{(+)}(\Omega_r)\frac{e^{+ik_a r}}{r}}_{\textit{elastic}\text{-scattered}} \Big\}$$

any *other* state: $\chi_b \neq \chi_a$

$$\psi_{\mathbf{b}}^{(+)}(\vec{r}) \sim A\Big\{ \underbrace{f_{\mathbf{a},b}^{(+)}(\Omega_r)\frac{e^{+ik_b r}}{r}}_{\textit{inelastic}\text{-scattered}} \Big\}_{b\neq a}$$

out-going spherical wave

- The probability currents are

$$J_{I\,(\mathbf{a})} = |A|^2\frac{\hbar k_a}{\mu} \qquad J_{S\,(\mathbf{a})} = |A|^2\frac{\hbar k_a}{\mu}\frac{1}{r^2}\big|f_{\mathbf{a},a}^{(+)}(\Omega_r)\big|^2 \qquad J_{S\,(\mathbf{b})} = |A|^2\frac{\hbar k_b}{\mu}\frac{1}{r^2}\big|f_{\mathbf{a},b}^{(+)}(\Omega_r)\big|^2$$

- …and the cross-sections

$$\sigma_{\mathbf{a},a}(\theta_r) = \big|f_{\mathbf{a},a}^{(+)}(\Omega_r)\big|^2 \qquad \sigma_{\mathbf{a},b}(\theta_r) = \frac{k_b}{k_a}\big|f_{\mathbf{a},b}^{(+)}(\Omega_r)\big|^2$$

- Aim to compute the scattering amplitudes

Quantum Scattering

General Theory

- These wave-functions, after all, satisfy a Sturm-Liouville type 2nd order partial differential equation
  - …and there is always the "other" solution

$$\underbrace{\frac{e^{+ik_a r}}{r}}_{out} \leftrightarrow \underbrace{e^{ikx}}_{right} \quad \text{just as} \quad \underbrace{e^{-ikx}}_{left} \leftrightarrow \underbrace{\frac{e^{-ik_a r}}{r}}_{in}$$

$$e^{-i\omega t}e^{ikx} = \underbrace{e^{-i(\omega t - kx)}}_{\text{stationary } @\, x = \frac{\omega}{k}t}$$

as t grows, so does x

- So, include

$$\widehat{H}\,\Psi_{\mathbf{b}}^{(-)} = E_{\mathbf{b}}\,\Psi_{\mathbf{b}}^{(-)} \qquad \Psi_{\mathbf{b}}^{(-)} = \sum_{b'} \psi_{\mathbf{b},b'}^{(-)}(\vec{r})\,\chi_{b'} \qquad \mathbf{b} := (\vec{k}_b, b)$$

$$\psi_{\mathbf{b}}^{(-)}(\vec{r}) \sim A\Big\{\underbrace{e^{i\vec{k}_b\cdot\vec{r}}}_{\text{plane}} + \underbrace{f_{\mathbf{b},b}^{(-)}(\Omega_r)\frac{e^{-ik_b r}}{r}}_{\text{captured}}\Big\}$$

$$\psi_{\mathbf{c}}^{(-)}(\vec{r}) \sim A\Big\{\underbrace{f_{\mathbf{b},c}^{(-)}(\Omega_r)\frac{e^{-ik_c r}}{r}}_{\text{captured}}\Big\}_{c\neq b} \qquad \chi_c \neq \chi_b$$

in-coming spherical wave

Use

$$\{\,\Psi_{\mathbf{a}}^{(+)}\,\} \quad \& \quad \{\,\Psi_{\mathbf{b}}^{(-)}\,\}$$

Quantum Scattering

Scattering Amplitude Theorem

- Scattering states are not "finitely normalized"

$$\langle\chi_a|\chi_b\rangle = \delta_{a,b} \quad \text{but} \quad \langle\psi_{\mathbf{a}}^{(+)}|\psi_{\mathbf{b}}^{(+)}\rangle \propto \delta(\mathbf{a}-\mathbf{b}) \quad \langle\psi_{\mathbf{a}}^{(+)}|\psi_{\mathbf{a}}^{(+)}\rangle = \infty$$

$$\langle\psi_{\mathbf{a}}^{(+)}|\psi_{\mathbf{b}}^{(+)}\rangle_R := \int_0^R r^2 \mathrm{d}r \int \mathrm{d}^2\Omega\, \psi_{\mathbf{a}}^{*(+)}(\vec{r})\, \psi_{\mathbf{b}}^{(+)}(\vec{r})$$

- But, scattering states do not vanish at the boundary *(neither finite nor infinite!)*
  - …and so the Laplacian is not Hermitian b/c of boundary terms
  - Derive the consequences!

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Quantum Scattering

Scattering Amplitude Theorem

- Scattering states are not “finitely normalized”

$$\langle\chi_a|\chi_b\rangle = \delta_{a,b} \quad \text{but} \quad \langle\psi_{\mathbf{a}}^{(+)}|\psi_{\mathbf{b}}^{(+)}\rangle \propto \delta(\mathbf{a}-\mathbf{b}) \qquad \langle\psi_{\mathbf{a}}^{(+)}|\psi_{\mathbf{a}}^{(+)}\rangle = \infty$$

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- But, scattering states do not vanish at the boundary *(neither finite nor infinite!)*
  - …and so the Laplacian is not Hermitian b/c of boundary terms
  - Derive the consequences!
- Consider: $\widehat{H} = \widehat{H}_0 + \widehat{W} \qquad \widehat{H}' = \widehat{H}_0 + \widehat{W}'$

$$\widehat{H}\,\Psi_{\mathbf{a}}^{(+)} = E_{\mathbf{a}}\,\Psi_{\mathbf{a}}^{(+)} \qquad E_{\mathbf{b}} = E = E_{\mathbf{a}} \qquad \widehat{H}'\,\Psi_{\mathbf{b}}'^{(-)} = E_{\mathbf{b}}\,\Psi_{\mathbf{b}}'^{(-)}$$

$$\frac{\hbar^2}{2\mu}\vec{\nabla}^2\Psi_{\mathbf{a}}^{(+)}(\vec{r}) = \big[\hat{h}_1 + \hat{h}_2 + \widehat{W} - E\big]\Psi_{\mathbf{a}}^{(+)}(\vec{r}) \qquad ⚡\ \Psi_{\mathbf{b}}'^{*(-)}(\vec{r})\cdot$$

$$\frac{\hbar^2}{2\mu}\vec{\nabla}^2\Psi_{\mathbf{b}}'^{(-)}(\vec{r}) = \big[\hat{h}_1 + \hat{h}_2 + \widehat{W}' - E\big]\Psi_{\mathbf{b}}'^{(-)}(\vec{r}) \qquad ⚡\ \Psi_{\mathbf{a}}^{*(+)}(\vec{r})\cdot$$

Quantum Scattering

Scattering Amplitude Theorem

- After pre-multiplying

$$\frac{\hbar^2}{2\mu}\Psi'^{*(-)}_{\mathbf{b}}(\vec r)\,\vec\nabla^2\Psi^{(+)}_{\mathbf{a}}(\vec r) = \Psi'^{*(-)}_{\mathbf{b}}(\vec r)\left[\widehat{h}_1+\widehat{h}_2+\widehat{W}-E\right]\Psi^{(+)}_{\mathbf{a}}(\vec r)$$
$$= \Psi'^{*(-)}_{\mathbf{b}}(\vec r)\left[\epsilon_a+\widehat{W}-E\right]\Psi^{(+)}_{\mathbf{a}}(\vec r)$$

Hermitian!

$$\frac{\hbar^2}{2\mu}\Psi^{*(+)}_{\mathbf{a}}(\vec r)\,\vec\nabla^2\Psi'^{(-)}_{\mathbf{b}}(\vec r) = \Psi^{*(+)}_{\mathbf{a}}(\vec r)\left[\widehat{h}_1+\widehat{h}_2+\widehat{W}'-E\right]\Psi'^{(-)}_{\mathbf{b}}(\vec r)$$
$$= \Psi^{*(+)}_{\mathbf{a}}(\vec r)\left[\epsilon_a+\widehat{W}'-E\right]\Psi'^{(-)}_{\mathbf{b}}(\vec r)$$

- The l.h.s. diff. would vanish if the Laplacian was Hermitian
- The $r \leq R$ integrated difference is

$$\frac{\hbar^2}{2\mu}\left\{\langle\Psi^{(-)}_{\mathbf{b}}|\vec\nabla^2\Psi^{(+)}_{\mathbf{a}}\rangle_R-\langle\Psi^{(+)}_{\mathbf{a}}|\vec\nabla^2\Psi^{(-)}_{\mathbf{b}}\rangle_R\right\} = \langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}|\Psi^{(+)}_{\mathbf{a}}\rangle_R-\langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}'|\Psi^{(+)}_{\mathbf{a}}\rangle_R$$

- Use:

$$u\vec\nabla^2 v - v\vec\nabla^2 u = \left(\vec\nabla\cdot(u\vec\nabla v)-\cancel{(\vec\nabla u)\cdot(\vec\nabla v)}\right)-\left(\vec\nabla\cdot(v\vec\nabla u)-\cancel{(\vec\nabla v)\cdot(\vec\nabla u)}\right)$$
$$= \vec\nabla\cdot\left((u\vec\nabla v)-(v\vec\nabla u)\right)$$

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Quantum Scattering

Scattering Amplitude Theorem

- Using Gauss's theorem $\int_V \mathrm{d}^3\vec{r}\,\vec{\nabla}\cdot\vec{A} = \oint_{S=\partial V} \mathrm{d}^2\vec{\sigma}\cdot\vec{A}$

$$\frac{\hbar^2}{2\mu}\sum_c \oint_R \mathrm{d}^2\vec{s}\cdot\Big\{(\psi'^{(-)}_{\mathbf{b},c})^*\vec{\nabla}\psi^{(+)}_{\mathbf{a},c} - \psi^{(+)}_{\mathbf{a},c}(\vec{\nabla}\psi'^{(-)}_{\mathbf{b},c})^*\Big\} \qquad \mathrm{d}^2\vec{\sigma} \propto \hat{\mathbf{e}}_r$$

$$= \frac{\hbar^2 R^2}{2\mu}\sum_c \oint_R \mathrm{d}^2\Omega_R\,\Big\{(\psi'^{(-)}_{\mathbf{b},c})^*\frac{\partial\psi^{(+)}_{\mathbf{a},c}}{\partial r} - \Big(\frac{\partial\psi'^{(-)}_{\mathbf{b},c}}{\partial r}\Big)^*\psi^{(+)}_{\mathbf{a},c}\Big\}$$ <= work on this, l.h.s.

$$= \langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}|\Psi^{(+)}_{\mathbf{a}}\rangle_R - \langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}'|\Psi^{(+)}_{\mathbf{a}}\rangle_R$$

$$(\psi'^{(-)}_{\mathbf{b},c})^*\frac{\partial\psi^{(+)}_{\mathbf{a},c}}{\partial r} - \Big(\frac{\partial\psi'^{(-)}_{\mathbf{b},c}}{\partial r}\Big)^*\psi^{(+)}_{\mathbf{a},c} \qquad \vartheta := \measuredangle(\vec{k}_c,\vec{r})$$

$$= |A|^2\Big\{\Big(e^{ik_c r\cos\vartheta} + f'^{(-)}_{\mathbf{b},c}\frac{e^{-ik_c r}}{r}\Big)^*\frac{\partial}{\partial r}\Big(e^{ik_c r\cos\vartheta} + f^{(+)}_{\mathbf{b},c}\frac{e^{ik_c r}}{r}\Big)$$

$$- \Big[\frac{\partial}{\partial r}\Big(e^{ik_c r\cos\vartheta} + f'^{(-)}_{\mathbf{b},c}\frac{e^{-ik_c r}}{r}\Big)^*\Big]\Big(e^{ik_c r\cos\vartheta} + f^{(+)}_{\mathbf{b},c}\frac{e^{ik_c r}}{r}\Big)\Big\}_{r\to R}$$

Quantum Scattering

Scattering Amplitude Theorem

- Computing the derivatives, conjugating and substituting R

$$(\psi'^{(-)}_{\mathbf{b},c})^* \frac{\partial \psi^{(+)}_{\mathbf{a},c}}{\partial r} - \left(\frac{\partial \psi'^{(-)}_{\mathbf{b},c}}{\partial r}\right)^* \psi^{(+)}_{\mathbf{a},c}$$

$$= |A|^2 \Bigg\{ \left(e^{ik_c r\cos\vartheta} + f'^{(-)}_{\mathbf{b},c}\frac{e^{-ik_c r}}{r}\right)^* \frac{\partial}{\partial r}\left(e^{ik_c r\cos\vartheta}+f^{(+)}_{\mathbf{b},c}\frac{e^{ik_c r}}{r}\right)$$

$$- \left[\frac{\partial}{\partial r}\left(e^{ik_c r\cos\vartheta}+f'^{(-)}_{\mathbf{b},c}\frac{e^{-ik_c r}}{r}\right)^*\right]\left(e^{ik_c r\cos\vartheta}+f^{(+)}_{\mathbf{b},c}\frac{e^{ik_c r}}{r}\right)\Bigg\}_{r\to R}$$

$$= |A|^2\Bigg\{\left(e^{-ik_cR\cos\vartheta}+\bar{f}'^{(-)}_{\mathbf{b},c}\frac{e^{ik_cR}}{R}\right)\left(ik_c\cos\vartheta\, e^{ik_cR\cos\vartheta}+f^{(+)}_{\mathbf{b},c}e^{ik_cR}\frac{ik_cR-1}{R^2}\right)$$

$$- \left(-ik_c\cos\vartheta\, e^{-ik_cR\cos\vartheta}+\bar{f}'^{(-)}_{\mathbf{b},c}e^{ik_cR}\frac{ik_cR-1}{R^2}\right)\left(e^{ik_cR\cos\vartheta}+f^{(+)}_{\mathbf{b},c}\frac{e^{ik_cR}}{R}\right)\Bigg\}$$

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Quantum Scattering

Scattering Amplitude Theorem

- Expand and sort

$$
\begin{aligned}
&= |A|^2 \Big\{ \Big(e^{-ik_c R\cos\vartheta} + \bar{f}'^{(-)}_{\mathbf{b},c} \frac{e^{ik_c R}}{R} \Big) \Big(ik_c \cos\vartheta\, e^{ik_c R\cos\vartheta} + f^{(+)}_{\mathbf{b},c}\, e^{ik_c R} \frac{ik_c R - 1}{R^2} \Big) \\
&\qquad - \Big(-ik_c \cos\vartheta\, e^{-ik_c R\cos\vartheta} + \bar{f}'^{(-)}_{\mathbf{b},c}\, e^{ik_c R} \frac{ik_c R - 1}{R^2} \Big) \Big(e^{ik_c R\cos\vartheta} + f^{(+)}_{\mathbf{b},c} \frac{e^{ik_c R}}{R} \Big) \Big\} \\
&= |A|^2 \Big\{ 2ik_c \cos\vartheta \;+\; \bar{f}'^{(-)}_{\mathbf{b},c} f^{(+)}_{\mathbf{b},c}\, e^{2ik_c R} \frac{ik_c R - 1}{R^3} \boxed{(1-1) = 0} \\
&\qquad + \bar{f}'^{(-)}_{\mathbf{b},c} \Big(\frac{e^{ik_c R}}{R} \big(ik_c \cos\vartheta\, e^{ik_c R\cos\vartheta} \big) - e^{ik_c R} \frac{ik_c R - 1}{R^2} e^{ik_c R\cos\vartheta} \Big) \\
&\qquad + f^{(+)}_{\mathbf{b},c} \Big(e^{-ik_c R\cos\vartheta}\, e^{ik_c R} \frac{ik_c R - 1}{R^2} + \big(ik_c \cos\vartheta\, e^{-ik_c R\cos\vartheta} \big) \frac{e^{ik_c R}}{R} \Big) \Big\}
\end{aligned}
$$

- Tidy up: factor common terms and simplify

Quantum Scattering

Scattering Amplitude Theorem

- This produces under the integral

$$= |A|^2\Big\{2ik_c\cos\vartheta \;+ \bar{f}'^{(-)}_{\mathbf{b},c}\, e^{ik_cR(\cos\vartheta+1)}\Big[\frac{ik_c}{R}(\cos\vartheta-1)+\frac{1}{R^2}\Big]$$
$$+ f^{(+)}_{\mathbf{b},c}\, e^{ik_cR(\cos\vartheta-1)}\Big[\frac{ik_c}{R}(\cos\vartheta+1)-\frac{1}{R^2}\Big]\Big\}$$

- The l.h.s. thus becomes

$$\frac{\hbar^2R^2}{2\mu}\sum_c\oint_R \mathrm{d}^2\Omega_R\left\{(\psi'^{(-)}_{\mathbf{b},c})^*\frac{\partial\psi^{(+)}_{\mathbf{a},c}}{\partial r}-\Big(\frac{\partial\psi'^{(-)}_{\mathbf{b},c}}{\partial r}\Big)^*\psi^{(+)}_{\mathbf{a},c}\right\}$$

$$=\frac{\hbar^2|A|^2}{2\mu}\sum_c\oint_R \mathrm{d}^2\Omega_R\Big\{2ik_cR^2\cos\vartheta+\bar{f}'^{(-)}_{\mathbf{b},c}(\Omega_R)\,e^{ik_cR(\cos\vartheta+1)}\big[ik_cR(\cos\vartheta-1)+1\big]$$
$$+f^{(+)}_{\mathbf{b},c}(\Omega_R)\,e^{ik_cR(\cos\vartheta-1)}\big[ik_cR(\cos\vartheta+1)-1\big]\Big\}$$

$\vdots$ see Ballentine, Eqs.(16.52)–(16.56)

$$=\frac{2\pi\hbar^2}{\mu}|A|^2\Big\{\bar{f}'^{(-)}_{\mathbf{b},a}(-\Omega_{ka})-f^{(+)}_{\mathbf{a},b}(\Omega_{kb})\Big\}$$
$$=\langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}|\Psi^{(+)}_{\mathbf{a}}\rangle_R-\langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}'|\Psi^{(+)}_{\mathbf{a}}\rangle_R$$

This equality is the scattering amplitude theorem

Quantum Scattering

Applications & the Born Approximation

$$\frac{2\pi\hbar^2}{\mu}|A|^2\Big\{\bar{f}'^{(-)}_{\mathbf{b},a}(-\Omega_{ka}) - f^{(+)}_{\mathbf{a},b}(\Omega_{kb})\Big\} = \langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}|\Psi^{(+)}_{\mathbf{a}}\rangle_R - \langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}'|\Psi^{(+)}_{\mathbf{a}}\rangle_R$$

- Set $\widehat{W}' \to \widehat{W} \quad \Rightarrow \quad \bar{f}'^{(-)}_{\mathbf{b},a}(-\Omega_{ka}) = f^{(+)}_{\mathbf{a},b}(\Omega_{kb})$ (over-bar = conjugation)

  capturing — scattering

- Set $\widehat{W}' \to 0 \quad \widehat{H}' \to \widehat{H}_0 \quad \widehat{H}'\Phi_{\mathbf{b}} = E\Phi_{\mathbf{b}} \quad \Phi_{\mathbf{b}} = A\,e^{i\vec{k}_b\cdot\vec{r}}\chi_b$

  plane-wave w/int. dynamics

$$f^{(+)}_{\mathbf{a},b}(\Omega_{kb}) = -\frac{\mu}{2\pi\hbar^2}\langle\Phi_{\mathbf{b}}|\widehat{W}|\Psi^{(+)}_{\mathbf{a}}\rangle$$

$$= -\frac{\mu A^*}{2\pi\hbar^2}\int \mathrm{d}^3\vec{r}\,e^{-i\vec{k}_b\cdot\vec{r}}\langle\chi_b|\widehat{W}(\vec{r})|\chi_a\rangle\,\psi^{(+)}_{\mathbf{a},a}(\vec{r})$$

indeed a 3D Fourier transform!

$$= -\frac{\mu A^*}{2\pi\hbar^2}\int \mathrm{d}^3\vec{r}\,e^{-i\vec{k}_b\cdot\vec{r}}W_{ab}(\vec{r})\,\psi^{(+)}_{\mathbf{a},a}(\vec{r}) \qquad W_{ab}(\vec{r}) := \langle\chi_b|\widehat{W}(\vec{r})|\chi_a\rangle$$

no summation over a & b !

The detector is far away, where the scattered spherical wave is well approximated by a plane wave

Quantum Scattering

Applications & the Born Approximation

$$f^{(+)}_{\mathbf{a},b}(\Omega_{kb}) \approx -\frac{\mu|A|^2}{2\pi\hbar^2}\int \mathrm{d}^3\vec{r}\, e^{-i\vec{k}_b\cdot\vec{r}}\, W_{ab}(\vec{r})\, e^{+i\vec{k}_a\cdot\vec{r}}$$

- The difference in linear momenta is the momentum transfer

$$f^{(+)}_{\mathbf{a},b}(\Omega_{kb}) \approx -\frac{\mu|A|^2}{2\pi\hbar^2}\int \mathrm{d}^3\vec{r}\, e^{i\vec{q}\cdot\vec{r}}\, W_{ab}(\vec{r}) \qquad \vec{q} := \vec{k}_a - \vec{k}_b$$

- For central potentials $\widehat{W}(\vec{r}) = \widehat{W}(r) \quad \Rightarrow \quad W_{ab}(\vec{r}) = W_{ab}(r)$

$$f^{(+)}_{\mathbf{a},b}(\Omega_{kb}) \approx -\frac{\mu|A|^2}{2\pi\hbar^2}\int_0^\infty r^2\mathrm{d}r\, W_{ab}(r)\int_0^\pi \sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\; e^{i|\vec{q}|r\cos\theta}$$

$$= -\frac{\mu|A|^2}{\hbar^2}\int_0^\infty r^2\mathrm{d}r\, W_{ab}(r)\int_{-1}^{1}\mathrm{d}u\; e^{i|\vec{q}|ru}$$

$$= -\frac{2\mu}{\hbar^2}\frac{|A|^2}{|\vec{k}_a-\vec{k}_b|}\int_0^\infty r\mathrm{d}r\, W_{ab}(r)\sin(|\vec{k}_a-\vec{k}_b|r)$$

radial part of a cylindrical volume integral

a 1D sine-Fourier transform

Quantum Scattering

Distorted Wave Born Approximation

$$\frac{2\pi\hbar^2}{\mu}|A|^2\left\{\bar{f}'^{(-)}_{\mathbf{b},a}(-\Omega_{ka}) - f^{(+)}_{\mathbf{a},b}(\Omega_{kb})\right\} = \langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}|\Psi^{(+)}_{\mathbf{a}}\rangle_R - \langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}'|\Psi^{(+)}_{\mathbf{a}}\rangle_R$$

- Set $\widehat{W}' \to \widehat{W} + \widehat{W}_2$
- and use the 1st result of the Scattering Amplitude Theorem:

$$\bar{f}'^{(-)}_{\mathbf{b},a}(-\Omega_{ka}) = f'^{(+)}_{\mathbf{a},b}(\Omega_{kb})$$

- to obtain:

$$f'^{(+)}_{\mathbf{a},b}(\Omega_{kb}) = f^{(+)}_{\mathbf{a},b}(\Omega_{kb}) - \frac{\mu}{2\pi\hbar^2|A|^2}\langle\Psi'^{(-)}_{\mathbf{b}}|\widehat{W}_2|\Psi^{(+)}_{\mathbf{a}}\rangle_R \qquad \text{exact}$$

$$\approx f^{(+)}_{\mathbf{a},b}(\Omega_{kb}) - \frac{\mu}{2\pi\hbar^2|A|^2}\langle\Psi^{(-)}_{\mathbf{b}}|\widehat{W}_2|\Psi^{(+)}_{\mathbf{a}}\rangle_R \qquad \text{lowest order approximation}$$

- Use when $\widehat{H}' = \underbrace{\widehat{H}_0 + \widehat{W}_1}_{\widehat{H}\ \text{known}} + \widehat{W}_2$

1st order perturbation correction

(Note: the un-normalizable amplitude cancels out!)

Quantum Mechanics II

Now, go forth and calculate!!

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