

Quantum Mechanics II

# Scattering (2): The Quantum Theory

The General Theory;  
The Scattering Amplitude Theorem;  
Applications & the Born Approximation

Tristan Hübsch

*Department of Physics and Astronomy, Howard University, Washington DC*

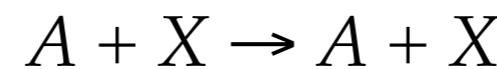
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# Quantum Scattering

## General Theory

### Scattering types:

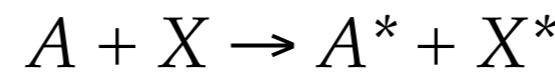
#### Elastic scattering:



- Kinetic energy is conserved; linear & angular momenta are conserved

- Ex.: billiard balls, marbles,

#### Inelastic scattering:



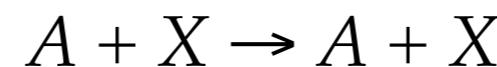
- Kinetic energy is not conserved, total energy is; must include internal energy

# Quantum Scattering

## General Theory

### Scattering types:

#### Elastic scattering:

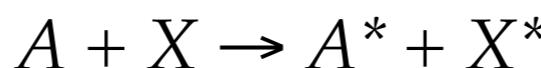


- Kinetic energy is conserved

- Ex.: billiard balls, marbles,

In fact, total energy-momentum  
and angular momentum  
are always conserved

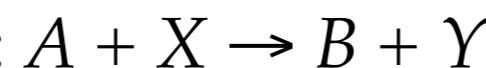
#### Inelastic scattering:



- Kinetic energy is not conserved, total energy is; must include internal energy

- Ex.: traffic collisions; vehicles and people absorb some of the energy

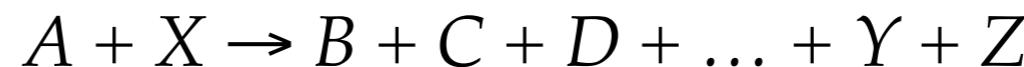
#### Rearrangement collision:



- Parts of the subsystems  $A$  and  $X$  get rearranged/exchanged

- Ex.: nuclear and chemical reactions

#### Particle production:



- The final state consists of more than two ( $n$ ) separate particles

- ( $n - 2$ ) of which are therefore *produced* by the collision

- Total energy-momentum being conserved, some of the kinetic energy of the colliding objects  $A$  and  $X$  is transformed into the masses of the created particles

# Quantum Scattering

## General Theory

- ➊ Focus on elastic and inelastic scattering / collisions
  - ➌ elastic scattering involves no exchange with internal energy
  - ➌ inelastic scattering involves some exchange with internal energy
- ➋ In- and out-states both involve two particles work in CM frame
  - ➌ For each of the different possible outcomes
  - ➌ ...calculate a separate scattering amplitude and cross-section
  - ➌ The total cross-section is the sum of these
  - ➌ The relative ratios are called branching ratios
- ➌ Set-up:  $\hat{H} = \underbrace{-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 + \hat{h}_1 + \hat{h}_2}_{=\hat{H}_0} + \hat{W}$        $\underbrace{[\hat{h}_1 + \hat{h}_2]\chi_a = \epsilon_a \chi_a}_{\text{"internal"}}$
- ➌ separation of variables:

$$\hat{H} \Psi_{\mathbf{a}}^{(+)} = E_{\mathbf{a}} \Psi_{\mathbf{a}}^{(+)} \quad \Psi_{\mathbf{a}}^{(+)} = \sum_{a'} \psi_{\mathbf{a},a'}^{(+)}(\vec{r}) \chi_{a'} \quad \mathbf{a} := (\vec{k}_a, a)$$

# Quantum Scattering

# General Theory

$$\Psi_{\mathbf{a}}^{(+)} = \sum_{a'} \psi_{\mathbf{a},a'}^{(+)}(\vec{r}) \chi_{a'}$$

- The general solution is expected in the form

the incident state:  
*same* state

$$\psi_{\mathbf{a}}^{(+)}(\vec{r}) \sim A \left\{ e^{i\vec{k}_a \cdot \vec{r}} + f_{\mathbf{a},a}^{(+)}(\Omega_r) \frac{e^{+ik_ar}}{r} \right\}$$

incident

*elastic*-scattered

$$\psi_{\mathbf{b}}^{(+)}(\vec{r}) \sim A \left\{ f_{\mathbf{a},b}^{(+)}(\Omega_r) \frac{e^{+ik_b r}}{r} \right\}_{b \neq a}$$

*inelastic*-scattered

any other state:

# out-going spherical wave

- # The probability currents are

$$J_{I(\mathbf{a})} = |A|^2 \frac{\hbar k_a}{\mu} \quad J_{S(\mathbf{a})} = |A|^2 \frac{\hbar k_a}{\mu} \frac{1}{r^2} \left| f_{\mathbf{a},a}^{(+)}(\Omega_r) \right|^2 \quad J_{S(\mathbf{b})} = |A|^2 \frac{\hbar k_b}{\mu} \frac{1}{r^2} \left| f_{\mathbf{a},b}^{(+)}(\Omega_r) \right|^2$$

- ...and the cross-sections

$$\sigma_{\mathbf{a},a}(\theta_r) = |f_{\mathbf{a},a}^{(+)}(\Omega_r)|^2$$

$$\sigma_{\mathbf{a},b}(\theta_r) = \frac{k_b}{k_a} |f_{\mathbf{a},b}^{(+)}(\Omega_r)|^2$$

- Aim to compute the scattering amplitudes

# Quantum Scattering

## General Theory

- These wave-functions, after all, satisfy a Sturm-Liouville type 2nd order partial differential equation
- ...and there is always the “other” solution

$$\frac{e^{+ik_ar}}{r} \leftrightarrow e^{ikx} \quad \text{just as} \quad e^{-ikx} \leftrightarrow \frac{e^{-ik_ar}}{r}$$

$e^{-i\omega t} e^{ikx} = \underbrace{e^{-i(\omega t - kx)}}_{\substack{\text{stationary} \\ @ x = \frac{\omega}{k}t \\ \text{as } t \text{ grows,} \\ \text{so does } x}}$

- So, include

$$\hat{H} \Psi_{\mathbf{b}}^{(-)} = E_{\mathbf{b}} \Psi_{\mathbf{b}}^{(-)}$$

$$\Psi_{\mathbf{b}}^{(-)} = \sum_{b'} \psi_{\mathbf{b}, b'}^{(-)}(\vec{r}) \chi_{b'}$$

$$\mathbf{b} := (\vec{k}_b, b)$$

$$\psi_{\mathbf{b}}^{(-)}(\vec{r}) \sim A \left\{ \underbrace{e^{i\vec{k}_b \cdot \vec{r}}}_{\text{plane}} + \underbrace{f_{\mathbf{b}, b}^{(-)}(\Omega_r) \frac{e^{-ik_b r}}{r}}_{\text{captured}} \right\}$$

in-coming  
spherical  
wave

$$\psi_{\mathbf{c}}^{(-)}(\vec{r}) \sim A \left\{ f_{\mathbf{b}, c}^{(-)}(\Omega_r) \frac{e^{-ik_c r}}{r} \right\}_{c \neq b} \quad \chi_c \neq \chi_b$$

Use  
 $\{ \Psi_{\mathbf{a}}^{(+)} \}$  &  $\{ \Psi_{\mathbf{b}}^{(-)} \}$

# Quantum Scattering

## Scattering Amplitude Theorem

- Scattering states are not “finitely normalized”

$$\langle \chi_a | \chi_b \rangle = \delta_{a,b} \quad \text{but} \quad \langle \psi_a^{(+)} | \psi_b^{(+)} \rangle \propto \delta(\mathbf{a} - \mathbf{b}) \quad \langle \psi_a^{(+)} | \psi_a^{(+)} \rangle = \infty$$

$$\langle \psi_a^{(+)} | \psi_b^{(+)} \rangle_R := \int_0^R r^2 dr \int d^2\Omega \psi_a^{*(+)}(\vec{r}) \psi_b^{(+)}(\vec{r})$$

- But, scattering states do not vanish at the boundary *(neither finite nor infinite!)*
  - ...and so the Laplacian is not Hermitian b / c of boundary terms
  - Derive the consequences!

# Quantum Scattering

## Scattering Amplitude Theorem

- Scattering states are not “finitely normalized”

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- But, scattering states do not vanish at the boundary *(neither finite nor infinite!)*
- ...and so the Laplacian is not Hermitian b/c of boundary terms
- Derive the consequences!

Consider:  $\hat{H} = \hat{H}_0 + \hat{W}$      $\hat{H}' = \hat{H}_0 + \hat{W}'$

$$\hat{H} \Psi_a^{(+)} = E_a \Psi_a^{(+)} \quad E_b = E = E_a \quad \hat{H}' \Psi_b'^{(-)} = E_b \Psi_b'^{(-)}$$

$$\frac{\hbar^2}{2\mu} \vec{\nabla}^2 \Psi_a^{(+)}(\vec{r}) = [\hat{h}_1 + \hat{h}_2 + \hat{W} - E] \Psi_a^{(+)}(\vec{r}) \quad \cancel{\Psi_b'^{(-)}(\vec{r})}.$$

$$\frac{\hbar^2}{2\mu} \vec{\nabla}^2 \Psi_b'^{(-)}(\vec{r}) = [\hat{h}_1 + \hat{h}_2 + \hat{W}' - E] \Psi_b'^{(-)}(\vec{r}) \quad \cancel{\Psi_a^{(+)}(\vec{r})}.$$

# Quantum Scattering

## Scattering Amplitude Theorem

- After pre-multiplying

$$\frac{\hbar^2}{2\mu} \Psi_{\mathbf{b}}'^*(-)(\vec{r}) \vec{\nabla}^2 \Psi_{\mathbf{a}}^{(+)}(\vec{r}) = \Psi_{\mathbf{b}}'^*(-)(\vec{r}) [\hat{h}_1 + \hat{h}_2 + \hat{W} - E] \Psi_{\mathbf{a}}^{(+)}(\vec{r})$$

$$= \Psi_{\mathbf{b}}'^*(-)(\vec{r}) [\epsilon_a + \hat{W} - E] \Psi_{\mathbf{a}}^{(+)}(\vec{r})$$

$$\frac{\hbar^2}{2\mu} \Psi_{\mathbf{a}}^{*(+)}(\vec{r}) \vec{\nabla}^2 \Psi_{\mathbf{b}}'^{(-)}(\vec{r}) = \Psi_{\mathbf{a}}^{*(+)}(\vec{r}) [\hat{h}_1 + \hat{h}_2 + \hat{W}' - E] \Psi_{\mathbf{b}}'^{(-)}(\vec{r})$$

$$= \Psi_{\mathbf{a}}^{*(+)}(\vec{r}) [\epsilon_a + \hat{W}' - E] \Psi_{\mathbf{b}}'^{(-)}(\vec{r})$$

*Hermitian!*

- The l.h.s. diff. would vanish if the Laplacian was Hermitian

- The  $r \leq R$  integrated difference is

$$\frac{\hbar^2}{2\mu} \left\{ \langle \Psi_{\mathbf{b}}^{(-)} | \vec{\nabla}^2 \Psi_{\mathbf{a}}^{(+)} \rangle_R - \langle \Psi_{\mathbf{a}}^{(+)} | \vec{\nabla}^2 \Psi_{\mathbf{b}}^{(-)} \rangle_R \right\} = \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W} | \Psi_{\mathbf{a}}^{(+)} \rangle_R - \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W}' | \Psi_{\mathbf{a}}^{(+)} \rangle_R$$

- Use:

$$\begin{aligned} u \vec{\nabla}^2 v - v \vec{\nabla}^2 u &= (\vec{\nabla} \cdot (u \vec{\nabla} v) - (\vec{\nabla} u) \cdot (\vec{\nabla} v)) - (\vec{\nabla} \cdot (v \vec{\nabla} u) - (\vec{\nabla} v) \cdot (\vec{\nabla} u)) \\ &= \vec{\nabla} \cdot ((u \vec{\nabla} v) - (v \vec{\nabla} u)) \end{aligned}$$

# Quantum Scattering

## Scattering Amplitude Theorem

Using Gauss's theorem  $\int_V d^3\vec{r} \cdot \vec{\nabla} \cdot \vec{A} = \oint_{S=\partial V} d^2\vec{\sigma} \cdot \vec{A}$

$$\begin{aligned} & \frac{\hbar^2}{2\mu} \sum_c \oint_R d^2\vec{s} \cdot \left\{ (\psi'_{\mathbf{b},c}^{(-)})^* \vec{\nabla} \psi_{\mathbf{a},c}^{(+)} - \psi_{\mathbf{a},c}^{(+)} (\vec{\nabla} \psi'_{\mathbf{b},c}^{(-)})^* \right\} \quad d^2\vec{\sigma} \propto \hat{\mathbf{e}}_r \\ &= \frac{\hbar^2 R^2}{2\mu} \sum_c \oint_R d^2\Omega_R \left\{ (\psi'_{\mathbf{b},c}^{(-)})^* \frac{\partial \psi_{\mathbf{a},c}^{(+)}}{\partial r} - \left( \frac{\partial \psi'_{\mathbf{b},c}^{(-)}}{\partial r} \right)^* \psi_{\mathbf{a},c}^{(+)} \right\} \quad \text{=< work on this, l.h.s.} \\ &= \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W} | \Psi_{\mathbf{a}}^{(+)} \rangle_R - \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W}' | \Psi_{\mathbf{a}}^{(+)} \rangle_R \end{aligned}$$

$$\begin{aligned} & (\psi'_{\mathbf{b},c}^{(-)})^* \frac{\partial \psi_{\mathbf{a},c}^{(+)}}{\partial r} - \left( \frac{\partial \psi'_{\mathbf{b},c}^{(-)}}{\partial r} \right)^* \psi_{\mathbf{a},c}^{(+)} \quad \vartheta := \angle(\vec{k}_c, \vec{r}) \\ &= |A|^2 \left\{ \left( e^{ik_c r \cos \vartheta} + f'_{\mathbf{b},c} \frac{e^{-ik_c r}}{r} \right)^* \boxed{\frac{\partial}{\partial r}} \left( e^{ik_c r \cos \vartheta} + f_{\mathbf{b},c}^{(+)} \frac{e^{ik_c r}}{r} \right) \right. \\ & \quad \left. - \left[ \boxed{\frac{\partial}{\partial r}} \left( e^{ik_c r \cos \vartheta} + f'_{\mathbf{b},c} \frac{e^{-ik_c r}}{r} \right)^* \right] \left( e^{ik_c r \cos \vartheta} + f_{\mathbf{b},c}^{(+)} \frac{e^{ik_c r}}{r} \right) \right\}_{r \rightarrow R} \end{aligned}$$

# Quantum Scattering

## Scattering Amplitude Theorem

- Computing the derivatives, conjugating and substituting  $R$

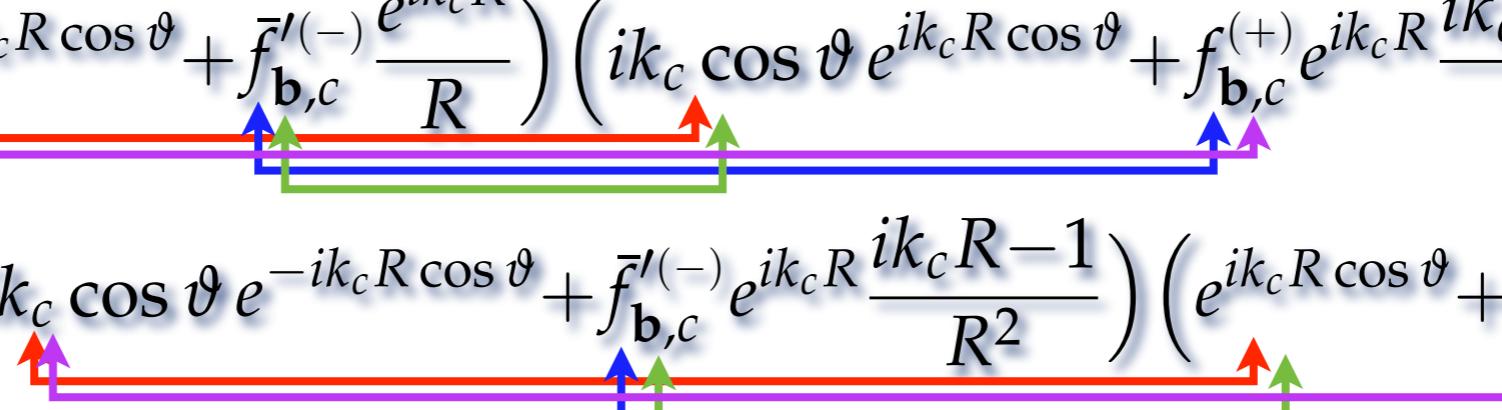
$$(\psi'_{\mathbf{b},c}^{(-)})^* \frac{\partial \psi_{\mathbf{a},c}^{(+)}}{\partial r} - \left( \frac{\partial \psi'_{\mathbf{b},c}^{(-)}}{\partial r} \right)^* \psi_{\mathbf{a},c}^{(+)}$$

$$\begin{aligned}
 &= |A|^2 \left\{ \left( e^{ik_c r \cos \vartheta} + f'_{\mathbf{b},c} \frac{e^{-ik_c r}}{r} \right)^* \frac{\partial}{\partial r} \left( e^{ik_c r \cos \vartheta} + f_{\mathbf{b},c}^{(+)} \frac{e^{ik_c r}}{r} \right) \right. \\
 &\quad \left. - \left[ \frac{\partial}{\partial r} \left( e^{ik_c r \cos \vartheta} + f'_{\mathbf{b},c} \frac{e^{-ik_c r}}{r} \right)^* \right] \left( e^{ik_c r \cos \vartheta} + f_{\mathbf{b},c}^{(+)} \frac{e^{ik_c r}}{r} \right) \right\}_{r \rightarrow R} \\
 &= |A|^2 \left\{ \left( e^{-ik_c R \cos \vartheta} + \bar{f}'_{\mathbf{b},c} \frac{e^{ik_c R}}{R} \right) \left( ik_c \cos \vartheta e^{ik_c R \cos \vartheta} + f_{\mathbf{b},c}^{(+)} e^{ik_c R} \frac{ik_c R - 1}{R^2} \right) \right. \\
 &\quad \left. - \left( -ik_c \cos \vartheta e^{-ik_c R \cos \vartheta} + \bar{f}'_{\mathbf{b},c} e^{ik_c R} \frac{ik_c R - 1}{R^2} \right) \left( e^{ik_c R \cos \vartheta} + f_{\mathbf{b},c}^{(+)} \frac{e^{ik_c R}}{R} \right) \right\}
 \end{aligned}$$

# Quantum Scattering

## Scattering Amplitude Theorem

- Expand and sort

$$\begin{aligned}
 &= |A|^2 \left\{ \left( e^{-ik_c R \cos \vartheta} + \bar{f}_{\mathbf{b},c}'^{(-)} \frac{e^{ik_c R}}{R} \right) \left( ik_c \cos \vartheta e^{ik_c R \cos \vartheta} + f_{\mathbf{b},c}^{(+)} e^{ik_c R} \frac{ik_c R - 1}{R^2} \right) \right. \\
 &\quad \left. - \left( -ik_c \cos \vartheta e^{-ik_c R \cos \vartheta} + \bar{f}_{\mathbf{b},c}'^{(-)} e^{ik_c R} \frac{ik_c R - 1}{R^2} \right) \left( e^{ik_c R \cos \vartheta} + f_{\mathbf{b},c}^{(+)} \frac{e^{ik_c R}}{R} \right) \right\} \\
 &= |A|^2 \left\{ 2ik_c \cos \vartheta + \bar{f}_{\mathbf{b},c}'^{(-)} f_{\mathbf{b},c}^{(+)} e^{2ik_c R} \frac{ik_c R - 1}{R^3} \boxed{(1 - 1) = 0} \right. \\
 &\quad + \bar{f}_{\mathbf{b},c}'^{(-)} \left( \frac{e^{ik_c R}}{R} (ik_c \cos \vartheta e^{ik_c R \cos \vartheta}) - e^{ik_c R} \frac{ik_c R - 1}{R^2} e^{ik_c R \cos \vartheta} \right) \\
 &\quad \left. + f_{\mathbf{b},c}^{(+)} \left( e^{-ik_c R \cos \vartheta} e^{ik_c R} \frac{ik_c R - 1}{R^2} + (ik_c \cos \vartheta e^{-ik_c R \cos \vartheta}) \frac{e^{ik_c R}}{R} \right) \right\}
 \end{aligned}$$


- Tidy up: factor common terms and simplify

# Quantum Scattering

## Scattering Amplitude Theorem

- This produces under the integral

$$= |A|^2 \left\{ 2ik_c \cos \vartheta + \bar{f}_{\mathbf{b},c}'^{(-)} e^{ik_c R(\cos \vartheta + 1)} \left[ \frac{ik_c}{R} (\cos \vartheta - 1) + \frac{1}{R^2} \right] \right. \\ \left. + f_{\mathbf{b},c}^{(+)} e^{ik_c R(\cos \vartheta - 1)} \left[ \frac{ik_c}{R} (\cos \vartheta + 1) - \frac{1}{R^2} \right] \right\}$$

- The l.h.s. thus becomes

$$\frac{\hbar^2 R^2}{2\mu} \sum_c \oint_R d^2\Omega_R \left\{ (\psi_{\mathbf{b},c}'^{(-)})^* \frac{\partial \psi_{\mathbf{a},c}^{(+)}}{\partial r} - \left( \frac{\partial \psi_{\mathbf{b},c}'^{(-)}}{\partial r} \right)^* \psi_{\mathbf{a},c}^{(+)} \right\} \\ = \frac{\hbar^2 |A|^2}{2\mu} \sum_c \oint_R d^2\Omega_R \left\{ 2ik_c R^2 \cos \vartheta + \bar{f}_{\mathbf{b},c}'^{(-)}(\Omega_R) e^{ik_c R(\cos \vartheta + 1)} [ik_c R(\cos \vartheta - 1) + 1] \right. \\ \left. + f_{\mathbf{b},c}^{(+)}(\Omega_R) e^{ik_c R(\cos \vartheta - 1)} [ik_c R(\cos \vartheta + 1) - 1] \right\}$$

∴ see Ballentine, Eqs.(16.52)–(16.56)

$$= \boxed{\frac{2\pi\hbar^2}{\mu} |A|^2 \left\{ \bar{f}_{\mathbf{b},a}'^{(-)}(-\Omega_{ka}) - f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) \right\}} \\ = \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W} | \Psi_{\mathbf{a}}^{(+)} \rangle_R - \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W}' | \Psi_{\mathbf{a}}^{(+)} \rangle_R$$

This equality is the scattering amplitude theorem

# Quantum Scattering

## Applications & the Born Approximation

$$\frac{2\pi\hbar^2}{\mu} |A|^2 \left\{ \bar{f}_{\mathbf{b},a}'(-\Omega_{ka}) - f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) \right\} = \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W} | \Psi_{\mathbf{a}}^{(+)} \rangle_R - \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W}' | \Psi_{\mathbf{a}}^{(+)} \rangle_R$$

Set  $\hat{W}' \rightarrow \hat{W}$   $\Rightarrow$   $\bar{f}_{\mathbf{b},a}'(-\Omega_{ka}) = f_{\mathbf{a},b}^{(+)}(\Omega_{kb})$  (over-bar = conjugation)  
 capturing scattering

Set  $\hat{W}' \rightarrow 0$   $\hat{H}' \rightarrow \hat{H}_0$   $\hat{H}'\Phi_{\mathbf{b}} = E\Phi_{\mathbf{b}}$   $\Phi_{\mathbf{b}} = A e^{i\vec{k}_b \cdot \vec{r}} \chi_b$   
 $f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) = -\frac{\mu}{2\pi\hbar^2} \langle \Phi_{\mathbf{b}} | \hat{W} | \Psi_{\mathbf{a}}^{(+)} \rangle$   
 $= -\frac{\mu A^*}{2\pi\hbar^2} \int d^3\vec{r} e^{-i\vec{k}_b \cdot \vec{r}} \langle \chi_b | \hat{W}(\vec{r}) | \chi_a \rangle \psi_{\mathbf{a},a}^{(+)}(\vec{r})$  indeed a 3D Fourier transform!  
 $= -\frac{\mu A^*}{2\pi\hbar^2} \int d^3\vec{r} e^{-i\vec{k}_b \cdot \vec{r}} W_{ab}(\vec{r}) \psi_{\mathbf{a},a}^{(+)}(\vec{r})$   $W_{ab}(\vec{r}) := \langle \chi_b | \hat{W}(\vec{r}) | \chi_a \rangle$   
 no summation over  $a$  &  $b$ !

The detector is far away, where the scattered spherical wave is well approximated by a plane wave

# Quantum Scattering

## Applications & the Born Approximation

$$f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) \approx -\frac{\mu|A|^2}{2\pi\hbar^2} \int d^3\vec{r} e^{-i\vec{k}_b \cdot \vec{r}} W_{ab}(\vec{r}) e^{+i\vec{k}_a \cdot \vec{r}}$$

- The difference in linear momenta is the momentum transfer

$$f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) \approx -\frac{\mu|A|^2}{2\pi\hbar^2} \int d^3\vec{r} e^{i\vec{q} \cdot \vec{r}} W_{ab}(\vec{r}) \quad \vec{q} := \vec{k}_a - \vec{k}_b$$

- For central potentials  $\hat{W}(\vec{r}) = \hat{W}(r) \Rightarrow W_{ab}(\vec{r}) = W_{ab}(r)$

$$f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) \approx -\frac{\mu|A|^2}{2\pi\hbar^2} \int_0^\infty r^2 dr W_{ab}(r) \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi e^{i|\vec{q}|r \cos\theta}$$

$$= -\frac{\mu|A|^2}{\hbar^2} \int_0^\infty r^2 dr W_{ab}(r) \int_{-1}^1 du e^{i|\vec{q}|ru}$$

$$= -\frac{2\mu}{\hbar^2} \frac{|A|^2}{|\vec{k}_a - \vec{k}_b|} \int_0^\infty r dr W_{ab}(r) \sin(|\vec{k}_a - \vec{k}_b|r)$$

radial part of a cylindrical volume integral

a 1D sine-Fourier transform

# Quantum Scattering

## Distorted Wave Born Approximation

$$\frac{2\pi\hbar^2}{\mu}|A|^2 \left\{ \bar{f}'_{\mathbf{b},a}^{(-)}(-\Omega_{ka}) - f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) \right\} = \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W} | \Psi_{\mathbf{a}}^{(+)} \rangle_R - \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W}' | \Psi_{\mathbf{a}}^{(+)} \rangle_R$$

Set  $\hat{W}' \rightarrow \hat{W} + \hat{W}_2$

and use the 1<sup>st</sup> result of the Scattering Amplitude Theorem:

$$\bar{f}'_{\mathbf{b},a}^{(-)}(-\Omega_{ka}) = f_{\mathbf{a},b}^{(+)}(\Omega_{kb})$$

to obtain:

$$f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) = f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) - \frac{\mu}{2\pi\hbar^2|A|^2} \langle \Psi_{\mathbf{b}}'^{(-)} | \hat{W}_2 | \Psi_{\mathbf{a}}^{(+)} \rangle_R \quad \text{exact}$$

$$\approx f_{\mathbf{a},b}^{(+)}(\Omega_{kb}) - \frac{\mu}{2\pi\hbar^2|A|^2} \langle \Psi_{\mathbf{b}}^{(-)} | \hat{W}_2 | \Psi_{\mathbf{a}}^{(+)} \rangle_R \quad \text{lowest order approximation}$$

Use when  $\hat{H}' = \underbrace{\hat{H}_0 + \hat{W}_1}_{\hat{H} \text{ known}} + \hat{W}_2$

1st order perturbation correction  
*(Note: the un-normalizable amplitude cancels out!)*

## Quantum Mechanics II

*Now, go forth and  
calculate!!*

Tristan Hübsch

*Department of Physics and Astronomy, Howard University, Washington DC*

<http://physics1.howard.edu/~thubsch/>