Quantum Mechanics II

WKB

Alpha-Decay
Gamow’s Simple Model
Some Improvements

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QMII

WKB

The Story So Far…

Extra!

- Focus on 1-dimensional physics: $\widehat{H}\psi = E\psi$

$$\widehat{H} = -\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + W(x) \qquad k(x) := \sqrt{\frac{2M}{\hbar^2}\left[E - W(x)\right]}$$

- The "standard wave-functions" are:

$$\psi_{\text{WKB}}(x) = \frac{A}{\sqrt{k(x)}}\,e^{+i\int dx\; k(x)} + \frac{B}{\sqrt{k(x)}}\,e^{-i\int dx\; k(x)} \quad \text{where } E > W(x)$$

$$\psi_{\text{WKB}}(x) = \frac{C}{\sqrt{\kappa(x)}}\,e^{-\int dx\; \kappa(x)} + \frac{D}{\sqrt{\kappa(x)}}\,e^{+\int dx\; \kappa(x)} \quad \text{where } E < W(x)$$

- Matching conditions:
  - The lower limit in the integrals in the exponents is the reference point in the matching condition specification

Barrier to Left

$$\begin{aligned} C &= (\vartheta^* A + \vartheta B) \\ D &= \tfrac{1}{2}(\vartheta A + \vartheta^* B) \\ A &= \tfrac{1}{2}\vartheta C + \vartheta^* D \\ B &= \tfrac{1}{2}\vartheta^* C + \vartheta D \end{aligned}$$

Barrier to Right

$$\begin{aligned} C &= \tfrac{1}{2}(\vartheta^* A + \vartheta B) \\ D &= (\vartheta A + \vartheta^* B) \\ A &= \vartheta C + \tfrac{1}{2}\vartheta^* D \\ B &= \vartheta^* C + \tfrac{1}{2}\vartheta D \end{aligned}$$

QM II

WKB

Applications

$$\psi_{\text{WKB}}(x) = \frac{C}{\sqrt{\kappa(x)}}\, e^{-\int \mathrm{d}x\ \kappa(x)} + \frac{D}{\sqrt{\kappa(x)}}\, e^{+\int \mathrm{d}x\ \kappa(x)}$$

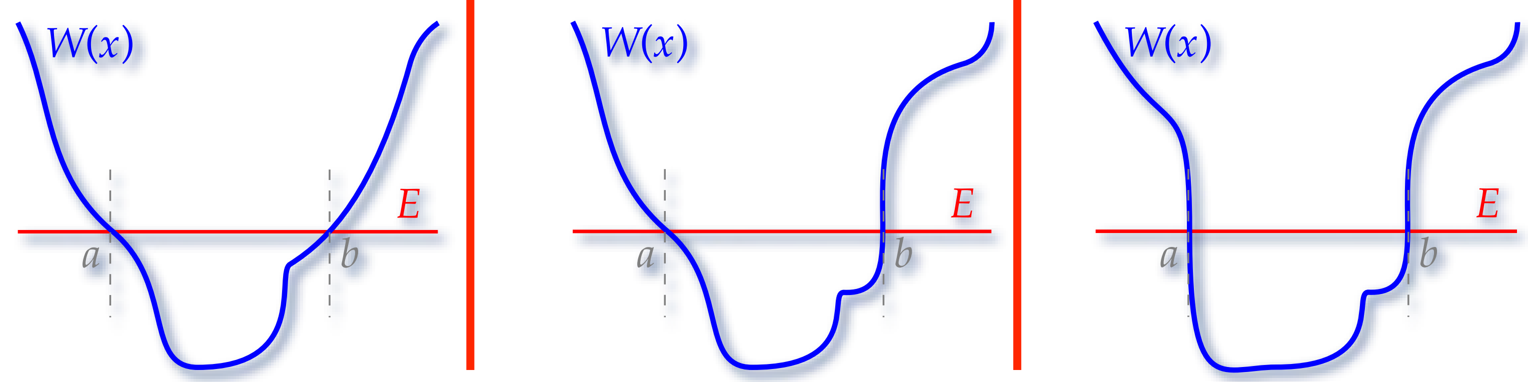
$$\psi_{\text{WKB}}(x) = \frac{A}{\sqrt{k(x)}}\, e^{+i\int \mathrm{d}x\ k(x)} + \frac{B}{\sqrt{k(x)}}\, e^{-i\int \mathrm{d}x\ k(x)}$$

- Potential well w/transition points $x_* = a$ and $x_* = b$
- The energy-quantization relation is:

$$\int_a^b \mathrm{d}x \sqrt{\frac{2M}{\hbar^2}[E - W(x)]} = \begin{cases}(2n+\frac{1}{2})\pi \\ (2n+\frac{3}{2})\pi\end{cases} = \begin{cases}(2n+\frac{3}{4})\pi \\ (2n+\frac{7}{4})\pi\end{cases} = \begin{cases}(2n+1)\pi \\ (2n+2)\pi\end{cases}$$

"symmetric"

"antisymmetric"



- Whenever $W(x)$ crosses E discontinuously

$$\triangle\psi(x) = 0 = \triangle\psi'(x)$$

- Whenever $W(x)$ crosses E continuously

WKB connection formulae

Barrier to Left

$$\begin{aligned} C &= (\vartheta^* A + \vartheta B) \\ D &= \tfrac{1}{2}(\vartheta A + \vartheta^* B) \\ A &= \tfrac{1}{2}\vartheta C + \vartheta^* D \\ B &= \tfrac{1}{2}\vartheta^* C + \vartheta D \end{aligned}$$

Barrier to Right

$$\begin{aligned} C &= \tfrac{1}{2}(\vartheta^* A + \vartheta B) \\ D &= (\vartheta A + \vartheta^* B) \\ A &= \vartheta C + \tfrac{1}{2}\vartheta^* D \\ B &= \vartheta^* C + \tfrac{1}{2}\vartheta D \end{aligned}$$

WKB: α-Decay

General Physics Facts

Extra!

- One typical α-decay starts with Uranium-234 (92 *p* & 142 *n*)
  - Too complicated: 234 3-vectors (702 equations of motion)
  - ...with C_2^{234} = 27,261 pairwise potential terms
  - ...representing the ***strong*** nuclear forces keeping the nucleus stable
  - ...where a (2*p*2*n*) subset form a subsystem
  - ...that escapes the strongly attractive potential of 230 other nucleons

$$ {}^{A+4}_{Z+2}\mathrm{X} \;\to\; {}^{4}_{2}\mathrm{He}^{++} + {}^{A}_{Z}\mathrm{Y}^{--}, \qquad \text{or} \qquad {}^{A+4}_{Z+2}\mathrm{X} \xrightarrow{\;\alpha\;} {}^{A}_{Z}\mathrm{Y}. $$

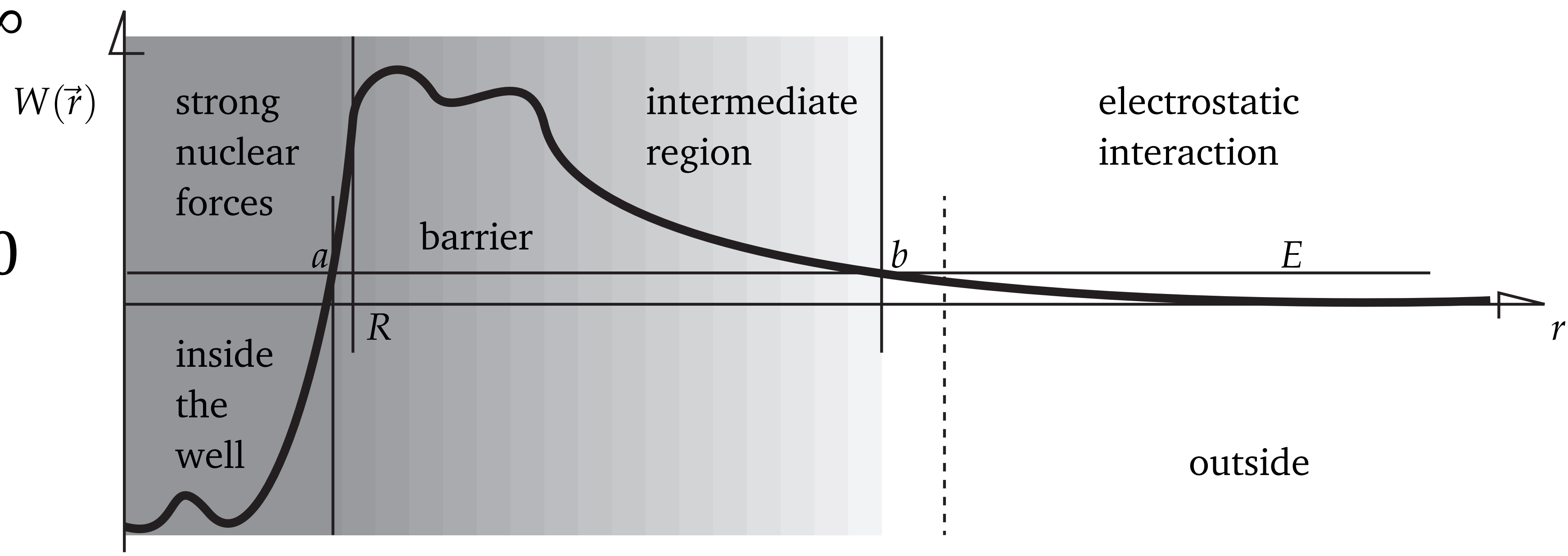
  - Special cases: $A–Z$ =(#*n*)= 2, 8, 20, 28, 50, 82, 126 ("magic numbers")
    - Lead-208 (#*p*=82 & #*n*=126) is doubly magical
    - *p*'s and *n*'s separately form "closed shells"
    - Polonium-212 → α + Lead-208; think Po-212 = [Pb-208+α], which decays
  - "Parent nucleus" = ["Daughter nucleus"+α] → "Daughter nucleus" + α
  - "Daughter nucleus"⇒ (classical) potential in which (quantum) α moves

WKB: α-Decay

General Physics Facts

Extra!

- The "daughter nucleus" potential:
  - is attractive $0 \le r \le (R \sim 1\text{fm})$ (strong interactions), where $W(\boldsymbol{r}) < 0$
  - away from the daughter nucleus, $r = R_\infty \gg (R \sim 1\text{fm})$, $W(r) = 2Ze'^2/r$ (E&M)
  - in-between, $R \le r \ll R_\infty$, $W(\boldsymbol{r})$ provides a barrier
  - For $\min[W(\boldsymbol{r})] \le E \le 0$, α is stably bound
  - For $0 \le E \le \max[W(\boldsymbol{r})]$, α is unstably bound and can decay (& "un-decay")
    - & there exist two points where $E_\alpha = W(a) = W(b)$
    - Classically allowed
      $0 \le r \le a \;\;\&\;\; b \le r \le \infty$
    - Classically forbidden
      $a \le r \le b$
  - For $\max[W(\boldsymbol{r})] \le E \le 0$
    α is free to move everywhere



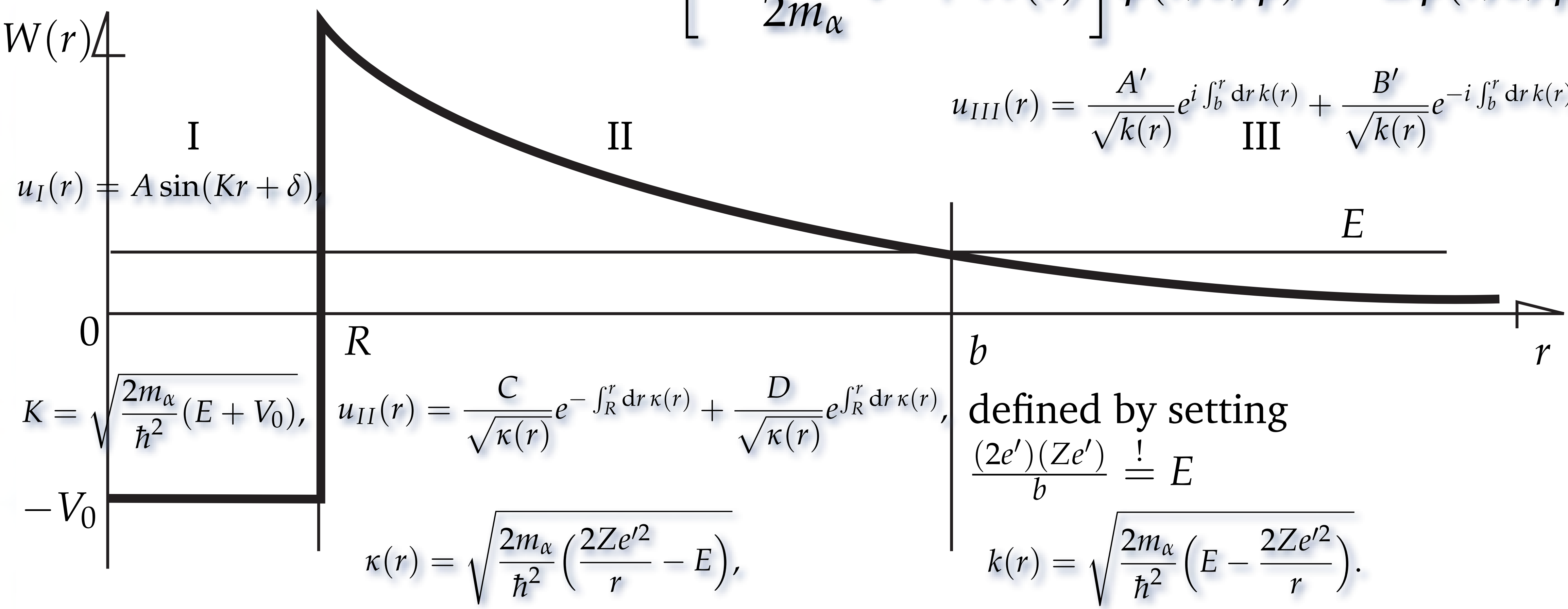
WKB: α-Decay

Gamow's Simple Model

Extra!

- 1928 (only 2 years after Schrödinger's equation), G.A. Gamow:
  - For $0 \le r \le R$, $W(\mathbf{r}) = -V_0 = const.$
  - For $R \le r < \infty$, $W(\mathbf{r}) = 2Ze'^2/r$

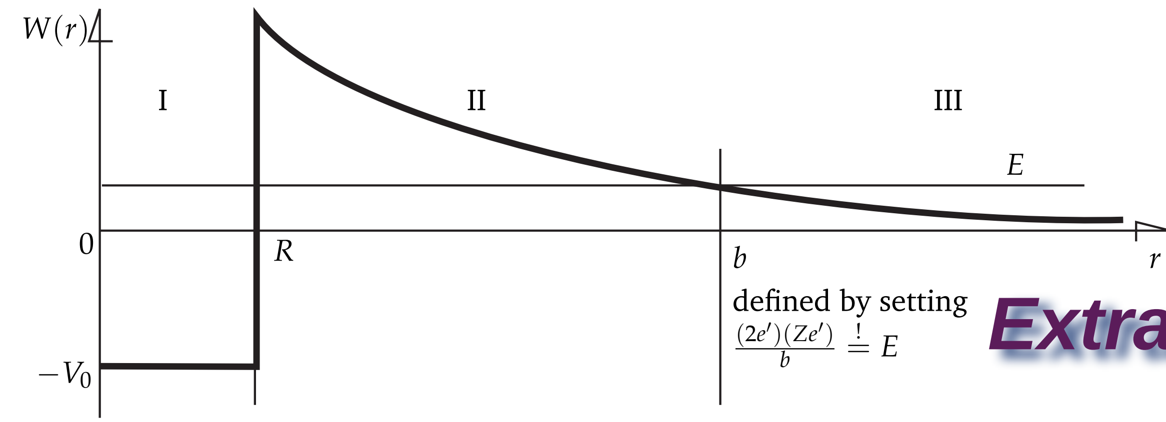
$$\left[-\frac{\hbar}{2m_\alpha}\vec{\nabla}^2 + W(\vec{r})\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$



$$\psi(r,\theta,\phi) = \frac{u(r)}{r}P_\ell^m(\cos\theta)e^{im\phi} \qquad \frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[\frac{2m_\alpha}{\hbar^2}[E - W(r)] - \frac{\ell(\ell+1)}{r^2}\right]u = 0$$

WKB: α-Decay

Gamow's Simple Model



Extra!

- @$r = R$, I ↔ II:

$$\lim_{r\to R-} u_I(r) = \lim_{r\to R+} u_{II}(r), \qquad \lim_{r\to R-} u_I'(r) = \lim_{r\to R+} u_{II}'(r).$$

$$A\sin(KR) = \frac{C+D}{\sqrt{\kappa_R}}, \qquad AK\cos(KR) = (-C+D)\sqrt{\kappa_R},$$

- solve for C, D:

$$C = \frac{A}{2\sqrt{\kappa_R}}\big[\kappa_R\sin(KR) - K\cos(KR)\big],$$

$$D = \frac{A}{2\sqrt{\kappa_R}}\big[\kappa_R\sin(KR) + K\cos(KR)\big].$$

- @ $r = b$, II ↔ III (for α-decay, $B' = 0$):

$$C = \vartheta^* e^{\sigma} A', \qquad \sigma = \int_R^b \mathrm{d}r\,\kappa_R = \sqrt{\frac{2m_\alpha}{\hbar^2}}\int_R^b \mathrm{d}r\sqrt{\frac{2Ze'^2}{r} - E}.$$

$$D = \tfrac{1}{2}\vartheta e^{-\sigma} A',$$

$$\vartheta = e^{i\pi/4},$$

WKB: α-Decay

Gamow's Simple Model

Extra!

- So, we have:

$$C = \frac{A}{2\sqrt{\kappa_R}}\left[\kappa_R \sin(KR) - K\cos(KR)\right], \quad = \vartheta^* e^{\sigma} A',$$

I↔II II↔III

$$D = \frac{A}{2\sqrt{\kappa_R}}\left[\kappa_R \sin(KR) + K\cos(KR)\right], \quad = \tfrac{1}{2}\vartheta e^{-\sigma} A',$$

- Dividing one by the other:

$$\vartheta^2 e^{-2\sigma} = 2\frac{\kappa_R \sin(KR) + K\cos(KR)}{\kappa_R \sin(KR) - K\cos(KR)}$$

imaginary $\vartheta^2 = i$ ***real***

- …which is horribly wrong!
- What have we done ???
  - Math: imposed boundary conditions left and right ($u_I(0) = 0$ & $B' = 0$)
  - Physics: assumed only α-decay, no α-capture (un-decay)

QM II

WKB: α-Decay

Gamow's Simple Model

Extra!

- Re-do, not assuming $B' = 0$:

$$A' = \frac{\vartheta e^{-\sigma} A}{4\sqrt{\kappa_R}}\left[\kappa_R \sin(KR) - K\cos(KR)\right] + \frac{\vartheta^* e^{\sigma} A}{4\sqrt{\kappa_R}}\left[\kappa_R \sin(KR) + K\cos(KR)\right],$$

$$= \frac{\vartheta e^{\sigma} A\cos(KR)}{4\sqrt{\kappa_R}}\Big[\kappa_R \tan(KR) - K - 2ie^{2\sigma}\left[\kappa_R \tan(KR) + K\right]\Big],$$

$$B' = \frac{\vartheta^* e^{-\sigma} A}{4\sqrt{\kappa_R}}\left[\kappa_R \sin(KR) - K\cos(KR)\right] + \frac{\vartheta e^{\sigma} A}{4\sqrt{\kappa_R}}\left[\kappa_R \sin(KR) + K\cos(KR)\right],$$

$$= \frac{\vartheta^* e^{\sigma} A\cos(KR)}{4\sqrt{\kappa_R}}\Big[\kappa_R \tan(KR) - K + 2ie^{2\sigma}\left[\kappa_R \tan(KR) + K\right]\Big].$$

- This implies that $|A'|^2 = |B'|^2$!
- For $B' = 0$, the real and imaginary parts must vanish separately:

$$\kappa_R \tan(KR) - K = 0 \quad \text{and} \quad \kappa_R \tan(KR) + K = 0, \quad K = 0,\ i.e.,\ E = -V_0.$$

WKB: α-Decay

Gamow's Simple Model

Extra!

- Consider the amplitude of the α-decay component:

$$A' = \frac{\vartheta e^{\sigma} A \cos(KR)}{4\sqrt{\kappa_R}} \Big[\kappa_R \tan(KR) - K - 2ie^{2\sigma}\big[\kappa_R \tan(KR) + K\big]\Big],$$

- This stems from the wave-function within the barrier

$$u_{II}(r) = \frac{C\,e^{-\sigma}}{\sqrt{\kappa(r)}} e^{-\int_b^r \mathrm{d}r\,\kappa(r)} + \frac{D\,e^{\sigma}}{\sqrt{\kappa(r)}} e^{\int_b^r \mathrm{d}r\,\kappa(r)}$$

- …where the C-term is exponentially suppressed by $e^{-\sigma}$.
- ***This*** is what allows the ***approximation***, which produces

$$A' \approx \frac{\vartheta e^{\sigma} AK \cos(KR)}{2\sqrt{\kappa_R}}$$

- and is the oft-quoted result [Gamow, 1928]
- …and which turns out to agree with experiments ***very well***!

WKB: α-Decay

Gamow's Simple Model

Extra!

- With this ***approximation***,

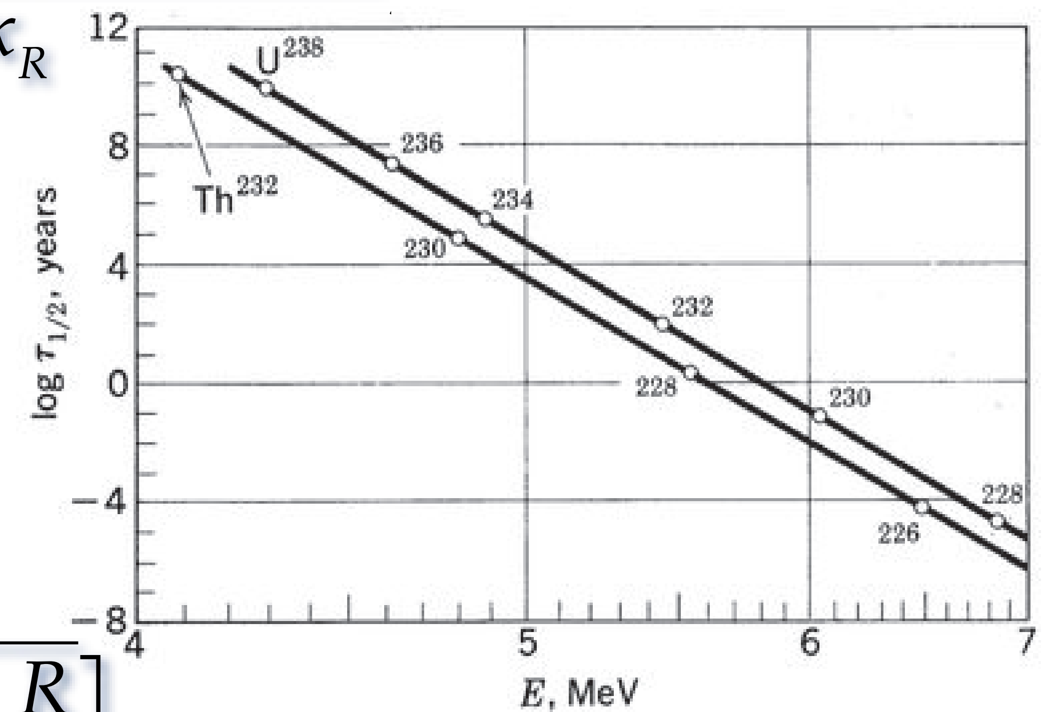
$$\lambda = \frac{4\pi\hbar|A'|}{m_\alpha} \approx \frac{4\pi\hbar K^2 |A'| e^{-2\sigma}\cos^2(KR)}{m_\alpha \kappa_R}$$

- ...and

$$\sigma = \sqrt{\frac{2m_\alpha}{\hbar^2}}\int_R^b \mathrm{d}r\,\sqrt{\frac{2Ze'^2}{r} - E},$$

$$= \sqrt{\frac{2m_\alpha E}{\hbar^2}}\int_R^b \mathrm{d}r\,\sqrt{\frac{b}{r} - 1},$$

$$= \sqrt{\frac{2m_\alpha E}{\hbar^2}}\, b\left[\arccos\sqrt{\frac{R}{b}} - \sqrt{\frac{R}{b}}\sqrt{1-\frac{R}{b}}\right]$$



$$\hbar\,\sigma \approx \tfrac{\pi}{2}\sqrt{2m_\alpha E}\,b - \sqrt{8m_\alpha E\,b\,R} + \tfrac{1}{3}\sqrt{2m_\alpha E\,R^3/b} + \frac{1}{10\sqrt{2}}\sqrt{\frac{m_\alpha E\,R^5}{b^3}} + \ldots$$

$R/b \ll 1$

WKB: α-Decay

Gamow's Model—Exactly

Extra!

- Return to the radial equation:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left[K^2 - \frac{\ell(\ell+1)}{r^2}\right] R = 0,$$

- The Bessel equation. Use $j_\ell(Kr)$—not $n_\ell(Kr)$— for $0 \le r \le R$.
- For $r > R$, substitute $R(r) = r^\ell e^{-\beta r} f(r)$

$$r f'' + \left[2(\ell+1) - 2\beta r\right] f' - \left[(2\beta(\ell+1) + 4Ze'^2 m_\alpha/\hbar^2) + (\beta^2 + 2m_\alpha E/\hbar^2) r\right] f = 0.$$

- Set $\beta^2 = -2m_\alpha E/\hbar$, to obtain

$$z f''(z) + \left[2(\ell+1) - z\right] f'(z) - \left[(\ell+1) + 2Ze'^2 m_\alpha/\beta\hbar^2)\right] f(z) = 0.$$

- The confluent hypergeometric equation.
- In fact, the Bessel equation is a(nother) special case of the CHEq.

$$R_{\text{out}}(r) = r^\ell e^{-\beta r}\left[B\,{}_1F_1\left({\ell+1+w \atop 2\ell+2}; 2\beta r\right) + C\,r^{-2\ell-1}\,{}_1F_1\left({w-\ell \atop -2\ell}; 2\beta r\right)\right].$$

WKB: α-Decay

Gamow's Model—Exactly

Extra!

- Using the asymptotic behavior of the confluent hypergeometric:

$$R_{\text{out}}(r) \sim \underbrace{F r^{-(w+1)}\, e^{-\beta r}}_{\text{incoming}} + \underbrace{G r^{w-1}\, e^{+\beta r}}_{\text{outgoing}} \qquad \beta^2 = -2m_\alpha E/\hbar < 0$$

$$F = B\left(\frac{e^{-i\pi}}{2\beta}\right)^{w+\ell+1} \frac{\Gamma(2\ell+2)}{\Gamma(\ell+1+w)} + C\left(\frac{e^{-i\pi}}{2\beta}\right)^{w-\ell} \frac{\Gamma(\ell+1+w)\sin[\pi(\ell+w)]}{\Gamma(2\ell+1)\sin[2\ell\pi]},$$

$$G = B(2\beta)^{w-\ell-1} \frac{\Gamma(2\ell+2)}{\Gamma(\ell+1+w)} + C(2\beta)^{w-\ell} \frac{\Gamma(\ell+1-w)\sin[\pi(\ell-w)]}{\Gamma(2\ell+1)\sin[2\ell\pi]}.$$

- For a purely outgoing wave, we need $F \approx 0$, so that

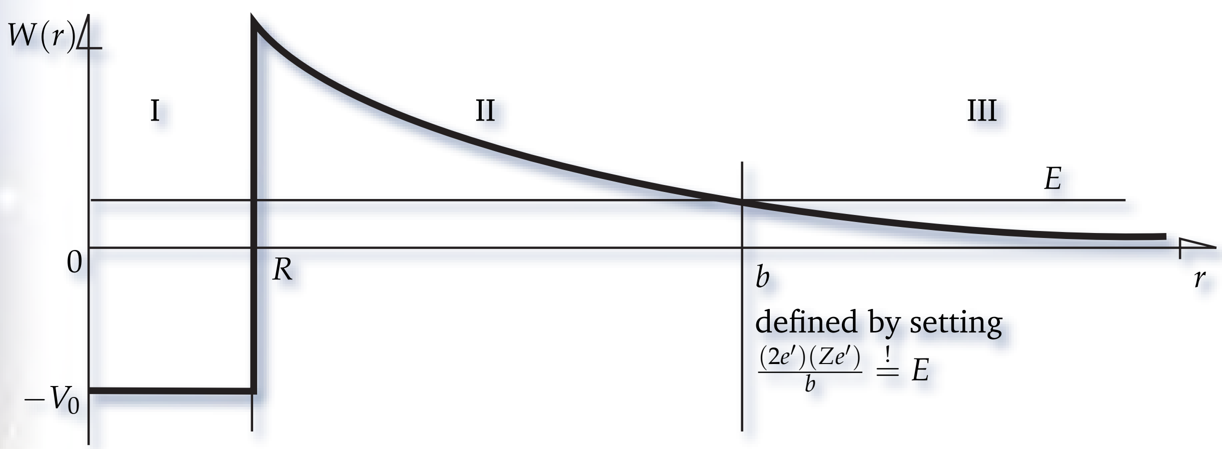
$$C \approx B \frac{(2\ell+1)}{(2\beta)^{2\ell+1}} \frac{\Gamma^2(2\ell+1)\sin[2\ell\pi]}{\Gamma^2(\ell+1+w)\sin[\pi(\ell+w)]},$$

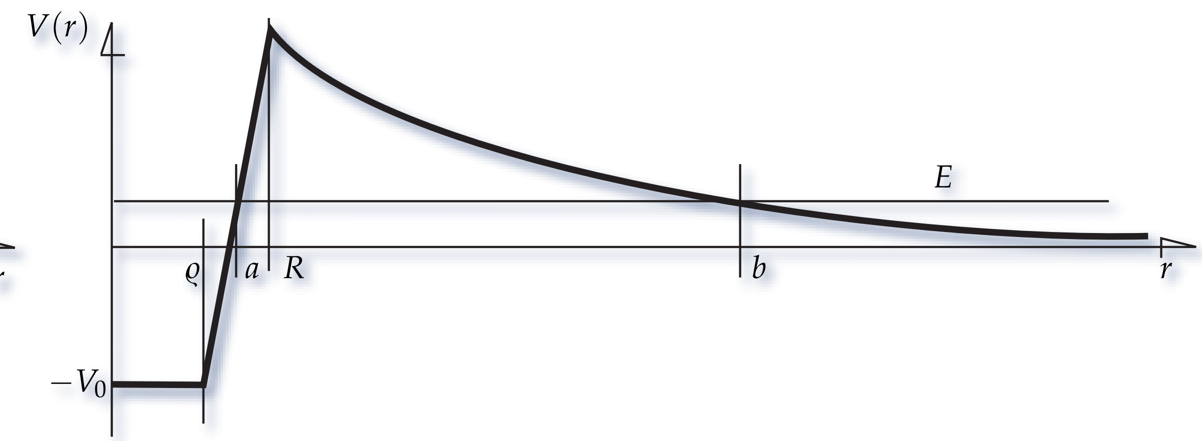
- Note that $w = 2Ze'^2 m_\alpha/\beta\hbar^2$ is imaginary
  - Use: $x^{ik} = e^{ik\ln(x)} = \cos[k\ln(x)] + i\sin[k\ln(x)]$…
  - Use $\Delta R(r) = 0 = \Delta R'(r)$ matching conditions @ $r = R$
  - Setting $F \approx 0$ leaves only 2 constants + E, w/2+1 constraints

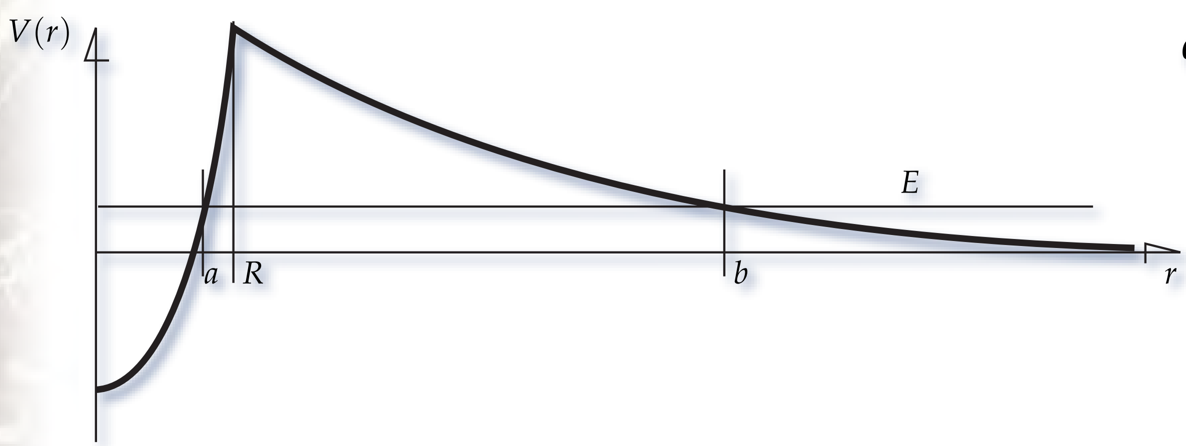
WKB: α-Decay

Improvements

- Modify the model:







$$\sigma = \sqrt{\frac{2m_\alpha}{\hbar^2}}\left[\int_a^R \mathrm{d}r\,\sqrt{\tfrac{1}{2}m_\alpha \omega r^2 - E} + \int_R^b \mathrm{d}r\,\sqrt{\frac{2Ze'^2}{r} - E}\right],$$

$$= \sqrt{\frac{2m_\alpha}{\hbar^2}}\left[\frac{R}{2}\sqrt{\tfrac{1}{2}m_\alpha\omega^2 R^2 - E} - \frac{a}{2}\sqrt{\tfrac{1}{2}m_\alpha\omega^2 a^2 - E} + \frac{E}{\omega\sqrt{2m_\alpha}}\ln\left(\frac{a m_\alpha \omega^2 + \omega\sqrt{2m_\alpha}\sqrt{\tfrac{1}{2}m_\alpha\omega^2 a^2 - E}}{R m_\alpha\omega^2 + \omega\sqrt{2m_\alpha}\sqrt{\tfrac{1}{2}m_\alpha\omega^2 R^2 - E}}\right)\right]$$

$$+ \sqrt{\frac{2m_\alpha E}{\hbar^2}}\, b\left(\arccos\sqrt{\frac{R}{b}} - \sqrt{\frac{R}{b}}\sqrt{1 - \frac{R}{b}}\right),$$

- They induce higher order corrections, improving the precision but not the overall behavior

Quantum Mechanics II

Now, go forth and calculate!!

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