Quantum Mechanics II

Discrete Physics

C: Charge Conjugation;
P: Space Reflections;
T: Time Reversal;

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Discrete Symmetries

C: Charge Conjugation

Extra!

- Charged particles interact with the EM field

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = \widehat{H}\,|\Psi\rangle = \big[\,\tfrac{1}{2M}\big(\widehat{\vec{P}} - \tfrac{q}{c}\vec{A}\big)^2 + \big(W + q\,\Phi\big)\,\big]\,|\Psi\rangle$$

- …and are subject to the gauge transformation
- $|\Psi\rangle \simeq e^{-i(q\chi/\hbar c)}\,|\Psi\rangle$, where q is the charge of the particle $|\Psi\rangle$.
- But then, $\langle\Psi| \simeq e^{+i(q\chi/\hbar c)}\,\langle\Psi| = e^{-i(-q)\chi/\hbar c}\,\langle\Psi|$.
  - So, $|\Psi\rangle$ and $\langle\Psi|$ interact with the EM field like particles with opposite electric charges

Observables are Hermitian.

- Hermitian conjugation flips the sign of the electric change
- and so is identified with Charge Conjugation ($\hat{C}$).
- What must the eigenvalues of $\hat{C}$ be? …………………………… ± 1
  - Why? …… b/c: $\hat{C}^2|\gamma\rangle = \hat{C}(\hat{C}|\gamma\rangle) = \hat{C}(\gamma|\gamma\rangle) = \gamma(\hat{C}|\gamma\rangle) = \gamma^2|\gamma\rangle = 1\,|\gamma\rangle$. *why?*
- What must the electric charges of the eigenstates be? ……… 0

We'll be back.

Discrete Symmetries

Π: Space Inversion

- First of all, space is 3-dimensional.
  - Reflect only one of the coordinates .... (planar, familiar) reflection (through the other two coordinate's plane)
  - Reflect two of the coordinates .... equivalent to a 180° rotation (through the third coordinate's axis line)
  - Reflect all three of the coordinates .... (coordinate) inversion (through the origin of the coordinate system)
- Furthermore, inversion = reflection + 180° rotation
- Can (& will) reflect through mutually non-orthogonal planes
- Just as with $\hat{C}$, $\widehat{\Pi}^2 = \mathbb{1}$ & $\widehat{\Pi}\,|\varpi\rangle = \varpi\,|\varpi\rangle \quad\Rightarrow\quad \varpi = \pm 1$
- What are the eigenstates?
- Built from coordinate eigenstates: $\widehat{\vec{Q}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$

$$\widehat{\Pi}\,|\vec{r}\rangle = |{-\vec{r}}\rangle \qquad \widehat{\Pi}\tfrac{1}{\sqrt{2}}\big(|\vec{r}\rangle \pm |{-\vec{r}}\rangle\big) = \pm\underbrace{\tfrac{1}{\sqrt{2}}\big(|\vec{r}\rangle \pm |{-\vec{r}}\rangle\big)}_{:=|\vec{r}\rangle_\pm}$$

Discrete Symmetries

Π: Space Inversion

- Actually, there's a small matter of legalese:
  - We *could* have chosen: $\widehat{\Pi}^2 = e^{i\vartheta}$ $\widehat{\Pi}\,|\vec{r}\rangle = e^{i\vartheta/2}\,|{-\vec{r}}\rangle$ $|\vec{r}\rangle_\pm := \frac{1}{\sqrt{2}}e^{i\vartheta/2}\big(\,|\vec{r}\rangle \pm |{-\vec{r}}\rangle\,\big)$
- Another caveat: is this a linear or an antilinear operator?
  - Linear operators: $\hat{O}\,(c_1|\Psi_1\rangle + c_2|\Psi_2\rangle) = c_1\,\hat{O}(|\Psi_1\rangle) + c_2\,\hat{O}(|\Psi_2\rangle)$.
  - Antilinear operators: $\hat{A}\,(c_1|\Psi_1\rangle + c_2|\Psi_2\rangle) = c_1^*\,\hat{A}(|\Psi_1\rangle) + c_2^*\,\hat{A}(|\Psi_2\rangle)$.
  - That is: linear operators commute with constants
    antilinear operators don't; they conjugate them in passing.
- Want: $\widehat{\Pi}^{-1}\widehat{Q}_\alpha\widehat{\Pi} = -\widehat{Q}_\alpha$ $\widehat{\Pi}^{-1}\widehat{P}_\alpha\widehat{\Pi} = -\widehat{P}_\alpha$ $\widehat{\Pi}^{-1}\widehat{G}_\alpha\widehat{\Pi} = -\widehat{G}_\alpha$

  ("true," "real," "polar") vectors

  $\widehat{\Pi}^{-1}\widehat{J}_\alpha\widehat{\Pi} = +\widehat{J}_\alpha$ — axial (pseudo-)vector

  $\widehat{\Pi}^{-1}\widehat{H}\widehat{\Pi} = +\widehat{H}$ — true scalar

  Linear! $\widehat{\Pi}^{-1}i\widehat{\Pi} = i$

- Then, $\widehat{\Pi}^{-1}\big[\widehat{Q}_\alpha, \widehat{P}_\alpha\big]\widehat{\Pi} = \widehat{\Pi}^{-1}i\hbar\widehat{\Pi} = \widehat{\Pi}^{-1}i\widehat{\Pi}\hbar = i\hbar$

  no sum on α

  $= \big[\widehat{\Pi}^{-1}\widehat{Q}_\alpha\widehat{\Pi}, \widehat{\Pi}^{-1}\widehat{P}_\alpha\widehat{\Pi}\big] = \big[(+\widehat{Q}_\alpha), (-\widehat{P}_\alpha)\big]$

Discrete Symmetries

Π: Space Reflections

- So, $\widehat{\Pi}^2 = \mathbb{1}$ $\quad \widehat{\Pi}\,|\vec{r}\rangle = |{-\vec{r}}\rangle$ $\quad |\vec{r}\rangle_\pm := \frac{1}{\sqrt{2}}\big(|\vec{r}\rangle \pm |{-\vec{r}}\rangle\big)$

$$\widehat{\Pi} = \widehat{\Pi}^{-1} = \widehat{\Pi}^\dagger \qquad \widehat{\Pi} = e^{i\pi \widehat{J}_x}\circ\widehat{\Pi}_x = e^{i\pi \widehat{J}_z}\circ\widehat{\Pi}_z = \dots$$

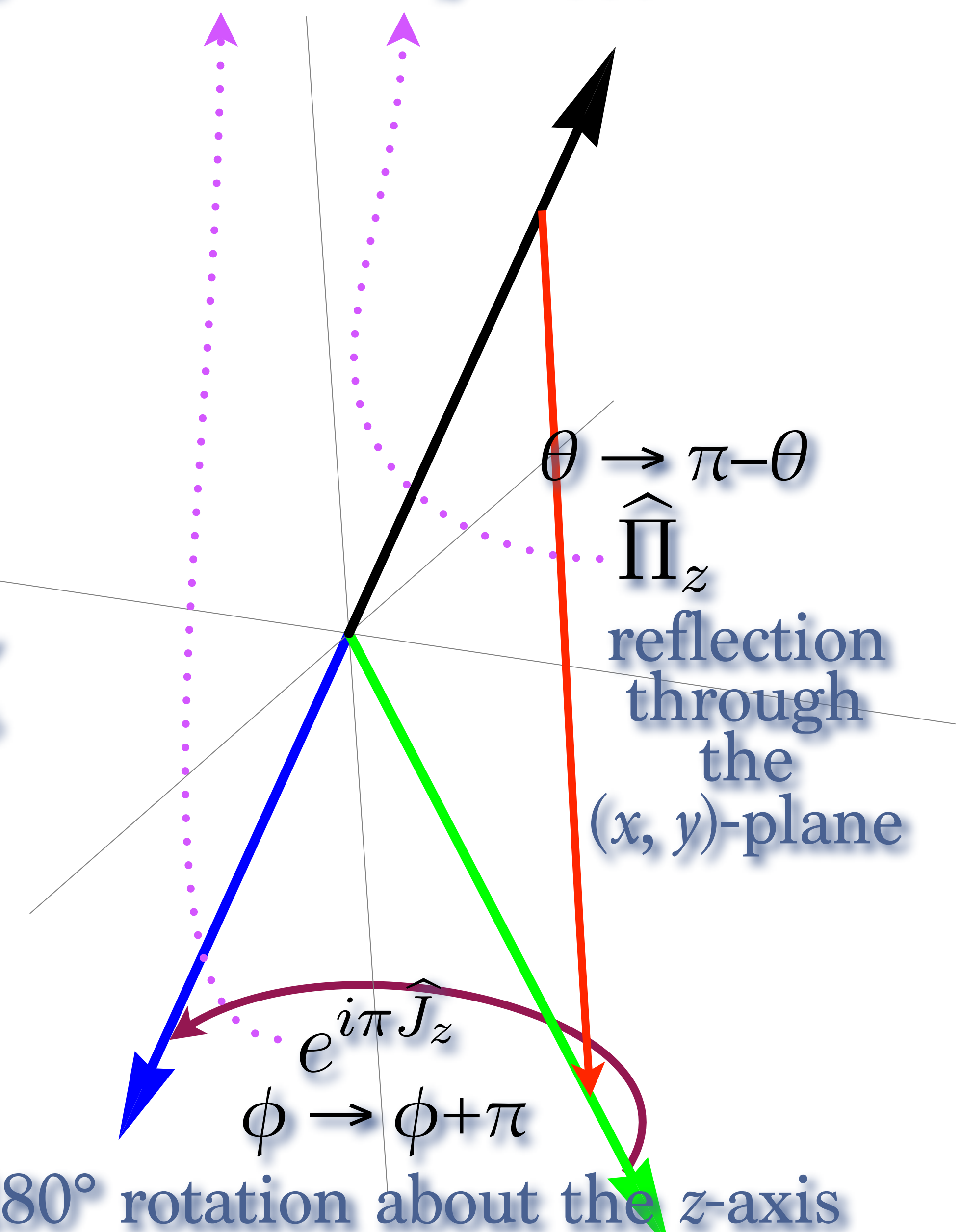
- In spherical coordinates?
  - Cannot map $\theta \to \theta + \pi$, since $0 \le \theta \le \pi$
  - Instead, map $(\theta, \phi) \to (\pi{-}\theta, \phi{+}\pi)$

$$\widehat{\Pi}\,Y_\ell^m(\theta,\phi) = Y_\ell^m(\pi{-}\theta,\phi{+}\pi)$$

$$\propto \underbrace{\left[(1{-}u^2)^{\frac{m}{2}}\frac{\mathrm{d}^{m+\ell}}{\mathrm{d}u^{m+\ell}}(u^2{-}1)^\ell\right]}_{P_\ell^m(-u),\quad u=\cos(\theta)=-\cos(\pi-\theta)} \times \underbrace{e^{-im(\phi+\pi)}}_{(-1)^m e^{-im\phi}}$$

$$(-1)^{m+\ell}P_\ell^m(u)$$

$$\widehat{\Pi}\;|\ell,m\rangle = (-1)^\ell\,|\ell,m\rangle$$



Discrete Symmetries

Π: Space Reflections

- However, $\widehat{\Pi}\,|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle=(-1)^{\ell_1+\ell_2}\,|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle$
- and $|\ell_1,\ell_2,j,m\rangle=\sum_{m_1+m_2=m}C^{j,m}_{\ell_1,\ell_2;m_1,m_2}\,|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle$
- so $\widehat{\Pi}\,|\ell_1,\ell_2,j,m\rangle=(-1)^{\ell_1+\ell_2}\,|\ell_1,\ell_2,j,m\rangle\neq(-1)^{j}\,|\ell_1,\ell_2,j,m\rangle$
- Why? .... 'cos $|\ell_1-\ell_2|\leq j\leq|\ell_1+\ell_2|$
- Electric dipole moment $\hat{\vec{d}}=\sum_k q_k\,\hat{\vec{Q}}_k$

$$\vec{d}:=\langle\Psi|\hat{\vec{d}}\,|\Psi\rangle=\langle\Psi|\widehat{\Pi}^{\dagger}\widehat{\Pi}\hat{\vec{d}}\,\widehat{\Pi}^{-1}\widehat{\Pi}|\Psi\rangle=\langle\Psi'|(-\hat{\vec{d}}\,)|\Psi'\rangle$$

$$\widehat{H}\,|\Psi\rangle=E_{\Psi}\,|\Psi\rangle \qquad \Big(\widehat{\Pi}\widehat{H}\widehat{\Pi}^{-1}\,|\Psi'\rangle=\widehat{H}\,|\Psi'\rangle\Big)=E_{\Psi}\,|\Psi'\rangle$$

$$|\Psi'\rangle=c\,|\Psi\rangle$$

if not degenerate!

$$\vec{d}=-\langle\Psi'|\hat{\vec{d}}\,|\Psi'\rangle=-|c|^2\langle\Psi|\hat{\vec{d}}\,|\Psi\rangle=-\vec{d}\;=0$$

if $\hat{H}$ is scalar and $|\Psi\rangle$ is not degenerate!

Discrete Symmetries

T: Time Reversal

- Seems just like space inversion / reflection: $t \to -t$.
- Want: $\widehat{T}^{-1}\widehat{Q}_\alpha\widehat{T} = +\widehat{Q}_\alpha \quad \widehat{T}^{-1}\widehat{P}_\alpha\widehat{T} = -\widehat{P}_\alpha \quad \widehat{T}^{-1}\widehat{J}_\alpha\widehat{T} = -\widehat{J}_\alpha$
- Then $\widehat{T}^{-1}[\widehat{Q}_\alpha, \widehat{P}_\alpha]\widehat{T} = \widehat{T}^{-1}i\hbar\widehat{T} = \widehat{T}^{-1}i\widehat{T}\hbar \qquad -i\hbar$

$$= [\widehat{T}^{-1}\widehat{Q}_\alpha\widehat{T}, \widehat{T}^{-1}\widehat{P}_\alpha\widehat{T}] = [(+\widehat{Q}_\alpha), (-\widehat{P}_\alpha)]$$

- It follows that $\widehat{T}^{-1}i\widehat{T} = -i$ time-reversal is antiunitary!
- Following Ballentine, let time-reversal act only from left.
- Complex conjugation is antilinear; depends on the basis on which it acts. (Ex.: if two bases differ only by phases.)

$$\widehat{T}\widehat{H}\widehat{T}^{-1}\widehat{T}|\Psi(t)\rangle = \widehat{T}i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = -i\hbar\widehat{T}\frac{\partial}{\partial t}|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\widehat{T}|\Psi(t)\rangle$$

$$\widehat{T}\widehat{H}\widehat{T}^{-1} = \widehat{H} \qquad \widehat{H}\big(\alpha|\Psi\rangle + \beta\widehat{T}|\Psi\rangle\big) = E_\Psi\big(\alpha|\Psi\rangle + \beta\widehat{T}|\Psi\rangle\big)$$

unphysical! Why?

Discrete Symmetries

$[\hat{S}_\alpha, \hat{S}_\beta] = i\hbar\, \varepsilon_{\alpha\beta}{}^{\gamma}\, \hat{S}_\gamma$

T: Time Reversal and Angular Momentum

- Consider a spin-½ system. Want: $\hat{S}_\alpha = \frac{1}{2}\hbar\hat{\boldsymbol{\sigma}}_\alpha \quad \hat{T}\hat{S}_\alpha\hat{T}^{-1} = -\hat{S}_\alpha$
- Know: $\hat{K}^{-1}\boldsymbol{\sigma}_x\hat{K} = \boldsymbol{\sigma}_x \quad \hat{K}^{-1}\boldsymbol{\sigma}_y\hat{K} = -\boldsymbol{\sigma}_y \quad \hat{K}^{-1}\boldsymbol{\sigma}_z\hat{K} = \boldsymbol{\sigma}_z$

  complex conj. real imaginary real

- Since $\hat{T} \neq \hat{K} = \hat{K}^{-1}$, write $\hat{T} = \hat{Y}\hat{K}$ $\quad \hat{Y} = \hat{K}\hat{T}$ is linear.
- Now, $\hat{Y}$ maps: $\hat{Y}^{-1}\hat{S}_x\hat{Y} = -\hat{S}_x$ and $\hat{Y}^{-1}\hat{S}_z\hat{Y} = -\hat{S}_z$, but $\hat{Y}^{-1}\hat{S}_y\hat{Y} = +\hat{S}_y$.
- Remember: reflection across the y-axis $= e^{-i\pi\hat{S}_y/\hbar} = \mathrm{Rot}_y(\pi)$!
- Thus, $\hat{T} = e^{-i\pi\hat{S}_y/\hbar}\,\hat{K}$. In turn, $\hat{T}^2 = e^{-2i\pi\hat{S}_y/\hbar} = e^{-2i\pi\hat{J}_y/\hbar}$
- …since $e^{-2i\pi\hat{L}_y/\hbar} = \mathbb{1}$. Therefore $\hat{T}^2\,|j,m\rangle = (-1)^{2j}\,|j,m\rangle$
- Kramer's Theorem: Time-reversal may be linearly dependent

  $\hat{T}\,|\Psi\rangle = c\,|\Psi\rangle \quad \hat{T}^2\,|\Psi\rangle = \hat{T}\,c\,|\Psi\rangle = c^*\hat{T}\,|\Psi\rangle = c^*c\,|\Psi\rangle = (-1)^{2j_\Psi}\,|\Psi\rangle$
- …only if j_Ψ is integral, and then $|c| = 1$.

  if $\hat{T}^{-1}\hat{H}\hat{T} = \hat{H}$
- For fermions, time-reversal produces a degenerate state.

Discrete Symmetries

A Summary & General Facts

- Charge conjugation equivalent to Hermitian conjugation
  - In relativistic physics w/fermions, Hermitian → Dirac conjugation
  - *Is not complex conjugation!* ($\hat{C}$: particles into antiparticles w/opp. ch.)
- Space inversion = reflection through a point (coord. origin)
  - Reflection through any straight line = 180° rotation about that line
  - Space inversion = reflection through any plane + 180° rotation in it
- Time-reversal is antilinear
  - Squares to $(-1)^{2j}$, not to **1**
  - Spin-dependent action
  - Recall: "spin" = "directional response to rotations"
    $|j, m\rangle$ means that the system intrinsically response to rotations
  - A "representation of the *Spin*(3) = *SU*(2) angular mom. algebra"
  - Particles (except scalars) do have a sense of direction to them.

$$\widehat{T} = e^{-i\pi\widehat{S}_i/\hbar}\,\widehat{K}$$

$$(\widehat{S}_i)^* = -\widehat{S}_i \;\;\&\;\; (\widehat{S}_j)^* = \widehat{S}_j, \;\; j \neq i$$

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Now, go forth and calculate!!

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