Quantum Mechanics II

Time-Dependent Physics

Energy-Time Indeterminacy;
Quantum Beats;
Kaons and Neutrinos

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Time-Dependent Phenomena

Energy-Time Indeterminacy

- Heisenberg's indeterminacy relations $\Delta_Q \Delta_P \geq \frac{1}{2}\hbar$
- Often, as "dual variables in Fourier transformation"
  - Cartesian coordinates, $\mathscr{F}$: $f(x) \to F(k_x)$, and $k_x := p_x/\hbar$. …*however*…
  - Typically, $\mathscr{F}$: $f(t) \to F(\omega)$, and $\omega := (E_2 - E_1)\hbar$, and so $\omega \neq E\hbar$
  - Ballentine: energy may be shifted $E \to E + E_0$, with $E_0 = const$. arbitrary
  - But, so can $p \to p + p_0$: for a fixed mass, this is just a Galilean boost.
- Sometimes, as "canonically conjugate variables"
  - No general precise derivation (dimensional analysis does check out)
  - Not general enough…

Time-Dependent Phenomena

General Indeterminacy Relations

- Robertson ('29), Schrödinger ('30), Jackiw & Carruthers+Nieto ('68):
- Given two Hermitian operators, define the third one as

$$\widehat{C} := -i[\,\widehat{A}\,,\,\widehat{B}\,] = -i[\,(\widehat{A}-\langle\widehat{A}\rangle)\,,\,(\widehat{B}-\langle\widehat{B}\rangle)\,]$$

$$0 \leqslant \Big\langle \Big([\widehat{A}-\langle\widehat{A}\rangle] - i\xi[\widehat{B}-\langle\widehat{B}\rangle]\Big)^{\dagger}\Big([\widehat{A}-\langle\widehat{A}\rangle] - i\xi[\widehat{B}-\langle\widehat{B}\rangle]\Big)\Big\rangle$$

$$= \Big\langle |\widehat{A}-\langle\widehat{A}\rangle|^2 \Big\rangle - i\xi \Big\langle [\widehat{A}-\langle\widehat{A}\rangle, \widehat{B}-\langle\widehat{B}\rangle] \Big\rangle + \xi^2 \Big\langle |\widehat{B}-\langle\widehat{B}\rangle|^2 \Big\rangle$$

$$= (\Delta_A)^2 + \xi\langle\widehat{C}\rangle + \xi^2(\Delta_B)^2$$

- True for all ξ, this is true for $\min(\xi) = -\langle\widehat{C}\rangle/2(\Delta_B)^2$

$$= (\Delta_A)^2 - \frac{\langle\widehat{C}\rangle^2}{2(\Delta_B)^2} + \frac{\langle\widehat{C}\rangle^2}{4(\Delta_B)^2} \quad = (\Delta_A)^2 - \frac{\langle\widehat{C}\rangle^2}{4(\Delta_B)^2}$$

$$(\Delta_A)^2 \geqslant \frac{\langle\widehat{C}\rangle^2}{4(\Delta_B)^2} \qquad (\Delta_A)^2(\Delta_B)^2 \geqslant \tfrac{1}{4}\underbrace{\langle\widehat{C}\rangle^2}_{\substack{\geq 0 \\ (\hat{C}^{\dagger} = \hat{C})}}$$

$$\boxed{\Delta_A\Delta_B \geqslant \tfrac{1}{2}\Big|\langle[\widehat{A},\widehat{B}]\rangle\Big|}$$

state-dependent

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  - But, so can $p \to p + p_0$: for a fixed mass, this is just a Galilean boost.
- Sometimes, as "canonically conjugate variables"
  - No general precise derivation (dimensional analysis does check out)
  - Not general enough…
- Nonrelativistic Quantum Mechanics
  - coordinates are eigenvalues (expectation values) of Hermitian operators
  - time is not.

Time is a parameter

…on which everything *else* depends

Time-Dependent Phenomena

Energy-Time Indeterminacy

- Wolfgang Pauli's "*TH*-theorem" (1933)
- Suppose time was the eigenvalue of a Hermitian operator, canonically conjugate to the Hamiltonian

$$[\hat{T}, \hat{H}] = i\hbar \qquad \hat{U}_\varepsilon := \exp\{-i\varepsilon\hat{T}/\hbar\}$$

$$[\hat{H}, \hat{U}_\varepsilon] = [\hat{H}, \hat{T}]\frac{\partial\hat{U}_\varepsilon}{\partial\hat{T}} = (-i\hbar)\hat{U}_\varepsilon(-i\varepsilon/\hbar) = -\varepsilon\,\hat{U}_\varepsilon$$

- Then,

$$\hat{H}\,|E\rangle = E\,|E\rangle \quad \hat{H}(\hat{U}_\varepsilon\,|E\rangle\,) = (\hat{U}_\varepsilon\overset{E}{\hat{H}} - \varepsilon\hat{U}_\varepsilon)\,|E\rangle = (E-\varepsilon)(\hat{U}_\varepsilon\,|E\rangle\,)$$

Time-Dependent Phenomena

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$\hat{U}_\varepsilon\,|E\rangle = |E-\varepsilon\rangle$ **ε → ∞ ⇒ ∃ normalized state with E → -∞**

⇒ no ground-state with definite energy!

- Thus, time cannot be the eigenvalue of a Hermitian operator.

Time-Dependent Phenomena

Energy-Time Indeterminacy

- So, what *should* a relationship such as "$\Delta_E\,\Delta_\tau \geq i\hbar$" mean?
  - It is not time itself that is observed, but sequential variations in some other observable…
  - …which then serves as a clock.
- Consider then an observable that is not stationary

$$[\,\widehat{H}\,,\,\widehat{R}\,] \neq 0 \qquad \Delta_R\Delta_H \geqslant \tfrac{1}{2}\big|\langle[\widehat{H},\widehat{R}]\rangle\big| = \frac{\hbar}{2}\left|\frac{\mathrm{d}\langle\widehat{R}\rangle}{\mathrm{d}t}\right|$$

S. Mandelstam & I. Tamm

- Then, define:

$$\tau_R := \Delta_R\left|\frac{\mathrm{d}\langle\widehat{R}\rangle}{\mathrm{d}t}\right|^{-1} \qquad \tau_R\,\Delta_H \geqslant \tfrac{1}{2}\hbar$$

- This τ_R serves as a characteristic time (period) of any phenomenon in which variations in the observable R serve as a clock. Whence "clock-observable."

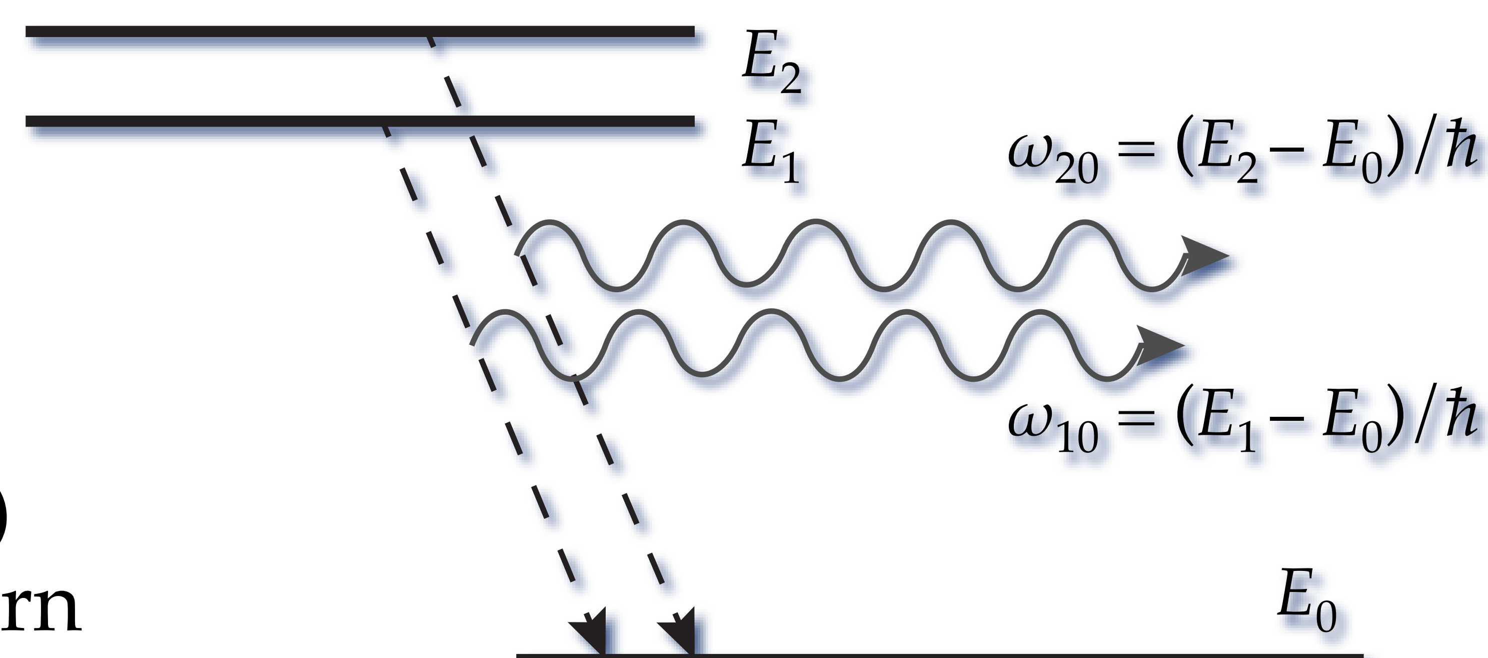
Time-Dependent Phenomena

Quantum Beats

- Recall the spin-½ system (particle) with $\boldsymbol{\mu} = \gamma \boldsymbol{S}$
  - …gaining/losing $\tfrac{1}{2}\gamma\hbar\boldsymbol{B}\cdot\boldsymbol{\sigma}$ energy in the $\boldsymbol{B}$-field
  - …with $\boldsymbol{\mu}$ precessing with frequency $\omega = \gamma\,|\boldsymbol{B}|$
- A linear combination of the two states is not stationary

$$\hat{U}_t \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = e^{-i\omega t \sigma_z} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} c_+ e^{-i\omega t} \\ c_- e^{+i\omega t} \end{bmatrix}$$

- Similar result for all 2-state systems
- Suppose this linear combination decays into a lower state
- The two emitted photons interfere
- Int. ∝ Prob. ∝ $\cos(\omega_{21}\, t)$ modulated decay pattern

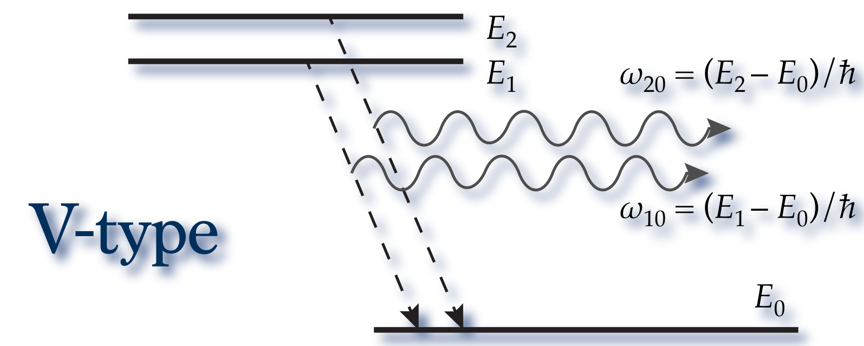


Time-Dependent Phenomena

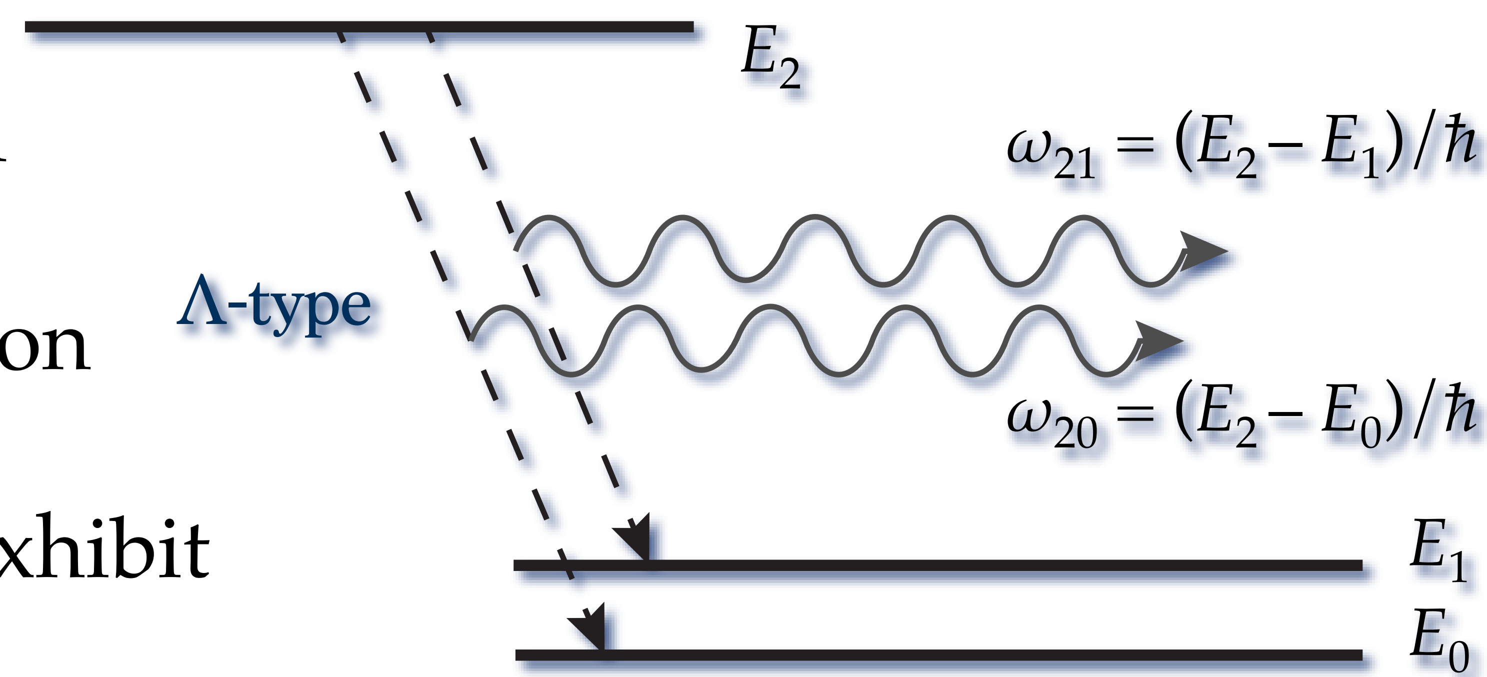
Quantum Beats

- Treating the EM radiation classically,

$$I \propto \sin(\omega_{20}t) + \sin(\omega_{10}t)$$
$$= \underbrace{2\sin\left(\frac{\omega_{20}+\omega_{10}}{2}t\right)}_{\text{“carrier”}} \underbrace{\sin\left(\frac{\omega_{20}-\omega_{10}}{2}t\right)}_{\text{“AM” = beats}}$$



- A similar effect should also exist in the flipped situation…
- …while the EM radiation is treated classically
- But, V-type atoms do exhibit beats as predicted…
- …while Λ-type atoms do not.



Time-Dependent Phenomena

Quantum Beats

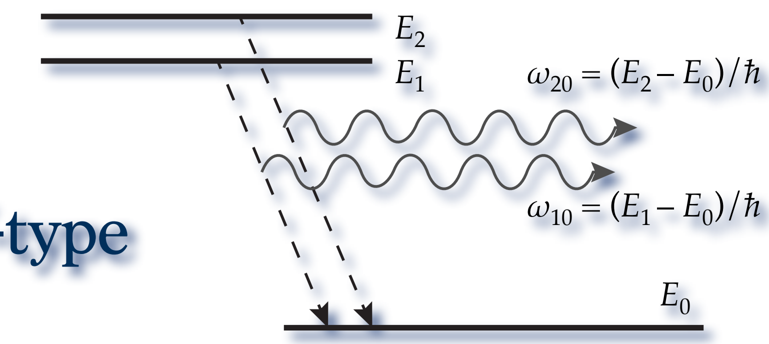
- The full quantum description of the two cases uses the final states:

$$\alpha_1 \left|E_0;\omega_{10}\right\rangle + \alpha_2 \left|E_0;\omega_{20}\right\rangle$$

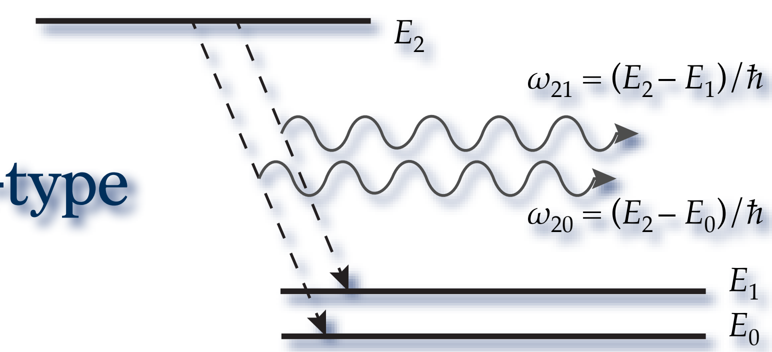
$$\beta_1 \left|E_1;\omega_{21}\right\rangle + \beta_2 \left|E_0;\omega_{20}\right\rangle$$

EM radiation

V-type



Λ-type



- Beats are caused by interference
- …for which the probability amplitudes are

$$\left\langle E_0;\omega_{10}\right| \widehat{H}_\gamma \left|E_0;\omega_{20}\right\rangle_{\text{V-type}} \qquad \left\langle E_1;\omega_{21}\right| \widehat{H}_\gamma \left|E_0;\omega_{20}\right\rangle_{\Lambda\text{-type}}$$

$$\underbrace{\left\langle E_0|E_0\right\rangle}_{=1} \cdot \left\langle\omega_{10}\right| \widehat{H}_\gamma \left|\omega_{20}\right\rangle \qquad \underbrace{\left\langle E_1|E_0\right\rangle}_{=0} \cdot \left\langle\omega_{21}\right| \widehat{H}_\gamma \left|\omega_{20}\right\rangle$$

beats   ~~beats~~

- Useful when E_1 and E_2 cannot be resolved experimentally

Time-Dependent Phenomena

Kaons

- Two neutral, spin-0 mesons
  - one decays into two pions, after 8.958×10^{-11} s $= K_S$ $CP = +1$
  - the other into three pions, after 5.114×10^{-8} s $= K_L$ $CP = -1$
  - although they are created in same collision processes
- So, K_S and K_L are decay eigenstates
  - The creation eigenstates are $K_0 = (K_S + K_L)$ and $\bar{K}_0 = (K_S - K_L)$
- Created 50%-50%, the ratio soon depletes

$$\frac{N(K_S)}{N(K_L)} = \frac{e^{-t/\tau_S}}{e^{-t/\tau_L}} = \exp\left\{-\frac{t}{\tau_S} + \frac{t}{\tau_L}\right\} \approx \exp\left\{-569.9\,\frac{t}{\tau_L}\right\}$$

  - which drops to 1.447×10^{-5} after just 1 ns!
  - However, virtual decay-undecay processes can oscillate $K_S \leftrightarrows K_L$
  - …and yield two-pion decays even after many seconds of flight.
  - This regeneration violates CP conservation *needed for Big-Bang baryogenesis*

Time-Dependent Phenomena

Neutrino Oscillations

- β-decay: ${}_{Z}\mathrm{X} \to {}_{Z\pm1}\mathrm{Y} + e^{\pm}$ (*i.e.*, $n^0 \to p^+ + e^-$ or $p^+ \to n^0 + e^+$) cannot satisfy both energy and momentum conservation
- W. Pauli (1930): a third (very light, neutral) particle
  E. Fermi: "small neutron" = neutrino
- $n^0 \to p^+ + e^- + \bar{\nu}$ or $p^+ \to n^0 + e^+ + \nu$
- In the next 1-2 decades, cosmic ray sources:

$$\begin{aligned}\pi^- &\to \mu^- + \bar{\nu}_\mu \\ &\to (e^- + \bar{\nu}_e + \nu_\mu) + \bar{\nu}_\mu \\ \pi^+ &\to \mu^+ + \nu_\mu \\ &\to (e^+ + \nu_e + \bar{\nu}_\mu) + \nu_\mu\end{aligned}$$

- There should be twice as many cosmic ν_μ's than ν_e's.
- Vertically, yes; ≈2:1. Horizontally (along the horizon), no; ≈1:1

Time-Dependent Phenomena

Neutrino Oscillations

- By 1938, Hans Bethe: "Carbon cycle" & "*pp*-process" H→He fusion in stars such as the Sun
- …with a detailed spectrum of neutrinos predicted
- …of which only ~⅓ ν_e's arrive to Earth (ν_μ's not detected)
- Creation & detection = $\hat{H}_I$-eigenstates, ν_e and ν_μ
- Propagation/evolution = $\hat{H}_0$-eigenstates, say "1" and "2"
  - …these are also the mass-eigenstates
- Created as, say, ν_e:

$$e^{-it\hat{H}_0/\hbar}\Big(|\nu_e;0\rangle := |\text{“}1{+}2\text{”};0\rangle \Big) = |\text{“}1{+}2\text{”};t\rangle$$

$$= C_1 e^{-iE_1 t/\hbar}\,|1\rangle + C_2 e^{-iE_2 t/\hbar}\,|2\rangle$$

$$P_\alpha := \Big|\big[\cos(\alpha)\langle 1| + \sin(\alpha)\langle 2|\big]\,|\text{“}1{+}2\text{”};t\rangle\Big|^2$$

$$= |C_1|^2\cos^2(\alpha) + |C_2|^2\sin^2(\alpha) + \sin(2\alpha)\,\Re e\left[C_1 C_2^* e^{-i(E_1-E_2)t/\hbar}\right]$$

Time-Dependent Phenomena

Neutrino Oscillations

$$P_\alpha := \Big|\big[\cos(\alpha)\langle 1| + \sin(\alpha)\langle 2|\big]|\text{“}1+2\text{”};t\rangle\Big|^2$$
$$= |C_1|^2\cos^2(\alpha) + |C_2|^2\sin^2(\alpha) + \sin(2\alpha)\,\Re e\left[C_1 C_2^* e^{-i(E_1-E_2)t/\hbar}\right]$$

- So, if the neutrino was initially in the “opposite” linear combination, $-\sin(\alpha)|1\rangle + \cos(\alpha)|2\rangle$, (the $\alpha + \pi/2$ state)

$$P_{|\alpha+\frac{\pi}{2}\rangle\to|\alpha\rangle} = \sin^2(2\alpha)\sin^2(\tfrac{1}{2}\omega_{12}t), \qquad \omega_{12} := \frac{E_1-E_2}{\hbar}$$

- The neutrino oscillates

$$|\nu_e\rangle := \quad |\alpha\rangle = \cos(\alpha)\,|1\rangle + \sin(\alpha)\,|2\rangle$$
$$|\nu_\mu\rangle := \left|\alpha+\tfrac{\pi}{2}\right\rangle = -\sin(\alpha)\,|1\rangle + \cos(\alpha)\,|2\rangle$$

- …provided:
  - The two stationary states are not degenerate, $E_1 \neq E_2$, $\omega_{12} \neq 0$
  - The interaction eigenstates are not equal to the stationary states, $\alpha \neq 0$

and $E_1 - E_2 = \sqrt{|\vec{p}|^2c^2 + m_1^2c^4} - \sqrt{|\vec{p}|^2c^2 + m_2^2c^4} \approx \dfrac{(m_1^2 - m_2^2)c^4}{2\bar{E}}$

and, there are really three distinct neutrinos…

Quantum Mechanics II

Now, go forth and calculate!!

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