Quantum Mechanics II

Time-Dependent Physics

Exponential vs. Non-Exponential Decay; A Typo and an Integral; Decay Paradoxes

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Time-Dependent Phenomena

Decay Probability: A Century-Old Result

- Many systems are unstable; $|\Psi|^2$ need not be conserved.
- "State decay" is really state transition
- General expectation: $\text{Prob}(u(t_2)\,|\,u(t_1)) = \text{Prob}(t_2 - t_1)$ if $t_2 > t_1$.
- Also, expect: $\text{Prob}(u(t_3)\ \&\ u(t_2)\,|\,u(t_1)) = \text{Prob}(u(t_3)\,|\,u(t_1))$
  - whenever $t_3 > t_2 > t_1$.
  - Assumption(!): if $u(t_3) = \text{true}$, then $u(t_2) = \text{true}$ also.
- Bayes' rule (re-read section 1.5):
  - $\text{Prob}(u(t_3)\ \&\ u(t_2)\,|\,u(t_1)) = \text{Prob}(u(t_3)\,|\,\cancel{u(t_2)}\ \&\ u(t_1)) \cdot \xcancel{\text{Prob}(u(t_2)\,|\,u(t_1))}$

Time-Dependent Phenomena

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  - whenever $t_3 > t_2 > t_1$.
  - Assumption(!): ~~if $u(t_3)$ = true, then $u(t_2)$ = true also.~~ Caution!
- Bayes' rule (re-read section 1.5):
  - $\text{Prob}(u(t_3)\ \&\ u(t_2)\,|\,u(t_1)) = \text{Prob}(u(t_3)\,|\,u(t_2)\ \&\ u(t_1)) \cdot \text{Prob}(u(t_2)\,|\,u(t_1))$
- The "general expectation" now implies that
  - $\text{Prob}(t_3 - t_1) = \text{Prob}(t_3 - t_2) \cdot \text{Prob}(t_2 - t_1)$
- Also, $\text{Prob}(\sim u(t_2)\,|\,u(t_1)) = 1 - \text{Prob}(u(t_2)\,|\,u(t_1))$
- Then: $\text{Prob}(u(t_2)\,|\,u(t_1)) = \exp\{-\lambda(t_2 - t_1)\}$ E. Rutherford

Exponential law of decay probabilities

Time-Dependent Phenomena

Decay Probability: the Quantum Theory

- A system evolves: $|\Psi(t)\rangle = \hat{U}_t|\Psi(0)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$
- The *amplitude* of probability that the system has not decayed
  - $A_0(t) = \langle\Psi(0)|e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$ & $P_u(t) = \text{Prob}(\Psi(0)|\Psi(0)) = |A_0(t)|^2$
- Denote $\langle\hat{H}\rangle = \langle\Psi(0)|\hat{H}|\Psi(0)\rangle$. Then

$$
\begin{aligned}
P_u(t) &:= \big|A_0(t)\big|^2 = \big|\langle\Psi(0)|e^{-i\widehat{H}t/\hbar}|\Psi(0)\rangle\big|^2 \\
&= \Big\langle 1 - i\tfrac{t}{\hbar}\widehat{H} - \tfrac{t^2}{2\hbar^2}\widehat{H}^2 + \dots \Big\rangle \cdot \Big\langle 1 - i\tfrac{t}{\hbar}\widehat{H} - \tfrac{t^2}{2\hbar^2}\widehat{H}^2 + \dots \Big\rangle^\dagger \\
&= \Big\langle 1 - i\tfrac{t}{\hbar}\widehat{H} - \tfrac{t^2}{2\hbar^2}\widehat{H}^2 + \dots \Big\rangle \cdot \Big\langle 1 + i\tfrac{t}{\hbar}\widehat{H} - \tfrac{t^2}{2\hbar^2}\widehat{H}^2 + \dots \Big\rangle \\
&= 1 - \tfrac{t^2}{\hbar^2}\Big\langle \big(\widehat{H} - \langle\widehat{H}\rangle\big)^2 \Big\rangle + \cdots
\end{aligned}
$$

$\neq e^{-\lambda t}$ **Huh?!**

- Where could Rutherford have gone wrong?
  - …after all, the exponential law is at the heart of the decay-clock
  - …vindicated by experiment, time and time again…

Time-Dependent Phenomena

Decay Probability: the Quantum Theory

- Well, write $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle = A_0(t)|\Psi(0)\rangle + |\Phi(t)\rangle$
  - …where $\langle\Psi(0)|\Phi(t)\rangle = 0$; $|\Phi(t)\rangle$ denotes "all else," $\perp$ to $|\Psi(0)\rangle$.
- Then, evolve from $t = 0$ to $t > 0$, to $(t'+t) > t > 0$.

$$A(t'+t) = \left\langle \Psi(0)|e^{-i\hat{H}t'/\hbar}\, e^{-i\hat{H}t/\hbar}|\Psi(0)\right\rangle$$
$$= \langle\Psi(0)|\, e^{-i\hat{H}t'/\hbar}\,\big(A_0(t)\,|\Psi(0)\rangle + |\Phi(t)\rangle\,\big)$$
$$= A_0(t)\,A_0(t') + \langle\Psi(0)|\, e^{-i\hat{H}t'/\hbar}\,|\Phi(t)\rangle$$

- So, if and only if $\langle\Psi(0)|e^{-i\hat{H}t'/\hbar}|\Phi(t)\rangle = 0$, $P_u(t) = e^{-\lambda t}$.
- Indeed, $\langle\Psi(0)|e^{-i\hat{H}t'/\hbar}|\Phi(t)\rangle = 0$ in nuclear decays: the probability that a nucleus spontaneously regenerates is nil.
  - …though, not unheard of: α-decay may be followed by α-(re)capture.
- The assumption "if $u(t_3)$ = true, then $u(t_2)$ = true also" need not hold
- This "intermediate (virtual) regeneration" turns $P_u(t) \neq e^{-\lambda t}$.

Time-Dependent Phenomena

Decay Probability: the Quantum Theory

- A "dual" description, in terms of energies
- Use an energy basis $\hat{H}\,|\,E_n\,\rangle = E_n\,|\,E_n\,\rangle$, where "$n$" is formal

$$|\Psi(0)\rangle = \sum_n |E_n\rangle\,\langle E_n|\Psi(0)\rangle$$

$$A(t) = \sum_n \langle\Psi(0)|\,e^{-i\hat{H}t/\hbar}\,|E_n\rangle\,\langle E_n|\Psi(0)\rangle$$

$$= \sum_n e^{-iE_nt/\hbar}\langle\Psi(0)|E_n\rangle\langle E_n|\Psi(0)\rangle$$

$$= \sum_n e^{-iE_nt/\hbar}\big|\langle E_n|\Psi(0)\rangle\big|^2$$

- This may be written as

$$A(t) = \int \mathrm{d}E\,\eta(E)\,e^{-iEt/\hbar}$$

$$\eta(E) \xrightarrow{\text{discrete}} \sum_n \big|\langle E_n|\Psi(0)\rangle\big|^2\,\delta(E-E_n)$$

Fourier transform

$$A(t) = \exp\left\{-\left(\tfrac{1}{2}\lambda + i\tfrac{E_0}{\hbar}\right)t\right\}$$

exponential law

$$\eta(E) = \frac{\frac{1}{2}\lambda\hbar/\pi}{(E-E_0)^2 + (\frac{1}{2}\lambda\hbar)^2}$$

Lorentzian distribution

one and only!!

QM II

Time-Dependent Phenomena

A Typo and an Integral

- In Ballentine's Eq. (12.26), inside the integral, $E_n \to E$. typo

  $A(t) = \int \mathrm{d}E\, \eta(E)\, e^{-iEt/\hbar}$ the correct(ed) formula

- He says "(12.26) could be evaluated as a contour integral…"
- Why a contour integral? Why not try *Mathematica*?
- Well, In[1]:= `Assuming[{e0 > 0, λ > 0, ħ > 0, t > 0},`

  $$\texttt{Integrate}\left[\frac{1}{\pi}\,\frac{\frac{1}{2}\lambda\hbar\, E^{-I\,e\,t/\hbar}}{(e-e0)^2+\left(\frac{1}{2}\lambda\hbar\right)^2},\ \{e, -\texttt{Infinity}, \texttt{Infinity}\}\right]\Big]$$

  Out[1]= $e^{-\frac{1}{2}t\left(\lambda+\frac{2\,i\,e0}{\hbar}\right)}$

  $$A(t) = \exp\left\{-\left(\tfrac{1}{2}\lambda + i\tfrac{E_0}{\hbar}\right)t\right\}$$

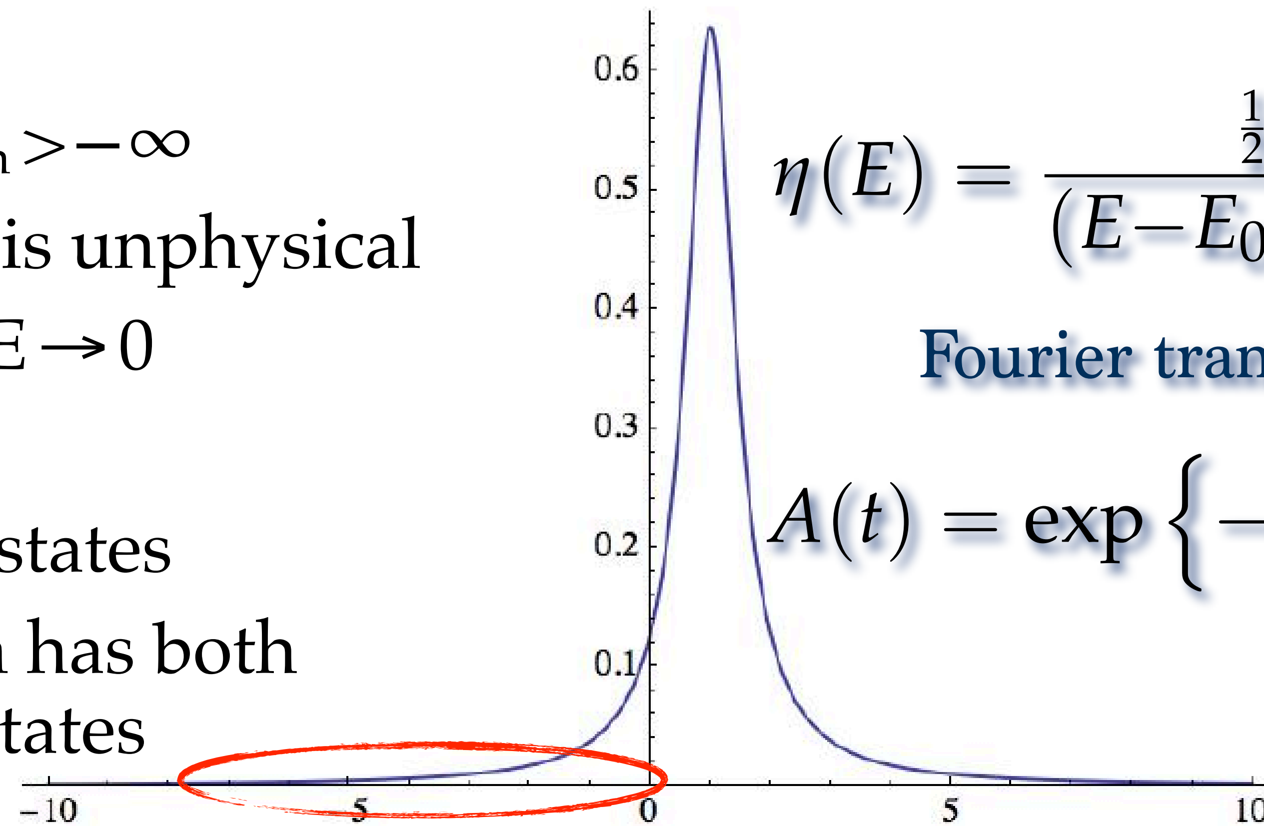
- This is indeed the standard result [Ballentine, (12.29)]
  - and provides for the exponential decay result: $P(t) = |A(t)|^2 = e^{-\lambda t}$

QMII

Time-Dependent Phenomena

A Typo and an Integral

- For the record, notice that the E-integration extends $(-\infty, +\infty)$
- The Lorentz distribution does extend over all of that region
  - However…
  - There must exist $E_{\min} > -\infty$
  - The $E < E_{\min}$ portion is unphysical
  - Typically, for $r \to \infty$, $E \to 0$
  - So $E_{\min} \leq 0$
  - If $E_{\min} = 0$, no bound states
  - If $E_{\min} < 0$, the system has both bound & scattering states



$$\eta(E) = \frac{\frac{1}{2}\lambda\hbar/\pi}{(E-E_0)^2 + (\frac{1}{2}\lambda\hbar)^2}$$

Fourier transform of

$$A(t) = \exp\left\{-\left(\tfrac{1}{2}\lambda + i\tfrac{E_0}{\hbar}\right)t\right\}$$

- The E-distribution (Lorentz or other) therefore <u>*should*</u> be cut-off below some $E_{\min}$.
  - This will result in a non-exponential decay law…
  - …but this also invites other modifications to the Lorentz distribution

Time-Dependent Phenomena

A Typo and an Integral

- Cutting the Lorentz distribution off below $E = 0$ produces

In[7]:= FullSimplify[Assuming[{e0 > 0, λ > 0, ℏ > 0, t > 0},

$$\text{Integrate}\left[\frac{1}{\pi}\,\frac{\frac{1}{2}\lambda\hbar\,\text{E}^{-\text{I}\,e\,t/\hbar}}{(e-e0)^2+\left(\frac{1}{2}\lambda\hbar\right)^2},\ \{e,\ 0,\ \text{Infinity}\}\right]]]$$

Out[7]=

$$\frac{1}{4\pi}e^{-\frac{1}{2}t\left(\lambda+\frac{2\,i\,e0}{\hbar}\right)}\left(\pi-2\,i\,\text{CosIntegral}\left[t\left(\frac{i\lambda}{2}-\frac{e0}{\hbar}\right)\right]-e^{t\lambda}\left(\pi-2\,i\,\text{CosIntegral}\left[t\left(-\frac{i\lambda}{2}-\frac{e0}{\hbar}\right)\right]+2\,\text{SinIntegral}\left[t\left(\frac{i\lambda}{2}+\frac{e0}{\hbar}\right)\right]\right)-2\,\text{SinIntegral}\left[\frac{i\,t\,\lambda}{2}-\frac{e0\,t}{\hbar}\right]\right)$$

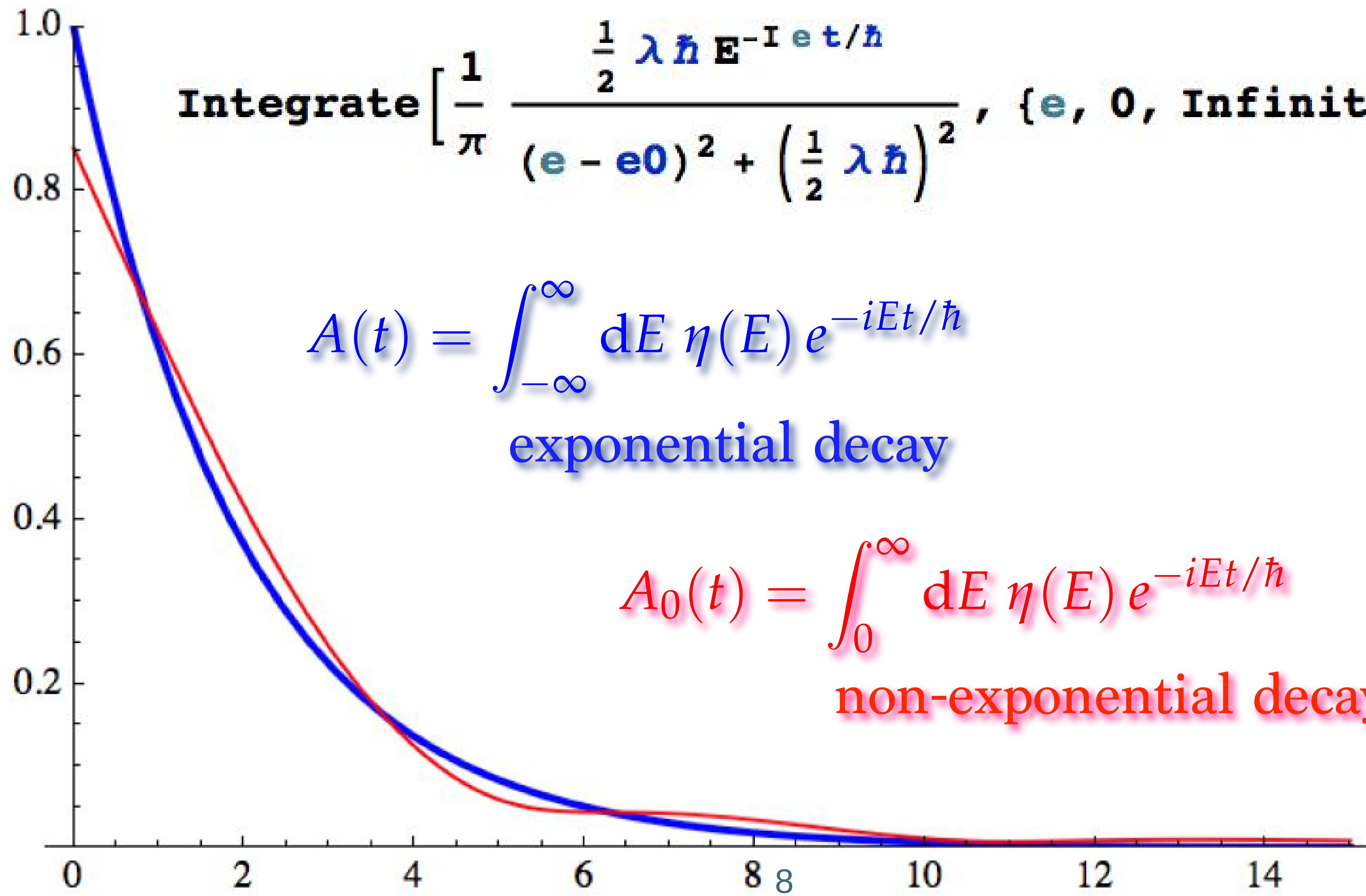
$$\text{Ci}(z) := -\int_z^\infty \frac{dt}{t}\cos(t) \qquad \text{Si}(z) := -\int_z^\infty \frac{dt}{t}\sin(t)$$

Time-Dependent Phenomena

A Typo and an Integral

- Cutting the Lorentz distribution off below $E = 0$ produces

```
In[7]:= FullSimplify[Assuming[{e0 > 0, λ > 0, ℏ > 0, t > 0},
    Integrate[1/π ((1/2) λ ℏ E^(-I e t/ℏ))/((e - e0)^2 + ((1/2) λ ℏ)^2), {e, 0, Infinity}]]]
```



$$A(t) = \int_{-\infty}^{\infty} \mathrm{d}E\, \eta(E)\, e^{-iEt/\hbar}$$

exponential decay

$$A_0(t) = \int_{0}^{\infty} \mathrm{d}E\, \eta(E)\, e^{-iEt/\hbar}$$

non-exponential decay

Time-Dependent Phenomena

A Typo and an Integral

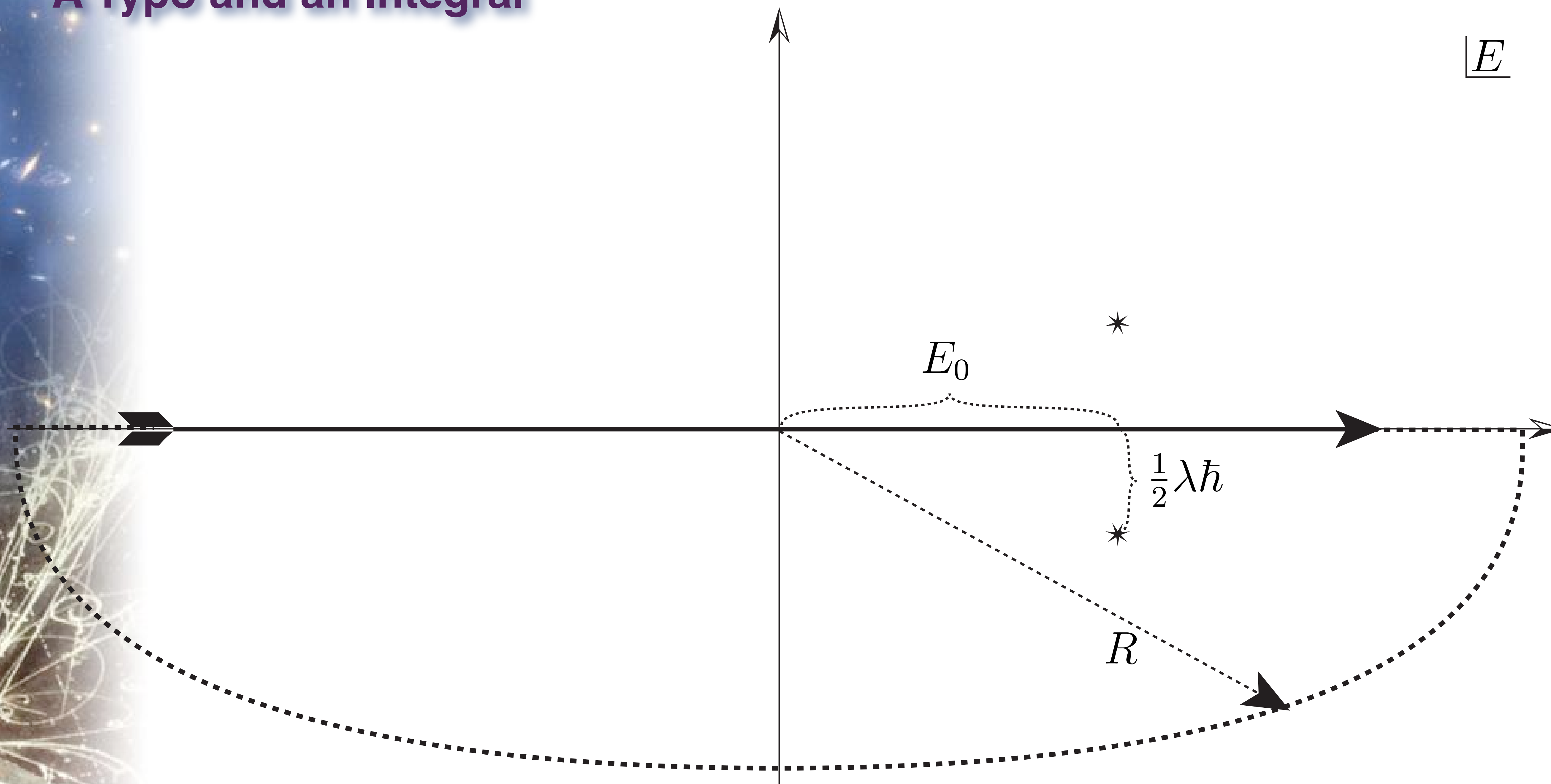
- However, do the (Math Methods *standard*) contour integral.
- We want: $$A(t) = \int_{-\infty}^{\infty} \mathrm{d}E\, \frac{(\frac{1}{2}\lambda\hbar)e^{-iEt/\hbar}}{(E-E_0)^2 + (\frac{1}{2}\lambda\hbar)^2}$$
- In the complex E-plane, this goes along the real axis.
- Where are the poles? $E = E_0 \pm \frac{i}{2}\lambda\hbar.$
- Not on the path of integration. No need for "principal parts."
- Which way can we close the contour?
  - Upper half-plane (with $\mathscr{Im}(E) > 0$), or
  - Lower half-plane (with $\mathscr{Im}(E) < 0$)?

$$e^{-i[\Re e(E)+i\,\Im m(E)]\,t/\hbar} = e^{-i\,\Re e(E)\,t/\hbar}e^{+\,\Im m(E)\,t/\hbar}$$

- This would diverge for $\mathscr{Im}(E) \to +\infty$.
- So, close the contour in the lower half-plane.

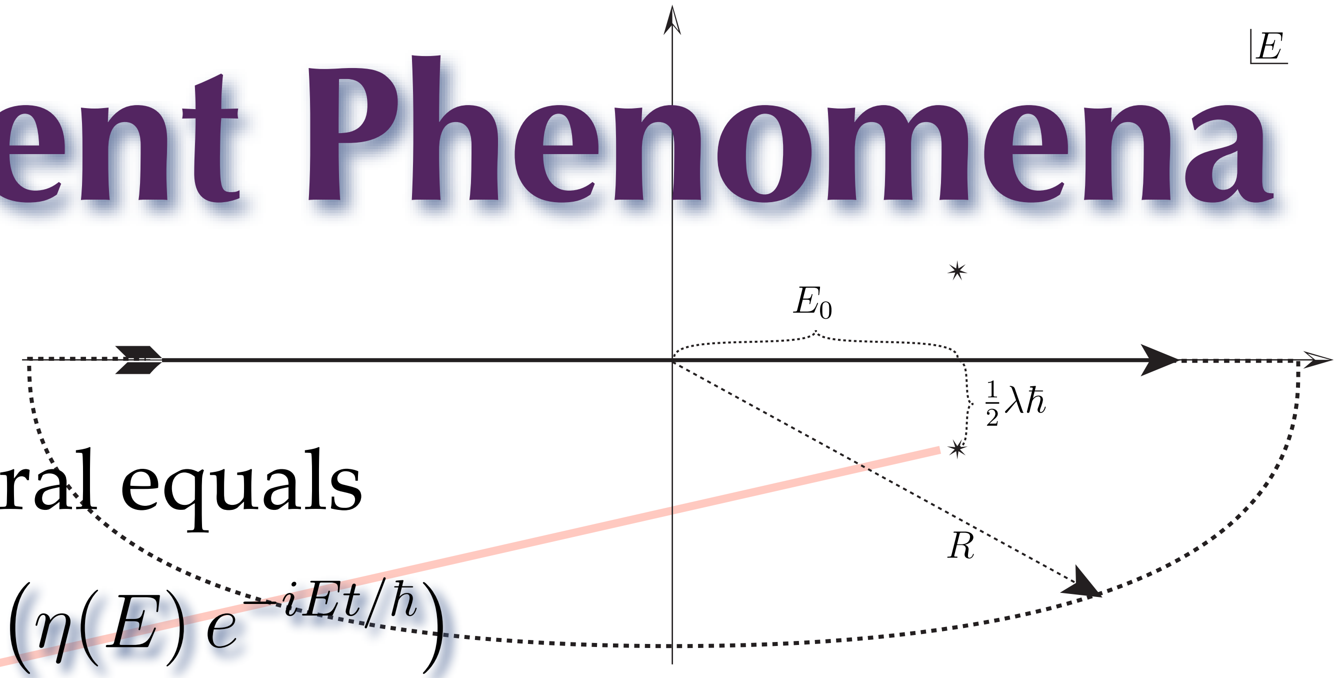
Time-Dependent Phenomena

A Typo and an Integral



Time-Dependent Phenomena

A Typo and an Integral



- So, the closed-contour integral equals

$$A(t) \;+\; I_C \;=\; -\,2\pi i \;\operatorname*{Res}_{E=E_-}\;\left(\eta(E)\,e^{-iEt/\hbar}\right)$$

- The arc-integral, I_C, is

$$I_C = \lim_{R\to\infty}\int_0^{-\pi} \mathrm{d}\theta\; \frac{e^{-iRe^{i\theta}t/\hbar}}{(Re^{i\theta}-E_0)^2+(\frac{1}{2}\lambda\hbar)^2}$$

$$= \lim_{R\to\infty}\int_0^{-\pi} \mathrm{d}\theta\; \frac{e^{-iR\cos(\theta)t/\hbar}\,e^{+R\sin(\theta)t/\hbar}}{(Re^{i\theta}-E_0)^2+(\frac{1}{2}\lambda\hbar)^2}.$$

- For $\theta \in (0, -\pi)$, $\sin(\theta) < 0$, so

$$\lim_{R\to\infty} e^{+R\sin(\theta)t/\hbar} \to 0, \qquad -\,1 < \lim_{R\to\infty} e^{-iR\cos(\theta)t/\hbar} < +1.$$

- and

$$A(t) = -2\pi i \operatorname*{Res}_{E=E_-}\left(\eta(E)\,e^{-iEt/\hbar}\right), \qquad E_- := E_0 - \tfrac{i}{2}\lambda\hbar,$$

Time-Dependent Phenomena

A Typo and an Integral

- The un-decaying amplitude for the Lorentzian energy distribution thus equals

$$
\begin{aligned}
A(t) &= -2\pi i \operatorname*{Res}_{E=E_-} \big(\eta(E)\, e^{-iEt/\hbar}\big), \qquad E_- := E_0 - \tfrac{i}{2}\lambda\hbar, \\
&= -2\pi i \lim_{E\to E_-} \Big((E-E_-)\, \big(\frac{1}{\pi}\frac{\frac{1}{2}\lambda\hbar\, e^{-iEt/\hbar}}{(E-E_0)^2+(\frac{1}{2}\lambda\hbar)^2}\big)\Big) \\
&= -i\lambda\hbar \lim_{E\to E_-} \Big(\cancel{(E-E_0+\tfrac{i}{2}\lambda\hbar)}\, \frac{e^{-iEt/\hbar}}{\cancel{(E-E_0+\tfrac{i}{2}\lambda\hbar)}(E-E_0-\tfrac{i}{2}\lambda\hbar)}\Big), \\
&= -i\lambda\hbar \lim_{E\to E_-} \Big(\frac{e^{-iEt/\hbar}}{(E-E_0-\tfrac{i}{2}\lambda\hbar)}\Big) = -i\lambda\hbar \frac{e^{-i(E_0-\frac{i}{2}\lambda\hbar)t/\hbar}}{(-i\lambda\hbar)}, \\
&= \exp\big(-\tfrac{1}{2}\lambda t - i\tfrac{E_0}{\hbar}t\big).
\end{aligned}
$$

- This is exactly the well-known result that Ballentine cites
- …and which leads to the exponential decay law, $P(t) = e^{-\lambda t}$.

Time-Dependent Phenomena

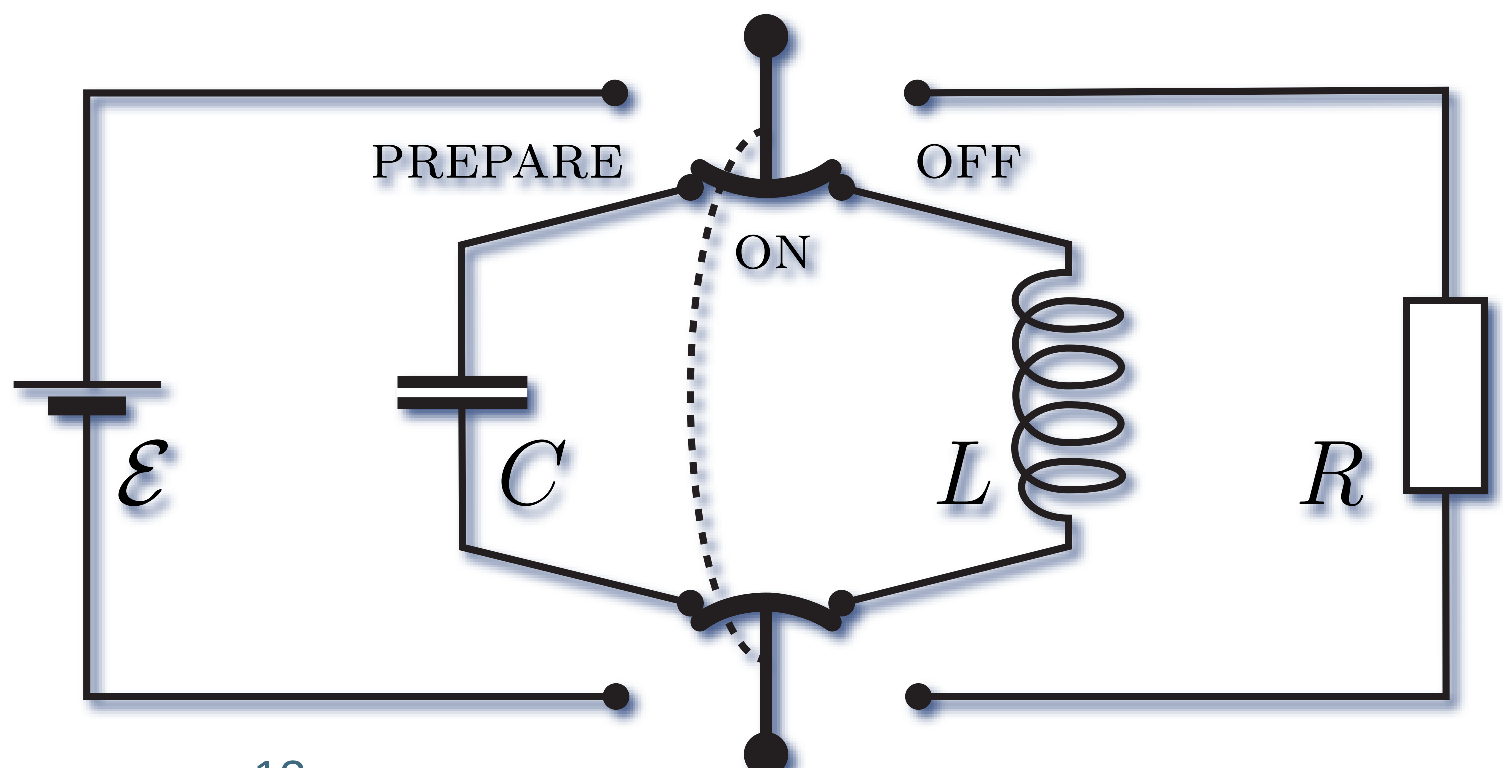
Decay Paradoxes: the Quantum Zeno Effect

- Remember, $P_u(t) \approx 1 - (t/\tau)^2$
  - …where $\tau = \hbar/\sigma$, and $\sigma^2 = \langle(\hat{H} - \langle\hat{H}\rangle)^2\rangle$
- Reasoning:
  - subdivide $[0, t] \to [0, (t/n), (2t/n), \ldots, t]$
  - $P_u(t) = [P_u(t/n)]^n = [1 - (t/n\tau)^2]^n$ and take the $n \to \infty$ limit
  - But then, $[1 - (t/n\tau)^2]^n \to 1$, whereas $[1 - (t/n\tau)]^n \to e^{-t/\tau}$.
  - So, short-time parabolic decay implies no decay at all for finite time?!?
- The (sleight of hand) limit $P_u(t) = [P_u(t/n)]^n$
  - …is really the product $P_u(t, (n-1)t/n) \cdot P_u((n-1)t/n, (n-2)t/n) \cdot \ldots \cdot P_u(2t/n, t/n) \cdot P_u(t/n, 0)$
  - …where the system is therefore *assumed* not to have decayed in all the intermediate points, $(t/n), (2t/n), \ldots, (n-1)t/n$.
  - When $n \to \infty$, "*assumed* not to have decayed infinitesimally before t" in which case it is no surprise it hasn't decayed infinitesimally later.

Time-Dependent Phenomena

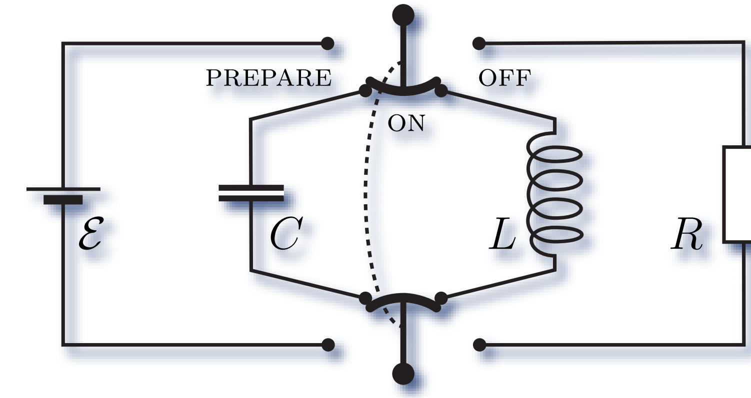
Decay Paradoxes: the Quantum Zeno Effect

- Nevertheless, the *meme* that "observation prevents a quantum system from evolving with $e^{-i\hat{H}t/\hbar}$" persists.
  - In science fiction ("weeping angels" in "Doctor Who" & many others)
  - In science: see references in http://arXiv.org/abs/0907.4361
  - Besides the quantum Zeno effect, there is also the anti-Zeno effect
  - Is this a bonanza of weirdness in quantum physics??
- No. The basic mechanism is trivial: it's all about meddling.
- It is possible to time the switching so that the discharge is caught
  - in an "early" phase
  - or in a "late" phase
- …and then iterated (while meddling)

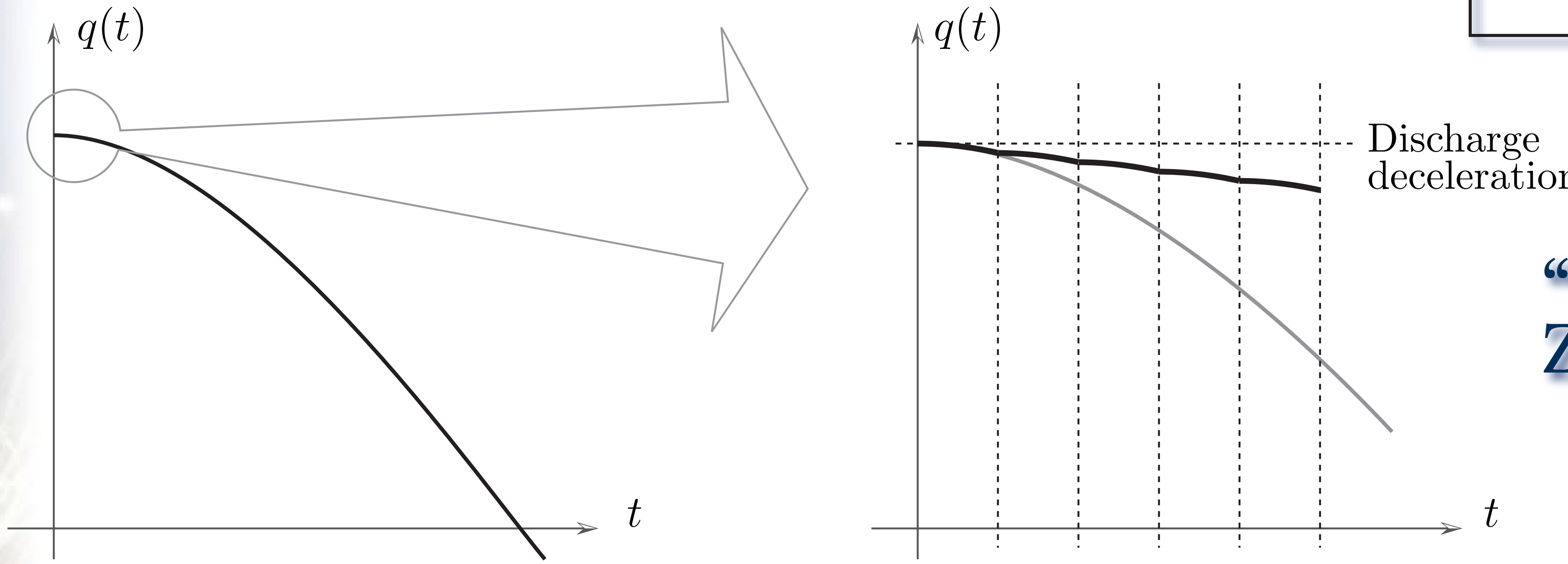


Time-Dependent Phenomena

Decay Paradoxes: the Quantum Zeno Effect

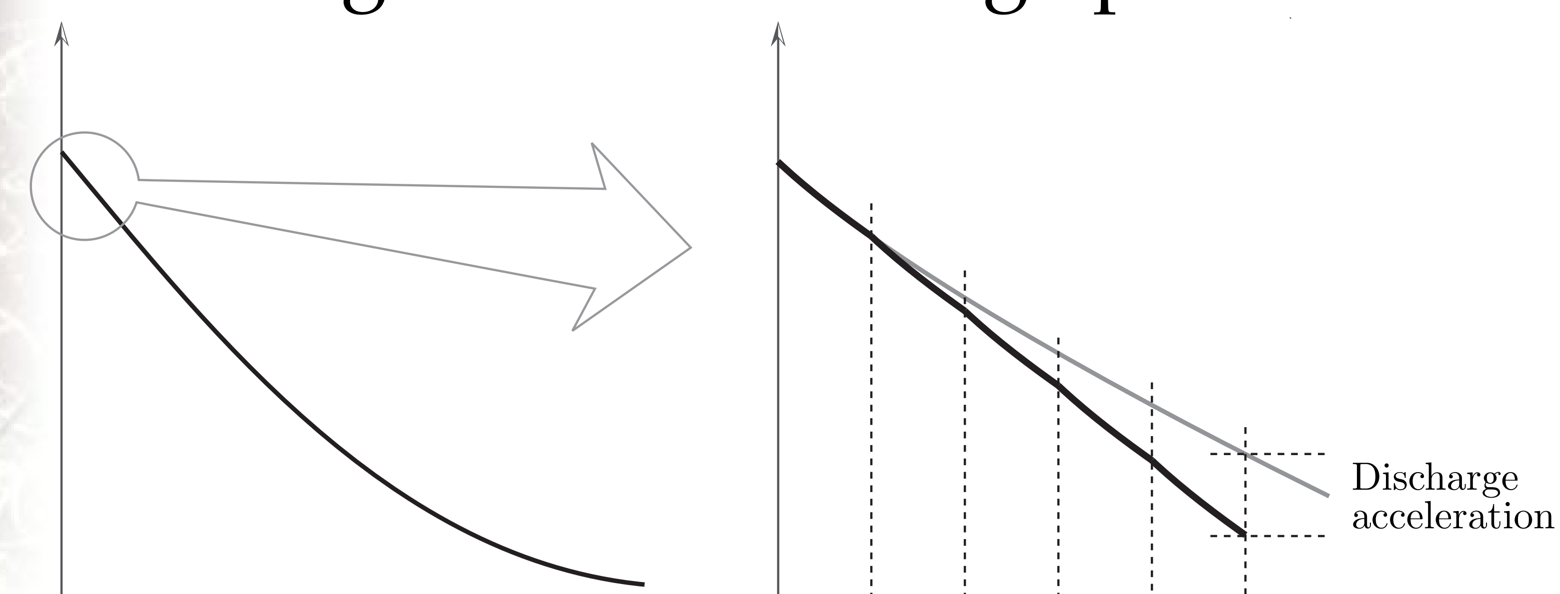


- If iterating the early discharge phase:



"quantum" Zeno effect

- If iterating the late discharge phase:



"quantum" anti-Zeno effect

Quantum Mechanics II

Now, go forth and calculate!!

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