Quantum Mechanics II

Time-Dependent Physics

Spin dynamics:
Interaction picture (a reminder)
Spin precession
Spin resonance

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QM II

Spin Dynamics

Interaction picture (a reminder)

- Separate: $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$ in $\quad i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\,\Psi, \quad \Psi \equiv \Psi(\vec{r},t)$
- Define: $\widehat{U}_0 := e^{-\frac{i(t-t_0)}{\hbar}\widehat{H}_0}$ and so $\widehat{U}_0^{-1} := e^{\frac{i(t-t_0)}{\hbar}\widehat{H}_0}$
- Then,

$$\frac{\partial}{\partial x}e^{F(x)} \equiv e^{F(x)}\,\frac{\partial F(x)}{\partial x} \qquad \frac{\partial}{\partial t}\,e^{\frac{i(t-t_0)}{\hbar}\widehat{H}_0} = e^{\frac{i(t-t_0)}{\hbar}\widehat{H}_0}\left(\tfrac{i}{\hbar}\widehat{H}_0\right)$$

Spin Dynamics

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$$\frac{\partial}{\partial t} e^{\frac{i(t-t_0)}{\hbar}\widehat{H}_0} = e^{\frac{i(t-t_0)}{\hbar}\widehat{H}_0}\left(\frac{i}{\hbar}\widehat{H}_0\right)$$

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- Then,

$$i\hbar\frac{\partial}{\partial t}\left(\widehat{U}_0^{-1}\Psi\right) = \left(i\hbar\frac{\partial \widehat{U}_0^{-1}}{\partial t}\right)\Psi + \widehat{U}_0^{-1}\left(i\hbar\frac{\partial \Psi}{\partial t}\right)$$

$$i\hbar\frac{\partial}{\partial t}\left(\widehat{U}_0^{-1}\Psi\right) = \left(\widehat{U}_0^{-1}(-\widehat{H}_0)\right)\Psi + \widehat{U}_0^{-1}\left([\widehat{H}_0 + \widehat{H}_1(t)]\Psi\right)$$

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Spin Dynamics

$$\frac{\partial}{\partial t} e^{\frac{i(t-t_0)}{\hbar}\widehat{H}_0} = e^{\frac{i(t-t_0)}{\hbar}\widehat{H}_0}\left(\tfrac{i}{\hbar}\widehat{H}_0\right)$$

Interaction picture (a reminder)

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$$\| \qquad\qquad \| \qquad\qquad \|$$

$$i\hbar\frac{\partial}{\partial t}\left(\widehat{U}_0^{-1}\Psi\right) = \left(\widehat{U}_0^{-1}(-\cancel{\widehat{H}_0})\right)\Psi + \widehat{U}_0^{-1}\left([\cancel{\widehat{H}_0} + \widehat{H}_1(t)]\Psi\right)$$

$$\|$$

$$i\hbar\frac{\partial}{\partial t}\left(\widehat{U}_0^{-1}\Psi\right) = \widehat{U}_0^{-1}\,\widehat{H}_1(t)\,\Psi = \left(\widehat{U}_0^{-1}\,\widehat{H}_1(t)\,\widehat{U}_0\right)\left(\widehat{U}_0^{-1}\Psi\right)$$

$$i\hbar\frac{\partial}{\partial t}\Psi_I(\vec{r},t) = \widehat{H}_I(t)\,\Psi_I(\vec{r},t) \quad \text{with} \quad \Psi_I(\vec{r},t) := \widehat{U}_0^{-1}\Psi(\vec{r},t),$$

focus on interactions

$$\widehat{H}_I := \widehat{U}_0^{-1}\widehat{H}_1\widehat{U}_0.$$

Spin Dynamics

Interaction picture (a reminder)

- Separate: $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$ in $\quad i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\,\Psi, \quad \Psi \equiv \Psi(\vec{r},t)$

$$i\hbar\frac{\partial}{\partial t}\Psi_I(\vec{r},t) = \hat{H}_I(t)\,\Psi_I(\vec{r},t) \quad \text{with} \quad \Psi_I(\vec{r},t) := \hat{U}_0^{-1}\Psi(\vec{r},t),$$

focus on interactions $\qquad \hat{H}_I := \hat{U}_0^{-1}\hat{H}_1\hat{U}_0.$

- Note: if $[\,\hat{H}_0\,, \hat{H}_1(t)\,] = 0$, then $[\,\hat{U}_0\,, \hat{H}_1(t)\,] = 0$, and $\hat{H}_I(t) = \hat{H}_1(t)$.
- Often, $\hat{H}_1$(t) is expressible as a nontrivial matrix w.r.t. some basis,
- …while $\hat{H}_0$ acts identically on all elements of that basis
  and is therefore proportional to the identity matrix.
- Then, $[\,\hat{H}_0\,, \hat{H}_1(t)\,] = 0$ and $\hat{H}_I(t) = \hat{H}_1(t)$, as noted above.
- (The diagonalizing transformation of $\hat{H}_1(t)$ cannot change $\hat{H}_0$.)

Spin Dynamics

Interaction picture (a reminder)

- So, splitting: $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$, we define $\hat{U}_0 := e^{-\frac{i(t-t_0)}{\hbar}\hat{H}_0}$
- and $\Psi_I(\vec{r},t) := \hat{U}_0^{-1}\Psi(\vec{r},t)$ $\quad \hat{H}_I := \hat{U}_0^{-1}\hat{H}_1\hat{U}_0 \quad \hat{R}_I := \hat{U}_0^{-1}\hat{R}\hat{U}_0$

$$i\hbar\frac{\partial \hat{R}_I}{\partial t} = \left(i\hbar\frac{\partial \hat{U}_0^{-1}}{\partial t}\right)\hat{R}\,\hat{U}_0 + i\hbar\,\hat{U}_0^{-1}\left(\frac{\partial \hat{R}}{\partial t}\right)\hat{U}_0 + \hat{U}_0^{-1}\,\hat{R}\left(i\hbar\frac{\partial \hat{U}_0}{\partial t}\right)$$

$$= \left(-\hat{H}_0\,\hat{U}_0^{-1}\right)\hat{R}\,\hat{U}_0 + i\hbar\,\hat{U}_0^{-1}\left(\frac{\partial \hat{R}}{\partial t}\right)\hat{U}_0 + \hat{U}_0^{-1}\,\hat{R}\left(\hat{H}_0\,\hat{U}_0\right)$$

$$= -\hat{H}_0\left(\hat{U}_0^{-1}\hat{R}\hat{U}_0\right) + \left(\hat{U}_0^{-1}\hat{R}\hat{U}_0\right)\hat{H}_0 + i\hbar\left(\hat{U}_0^{-1}\left(\frac{\partial \hat{R}}{\partial t}\right)\hat{U}_0\right)$$

$$= -\hat{H}_0\,\hat{R}_I + \hat{R}_I\hat{H}_0 + i\hbar\left(\frac{\partial \hat{R}}{\partial t}\right)_I$$

Complementary!

$$\frac{\partial \hat{R}_I}{\partial t} = \tfrac{i}{\hbar}\left[\hat{H}_0, \hat{R}_I\right] + \left(\frac{\partial \hat{R}}{\partial t}\right)_I \qquad\qquad i\hbar\frac{\partial}{\partial t}\Psi_I = \hat{H}_I(t)\,\Psi_I$$

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Spin Dynamics

Interaction picture (a reminder)

$$\frac{\partial \hat{R}_I}{\partial t} = \frac{i}{\hbar}\left[\hat{H}_0, \hat{R}_I\right] + \left(\frac{\partial \hat{R}}{\partial t}\right)_I \qquad i\hbar\frac{\partial}{\partial t}\Psi_I = \hat{H}_I(t)\,\Psi_I$$

- Special case: $\hat{H} = \hat{H}_1(t)$ with $\hat{H}_0 = 0$, so $\hat{U}_0 = \mathbb{1}$.
- Then

$$\hat{R}_S := \mathbb{1}\cdot\hat{R}\cdot\mathbb{1} \qquad \frac{\partial \hat{R}_S}{\partial t} = \mathbb{1}\cdot\frac{\partial \hat{R}}{\partial t}\cdot\mathbb{1} \qquad i\hbar\frac{\partial}{\partial t}\Psi_S = \hat{H}(t)\,\Psi_S$$

- …is the Schrödinger picture. All dynamics is in the wave-function.
- In turn, setting: $\hat{H} = \hat{H}_0(t)$ with $\hat{H}_1 = 0$, so $\hat{U}_0 = \hat{U}$.
- Then

$$\hat{R}_H := \hat{U}^{-1}\,\hat{R}\,\hat{U} \qquad \frac{\partial \hat{R}_H}{\partial t} = \frac{i}{\hbar}\left[\hat{H}, \hat{R}_H\right] + \hat{U}^{-1}\frac{\partial \hat{R}}{\partial t}\,\hat{U} \qquad i\hbar\frac{\partial}{\partial t}\Psi_H = 0\cdot\Psi_H$$

- …is the Heisenberg picture. All dynamics is in the observables.
- The interaction picture is clearly intermediate,
  - ignoring any dynamics generated by $\hat{H}_0$,
  - focusing on the dynamics generated by $\hat{H}_1(t)$.

Spin Dynamics

Spin precession

- Interaction picture for $\hat{H} = \hat{H}_0 + \hat{H}_1$, with $\hat{H}_1 = -\boldsymbol{\mu}\cdot\boldsymbol{B}$
- Reverse-engineered: $\boldsymbol{\mu} = \gamma\,\boldsymbol{S}$, where $\boldsymbol{S}$ is "spin"…
  - E.g.: $\gamma_e \approx -e/M_e c$, $\gamma_p = +2.79e/M_p c$, but $\gamma_n = -1.91e/M_p c$.
- For the spin-½ electron, $\boldsymbol{S} = \tfrac{1}{2}\hbar\,\boldsymbol{\sigma}$ and $\hat{H}_I = \hat{H}_1 = -\tfrac{1}{2}\gamma_e \hbar B_0\, \boldsymbol{\sigma}_z$.
  - Eigenvalues of $\boldsymbol{\sigma}_z$ are ±1, eigenvectors $|+\rangle$ and $|-\rangle$.
  - Use matrix representation ($\omega_0 = \gamma_e B_0$, set $t_0 \to 0$):

$$|+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |-\rangle = \begin{bmatrix}0\\1\end{bmatrix}, \qquad H_I = -\tfrac{1}{2}\hbar\omega_0 \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$|\Psi_I(t)\rangle = a_+(t)\,|+\rangle + a_-(t)\,|-\rangle = \begin{bmatrix}a_+(t)\\ a_-(t)\end{bmatrix}$$

$$U_I\,|\Psi_I(t)\rangle = e^{-\frac{it}{\hbar}\cdot[-\frac{1}{2}\hbar\omega_0\,\sigma_z]}\,|\Psi_I(t)\rangle = e^{\frac{i}{2}\omega_0 t\,\sigma_z}\,|\Psi_I(t)\rangle$$

$$= a_+(0)e^{\frac{i}{2}\omega_0 t}\,|+\rangle + a_-(0)e^{-\frac{i}{2}\omega_0 t}\,|-\rangle \;=\; \begin{bmatrix}a_+(0)\,e^{\frac{i}{2}\omega_0 t}\\ a_-(0)\,e^{-\frac{i}{2}\omega_0 t}\end{bmatrix}$$

initial conditions

Spin Dynamics

Spin precession

- Compute the expectation value of $\boldsymbol{\mu} = \gamma \boldsymbol{S}$:
- …component by component

$$\langle \boldsymbol{\mu} \rangle = \tfrac{1}{2}\gamma\hbar\left(\hat{\mathrm{e}}_x\langle \boldsymbol{\sigma}_x\rangle + \hat{\mathrm{e}}_y\langle \boldsymbol{\sigma}_y\rangle + \hat{\mathrm{e}}_z\langle \boldsymbol{\sigma}_z\rangle\right)$$

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Spin Dynamics

Spin precession

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Compute the expectation value of $\boldsymbol{\mu} = \gamma \boldsymbol{S}$:
- …component by component

$$\langle \boldsymbol{\mu} \rangle = \tfrac{1}{2}\gamma\hbar\big(\hat{\mathrm{e}}_x\langle\boldsymbol{\sigma}_x\rangle + \hat{\mathrm{e}}_y\langle\boldsymbol{\sigma}_y\rangle + \hat{\mathrm{e}}_z\langle\boldsymbol{\sigma}_z\rangle\big)$$

$$\langle\boldsymbol{\sigma}_x\rangle = \langle\Psi_I(t)|\,\boldsymbol{\sigma}_x\,|\Psi_I(t)\rangle$$

$$= \begin{bmatrix} a_+^*(0)\,e^{-i\omega_0 t/2} & a_-^*(0)\,e^{i\omega_0 t/2} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_+(0)\,e^{i\omega_0 t/2} \\ a_-(0)\,e^{-i\omega_0 t/2} \end{bmatrix}$$

$$= \begin{bmatrix} a_+^*(0)\,e^{-i\omega_0 t/2} & a_-^*(0)\,e^{i\omega_0 t/2} \end{bmatrix} \cdot \begin{bmatrix} a_-(0)\,e^{-i\omega_0 t/2} \\ a_+(0)\,e^{i\omega_0 t/2} \end{bmatrix}$$

$$= 2\,\Re e\,\big(a_+^*(0)a_-(0)\,e^{-i\omega_0 t}\big) \quad \to \cos(\omega_0 t)$$

$$a_\pm(0) \to \tfrac{1}{\sqrt{2}}$$

Valid also when $a_+(0)=0$ or $a_-(0)=0$

- Together:

$$\langle\boldsymbol{\mu}_x\rangle = \tfrac{1}{2}\gamma\hbar\,2\,\Re e\big(a_+^*(0)a_-(0)\,e^{-i\omega_0 t}\big) \quad \to \tfrac{1}{2}\gamma\hbar\cos(\omega_0 t),$$

$$\langle\boldsymbol{\mu}_y\rangle = \tfrac{1}{2}\gamma\hbar\,2\,\Im m\big(a_+^*(0)a_-(0)\,e^{-i\omega_0 t}\big) \quad \to -\tfrac{1}{2}\gamma\hbar\sin(\omega_0 t),$$

$$\langle\boldsymbol{\mu}_z\rangle = \tfrac{1}{2}\gamma\hbar\big(|a_+(0)|^2 - |a_-(0)|^2\big) \quad \to 0.$$

misalignment ⇒ precession

Spin Dynamics

Spin precession

$$[\widehat{H}_0, (\tfrac{1}{2}\gamma\hbar\sigma_\alpha \hat{e}^\alpha)] = 0$$

$$[(\tfrac{1}{2}\sigma_j), (\tfrac{1}{2}\sigma_k)] = i\,\varepsilon_{jk}{}^{\ell}(\tfrac{1}{2}\sigma_\ell)$$

- Consider the same, in the Heisenberg picture (the interaction picture with $\hat{H}_0 \to \hat{H}$, $\hat{H}_1 \to 0$)
- Then,

$$\frac{\partial \boldsymbol{\mu}}{\partial t} = \tfrac{i}{\hbar}[\widehat{H}_0 - \boldsymbol{\mu}\cdot\boldsymbol{B}, \boldsymbol{\mu}] = \tfrac{i}{\hbar}[\widehat{H}_0 - \mu_\beta B^\beta, \mu_\gamma \hat{e}^\gamma]$$

$$= -\tfrac{i}{\hbar}B^\beta[(\tfrac{1}{2}\gamma\hbar\sigma_\beta), (\tfrac{1}{2}\gamma\hbar\sigma_\gamma)]\hat{e}^\gamma = -i\gamma^2\hbar B^\beta[(\tfrac{1}{2}\sigma_\beta), (\tfrac{1}{2}\sigma_\gamma)]\hat{e}^\gamma$$

$$= -i\gamma^2\hbar B^\beta(i\varepsilon_{\beta\gamma}{}^{\alpha}(\tfrac{1}{2}\sigma_\alpha))\hat{e}^\gamma = \varepsilon^{\alpha\beta\gamma}(\tfrac{1}{2}\gamma\hbar\sigma_\alpha)(\gamma B_\beta)\,\hat{e}_\gamma$$

$$= \boldsymbol{\mu} \times (\gamma\boldsymbol{B})$$

- The precession frequency, $\omega_0 = \gamma|\boldsymbol{B}|$, is independent of $\boldsymbol{\mu}$,
- …and independent of the initial state (except when $\boldsymbol{\mu} \parallel \boldsymbol{B}$).
- The precession of $\boldsymbol{\mu}$ is detectable, but hard.

Spin Dynamics

Spin resonance

- Varying the ***B***-field enables energy transfer & state transition
- Use $\boldsymbol{B}(t) = B_1\big(\hat{e}_x\cos(\omega t)+\hat{e}_y\sin(\omega t)\big)+B_0\,\hat{e}_z$
- Move to a co-rotating frame, Rot(ωt): $\hat{U}_{\omega t} = \exp\{-i\omega t\,\hat{S}_z/\hbar\}$,

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Big(\hat{U}_{\omega t}^{-1}\,\Psi\Big) &= i\hbar\Big(\frac{\partial\hat{U}_{\omega t}^{-1}}{\partial t}\Big)\Psi + \hat{U}_{\omega t}^{-1}\Big(i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\,\Psi\Big)\\
&= i\hbar\hat{U}_{\omega t}^{-1}\Big(i\tfrac{\omega}{\hbar}\hat{S}_z\Big)\Psi + \hat{U}_{\omega t}^{-1}\,\hat{H}\,\Psi\\
&= \Big(\hat{U}_{\omega t}^{-1}\big[-\omega\hat{S}_z+\hat{H}\big]\hat{U}_{\omega t}\Big)\Big(\hat{U}_{\omega t}^{-1}\,\Psi\Big)
\end{aligned}
$$

$$
\begin{aligned}
\hat{U}_{\omega t}^{-1}\big[-\omega\hat{S}_z+\hat{H}\big]\hat{U}_{\omega t} &= \hat{U}_{\omega t}^{-1}\big[-\omega\hat{S}_z-\omega_0\hat{S}_z-\gamma B_1\hat{S}_{x'}\big]\hat{U}_{\omega t}\\
&= -\big[(\omega_0+\omega)\hat{U}_{\omega t}^{-1}\,\hat{S}_z\,\hat{U}_{\omega t}+\gamma B_1(\hat{U}_{\omega t}^{-1}\hat{S}_{x'}\hat{U}_{\omega t})\big]\\
&= (\omega_0+\omega)\hat{S}_z+\gamma B_1\hat{S}_x
\end{aligned}
$$

co-rotating Hamiltonian

$\hat{U}_{\omega t}$ implements a rotation about the z-axis, co-rotating with the ***B***-field.

Spin Dynamics

Spin resonance

- Co-rotating with $\boldsymbol{B}(t) = B_1\big(\hat{\mathrm{e}}_x \cos(\omega t) + \hat{\mathrm{e}}_y \sin(\omega t)\big) + B_0\,\hat{\mathrm{e}}_z$
- we have:

$$i\hbar\frac{\partial\Phi}{\partial t} = \underbrace{\big[(\omega_0+\omega)\widehat{S}_z + \omega_1\widehat{S}_x\big]}_{\text{constant}}\Phi, \qquad \Phi := \widehat{U}_{\omega t}^{-1}\Psi$$

- where $\omega_1 = \gamma\, B_1$.
- For a spin-½ system, this co-rotating, constant Schrödinger equation is

$$i\hbar\begin{bmatrix}\dot{\Phi}_+\\ \dot{\Phi}_-\end{bmatrix} = \begin{bmatrix}(\omega_0+\omega) & \omega_1\\ \omega_1 & (\omega_0+\omega)\end{bmatrix}\begin{bmatrix}\Phi_+\\ \Phi_-\end{bmatrix}$$

Not hard. Really…

- But, there is an easier, way—owing to Cayley-Hamilton's theorem.
- "Every matrix satisfies its own secular equation."
  Secular equation: det[**M** – λ **1**] = 0 (expanded as a polynomial in λ).
- For spin-j systems, use $(2j+1) \times (2j+1)$ matrices,
  the secular equation is of $(2j+1)$'th order.

Consider spin-½.

Spin Dynamics

$$\boldsymbol{B}(t) = B_1\big(\hat{e}_x\cos(\omega t) + \hat{e}_y\sin(\omega t)\big) + B_0\,\hat{e}_z$$
$$i\hbar\frac{\partial\Phi}{\partial t} = \big[(\omega_0+\omega)\hat{S}_z + \omega_1\hat{S}_x\big]\Phi$$

Spin resonance

- Check: $\{\boldsymbol{\sigma}_\alpha, \boldsymbol{\sigma}_\beta\} = \delta_{\alpha\beta}\mathbf{1}$, for all Pauli's matrices.
- So, $(\boldsymbol{\sigma}_z)^2 = \mathbf{1}$, $(\boldsymbol{\sigma}_x)^2 = \mathbf{1}$ and $\boldsymbol{\sigma}_x\boldsymbol{\sigma}_z + \boldsymbol{\sigma}_z\boldsymbol{\sigma}_x = 0$.
- Also, for any matrix $\mathbf{M}^2 = \mathbf{1}$, $\exp\{i\varphi\,\mathbf{M}\} = \cos(\varphi)\mathbf{1} + i\sin(\varphi)\mathbf{M}$.
- For a co-rotating spin-½ system, we have:

$$i\hbar\frac{\partial\Phi}{\partial t} = \big[\tfrac{1}{2}\hbar\boldsymbol{\Omega}\,\sigma_\Omega\big]\Phi \qquad \Psi := \hat{U}_{\omega t}\Phi = \hat{U}_{\omega t}\hat{U}_t^{(c)}\Phi(0)$$

- with

$$\boldsymbol{\sigma}_\Omega := \frac{(\omega_0+\omega)\boldsymbol{\sigma}_z + \omega_1\boldsymbol{\sigma}_x}{\Omega}, \qquad (\boldsymbol{\sigma}_\Omega)^2 = \mathbb{1}, \quad \Omega = \sqrt{(\omega_0+\omega)^2 + \omega_1{}^2}$$

mixed terms cancel

- so

$$\hat{U}_t^{(c)} = \begin{bmatrix} \cos(\tfrac{1}{2}\Omega t) + i\frac{\omega_0+\omega}{\Omega}\sin(\tfrac{1}{2}\Omega t) & i\frac{\omega_1}{\Omega}\sin(\tfrac{1}{2}\Omega t) \\ i\frac{\omega_1}{\Omega}\sin(\tfrac{1}{2}\Omega t) & \cos(\tfrac{1}{2}\Omega t) - i\frac{\omega_0+\omega}{\Omega}\sin(\tfrac{1}{2}\Omega t) \end{bmatrix}$$

$$\Psi(t) = \hat{U}_{\omega t}\hat{U}_t^{(c)}\begin{bmatrix}1\\0\end{bmatrix} = e^{-i\omega t\boldsymbol{\sigma}_z/2}\begin{bmatrix} \cos(\tfrac{1}{2}\Omega t) + i\frac{\omega_0+\omega}{\Omega}\sin(\tfrac{1}{2}\Omega t) \\ i\frac{\omega_1}{\Omega}\sin(\tfrac{1}{2}\Omega t) \end{bmatrix}$$

initial state

phases

$e^{-i\omega t/2}$

$e^{i\omega t/2}$

Spin Dynamics

Spin resonance

$$\Psi(t) = \widehat{U}_{\omega t}\widehat{U}_t^{(c)}\begin{bmatrix}1\\0\end{bmatrix} = e^{-i\omega t\boldsymbol{\sigma}_z/2}\begin{bmatrix}\cos(\frac{1}{2}\Omega t) + i\frac{\omega_0+\omega}{\Omega}\sin(\frac{1}{2}\Omega t)\\ i\frac{\omega_1}{\Omega}\sin(\frac{1}{2}\Omega t)\end{bmatrix}$$

- The spin-flip probability is the magnitude-squared of the lower term:

$$\text{Prob}\big(\widehat{S}_z \to -\tfrac{1}{2}| + \tfrac{1}{2}\big) = \frac{\omega_1^{\,2}}{\Omega^2}\sin^2(\tfrac{1}{2}\Omega t)$$

- The peak value becomes maximal when $\omega = -\omega_0$,

$$\text{peak Prob} = \frac{\omega_1^{\,2}}{(\omega_0+\omega)^2 + \omega_1^{\,2}} \xrightarrow{\omega\to-\omega_0} 1$$

$$\text{Prob} \xrightarrow{\omega\to-\omega_0} \sin^2(\tfrac{1}{2}\omega_1 t)$$

- By switching the rotating ***B***-field, the spin-flip probability throbs at half the co-rotating frequency.
- By turning the rotating ***B***-field off after a suitable time T, the spin-flip probability can be tuned to will.

Quantum Mechanics II

Now, go forth and calculate!!

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