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Quantum Mechanics II

Quizz #1 (31rd Jan. '95), Solution

Q. By charge conservation alone, determine the (one-pion, one-nucleon) state into which $\pi^0 p^+$ could scatter. Determine the ration of the cross-sections for these processes (given the isospin wave-function factors, and everything else being the same) when $M_{3/2} \gg M_{1/2}$. Recall, $M_{3/2} \stackrel{\rm def}{=} \langle \pi, N, \frac{3}{2} \| \hat{H} \| \pi, N, \frac{3}{2} \rangle$ and $M_{1/2} \stackrel{\rm def}{=} \langle \pi, N, \frac{1}{2} \| \hat{H} \| \pi, N, \frac{1}{2} \rangle$.

A. Quite clearly, either $\pi^0 p^+ \to \pi^0 p^+$, or $\pi^0 p^+ \to \pi^+ n^0$, these two being the only one-π-one-N states with total (electric) charge +1.

Now,

$$
\begin{aligned}
\sigma_1 \propto & \left| \Big[\sqrt{\tfrac{2}{3}}\, \langle \pi, N; \tfrac{3}{2}, \tfrac{1}{2} | + \sqrt{\tfrac{1}{3}}\, \langle \pi, N; \tfrac{1}{2}, \tfrac{1}{2} | \Big] \hat{H} \Big[\sqrt{\tfrac{2}{3}}\, | \pi, N; \tfrac{3}{2}, \tfrac{1}{2} \rangle + \sqrt{\tfrac{1}{3}}\, | \pi, N; \tfrac{1}{2}, \tfrac{1}{2} \rangle \Big] \right|^2 \\
&= \left| \tfrac{2}{3} \langle \pi, N; \tfrac{3}{2}, \tfrac{1}{2} | \hat{H} | \pi, N; \tfrac{3}{2}, \tfrac{1}{2} \rangle + \tfrac{1}{3} \langle \pi, N; \tfrac{1}{2}, \tfrac{1}{2} | \hat{H} | \pi, N; \tfrac{1}{2}, \tfrac{1}{2} \rangle \right|^2 \\
&= \left| \tfrac{2}{3} \langle \tfrac{3}{2}, 0, \tfrac{1}{2}, 0 | \tfrac{3}{2}, \tfrac{1}{2} \rangle M_{3/2} + \tfrac{1}{3} \langle \tfrac{1}{2}, 0, \tfrac{1}{2}, 0 | \tfrac{1}{2}, \tfrac{1}{2} \rangle M_{1/2} \right|^2 = \tfrac{1}{9} \left| 2 M_{3/2} + M_{1/2} \right|^2 .
\end{aligned}
\tag{1}
$$

The first equality follows from the orthogonality of the (total) isospin eigenfunctions $|\pi, N, I, I_3\rangle$. The second equality follows from applying the Wigner-Eckart theorem, reducing the full matrix elements

$$
\langle \pi, N; I', I'_3 | \hat{H} | \pi, N; I, I_3 \rangle \;=\; \langle I', 0, I'_3, 0 | I, I_3 \rangle \langle \pi, N \| \hat{H} \| \pi, N \rangle \;, \tag{2}
$$

and where $\vec{I}(\hat{H}) = 0$. Finally, we used that

$$
\langle I', 0, I'_3, 0 | I, I_3 \rangle \;=\; \delta_{I', I}\, \delta_{I'_3, I_3} \;, \tag{3}
$$

in agreement with the selection rules—just as with (ordinary) angular momentum.

The same calculation for the second process produces

$$
\begin{aligned}
\sigma_2 \propto & \left| \Big[\sqrt{\tfrac{1}{3}}\, \langle \pi, N; \tfrac{3}{2}, \tfrac{1}{2} | - \sqrt{\tfrac{2}{3}}\, \langle \pi, N; \tfrac{1}{2}, \tfrac{1}{2} | \Big] \hat{H} \Big[\sqrt{\tfrac{2}{3}}\, | \pi, N; \tfrac{3}{2}, \tfrac{1}{2} \rangle + \sqrt{\tfrac{1}{3}}\, | \pi, N; \tfrac{1}{2}, \tfrac{1}{2} \rangle \Big] \right|^2 \\
&= \left| \tfrac{\sqrt{2}}{3} \langle \tfrac{3}{2}, 0, \tfrac{1}{2}, 0 | \tfrac{3}{2}, \tfrac{1}{2} \rangle M_{3/2} - \tfrac{\sqrt{2}}{3} \langle \tfrac{1}{2}, 0, \tfrac{1}{2}, 0 | \tfrac{1}{2}, \tfrac{1}{2} \rangle M_{1/2} \right|^2 = \tfrac{2}{9} \left| M_{3/2} - M_{1/2} \right|^2 .
\end{aligned}
\tag{4}
$$

Thus, in the limit when $M_{3/2} \gg M_{1/2}$, the ratio (all other factors equal) becomes

$$
\frac{\sigma_2}{\sigma_1} = \frac{2 \left| M_{3/2} - M_{1/2} \right|^2}{\left| 2 M_{3/2} + M_{1/2} \right|^2} \;\xrightarrow{\;M_{3/2} \gg M_{1/2}\;}\; \tfrac{1}{2} \;. \tag{5}
$$

That is, the elastic colision $\pi^0 p^+ \to \pi^0 p^+$ is expected 66.67%, while $\pi^0 p^+ \to \pi^+ n^0$ only 33.33% of the time if $M_{3/2} \gg M_{1/2}$. Similarly, if $M_{3/2} \ll M_{1/2}$, precisely the opposite ratio is obtained: the second process is expected twice as often as the first one.