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The Theory of Alpha Decay

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by

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ABSTRACT

One of the relatively simple physical processes that can be described successfully in the framework of introductory quantum theory is alpha decay. In this process, two protons and two neutrons out of a relatively large nucleus dissociate from the rest of the nucleus, form a separate subsystem and subsequently depart. In its full complexity, this process is not be describable, however a reasonable simplified version of it is readily tractable and will be studied in this note. Several alternative models will also be described, for the interested reader to pursue in greater detail.

1 The Physical Description of the Problem

The process of α-decay starts with a nucleus, say ${}^{234}_{92}$U (Uranium). This nucleus has 92 protons and 142 neutrons, and is obviously too complicated for a many-body type of analysis. That is, one would need to set up $3\times 234 = 702$ equations of motion for the three components of the radius vector of each of the 234 particles, and these would involve $\binom{234}{2} = 27{,}261$ 2-particle potential terms to describe the pair-wise interactions.

At some point, among these 234 particles, somehow two protons and two neutrons end up forming a little subsystem within which the pair-wise binding forces are stronger than the binding forces to any of the other 230 nucleons. Gradually[1], the little subsystem becomes an entity which may be thought of as a separate particle trapped within the confines of the 'rest' of the nucleus. Eventually, the little subsystem will tunnel outside the reach of the strong nuclear forces, at which point the repulsive Coulomb force expels it as the α-particle which is detected well outside the nucleus. The remaining 230-nucleon nucleus is identified as ${}^{230}_{90}$Th (Thorium).

The ${}^{234}_{92}$U nucleus is called the parent nucleus, ${}^{230}_{90}$Th the daughter nucleus, and the general formula for the α-decay would be

$$ {}^{A+4}_{Z+2}\mathrm{X} \;\to\; {}^{4}_{2}\mathrm{He}^{++} \;+\; {}^{A}_{Z}\mathrm{Y}^{--}, \qquad \text{or} \qquad {}^{A+4}_{Z+2}\mathrm{X} \;\xrightarrow{\;\alpha\;}\; {}^{A}_{Z}\mathrm{Y}. \tag{1} $$

In the first formula, we have used that the α-particle is in fact the nucleus of Helium atom, which would have two electrons in its stable state. The double positive charge then indicates that only the nucleus appears on the right, and so leaves the daughter atom twice negatively charged. In nuclear physics, the same symbols would be used to denote merely the nuclei (with no concern about the electrons), and so no charges are indicated in the second form of the formula, and the decay process (arrow) is labeled by 'α'.

Exceptional cases occur when the number of protons, Z, and/or the number of neutrons, $(A-Z)$, equal $2, 8, 20, 28, 50, 82$ or 126 (the so-called magic numbers). If, say $Z = 82$ (as in ${}^{208}_{82}$Pb (lead), which in fact is doubly magical, since also $(A-Z) = 126$), the nuclear shell model predicts that the 82 protons (and here also the 126 neutrons) form a strongly bound closed 'shell'. Thus, the nearby Polonium nucleus ${}^{212}_{84}$Po is rather accurately described as a system of a doubly closed shell of 82 protons and 126 neutrons, plus two extra protons and two extra neutrons. In this exceptional case, the four extra nucleons are easily identifiable and it does make sense to think of the ${}^{212}_{84}$Po nucleus as an α-particle bound to a ${}^{208}_{82}$Pb 'shell'. Then, the α-decay is simply a dissociation of this bond and the subsequent escape of the α-particle. It should also be clear that this situation is extremely rare: most of α-radioactive nuclei will not be this simple.

Since the physical process of the α-decay is rather involved, we are forced to make an approximation. This will consist of two separate assumptions. The above discussion notwithstanding:

1. the α-particle will be considered *as if* a well-defined separate entity and simply trapped within the confines of the nucleus.

[1] The description of the process here is purely conceptual and qualitative. The α-decay process is ultimately governed by the so-called 'nuclear strong interaction', the characteristic time for which is 10^{-23}s; therefore, "gradually" here still happens incredibly swiftly when compared to typical macroscopic processes, such as a cell division or the burst of an automobile tire.

2. The dynamics of this α-particle will then be determined by the Schrödinger equation, where the potential $V(\vec{r})$ is determined by the daughter nucleus.

2 The Model

The potential $V(\vec{r})$, considered to be produced *collectively* by the nucleons of the daughter nucleus, will determine the dynamics of α-decay, and we now turn to a discussion of $V(\vec{r})$.

Generally speaking, there must exist at least three regions where $V(\vec{r})$ appears qualitatively different.

1. Within a radius $0 \leq r \leq R$, $V(\vec{r})$ must produce an attractive force for the α-particle to be quasi-bound[2] within the nucleus. This binding is produced by strong nuclear interaction, the range of which is of the order of $1\text{fm} = 10^{-15}\text{m}$ (also called 1 Fermi), where this interaction is orders of magnitude stronger than the electrostatic interaction. Thus, $V(\vec{r}) < 0$ for $0 \leq r \leq R$ where R is expected to be several Fermi's.
2. Well away from the nucleus, $r \gg 1\text{fm}$, the effect of strong nuclear interaction is many orders of magnitude weaker than the electrostatic interaction, and we must approximate $V(\vec{r}) = 2Ze'^2/r$ for $r \gg 1\text{fm}$.
3. There is an intermediate region, for $R \leq r$ but $R \not\ll r$, where the two types of interaction are comparable and the shape of $V(\vec{r})$ is determined by this balance. Since $V(\vec{r})$ decays as $1/r$ for very large r, it must be that $V(\vec{r})$ has the shape of a potential barrier in this intermediate region, to prohibit the α-particles to simply fall out of the nucleus.

It is mainly this barrier which determines the transition amplitude and so the probability of α-decay, or α-capture[3]—since the same analysis will apply equally well to the time-reversed process. Moreover, given a potential $V(\vec{r})$, it will be possible to find stationary states of a given energy E. Clearly, when $\min[V(\vec{r})] \leq E < 0$, the α-particle will be *strictly bound*: the appearance of a classical such α-particle outside the nucleus is impossible, and the wave-function must decay exponentially with r, for $r > R$. (To see this, write down the corresponding WKB wave-function.)

On the other hand, for stationary states with $0 < E \leq \max[V(\vec{r})]$, there have to be at least two points where $E = V(\vec{r})$; write $r = a$ for the smallest such value and $r = b$ for the largest such value. Then $E > V(\vec{r})$ for $r < a$ and $r > b$ and these are the two important classically allowed regions: the α-particle inside ($r < a$), and outside ($r > b$) the nucleus. Also, $E < V(\vec{r})$ at least for a part of the interval $a < r < b$, and this will describe the barrier through which the α-particle must pass (by tunneling where $E < V(\vec{r})$) to escape in an α-decay, or be captured in the reverse process.

[2] Since we are discussing the α-decay, the α-particle is not completely bound, *i.e.*, is not perfectly localized within the nucleus, *i.e.*, outside the nucleus the probability of finding the α-particle will not decay exponentially with r. In distinction to true bound states (such as those in a harmonic oscillator), these are often called quasi-bound states.

[3] α-capture was first achieved by Rutherford, in 1919.

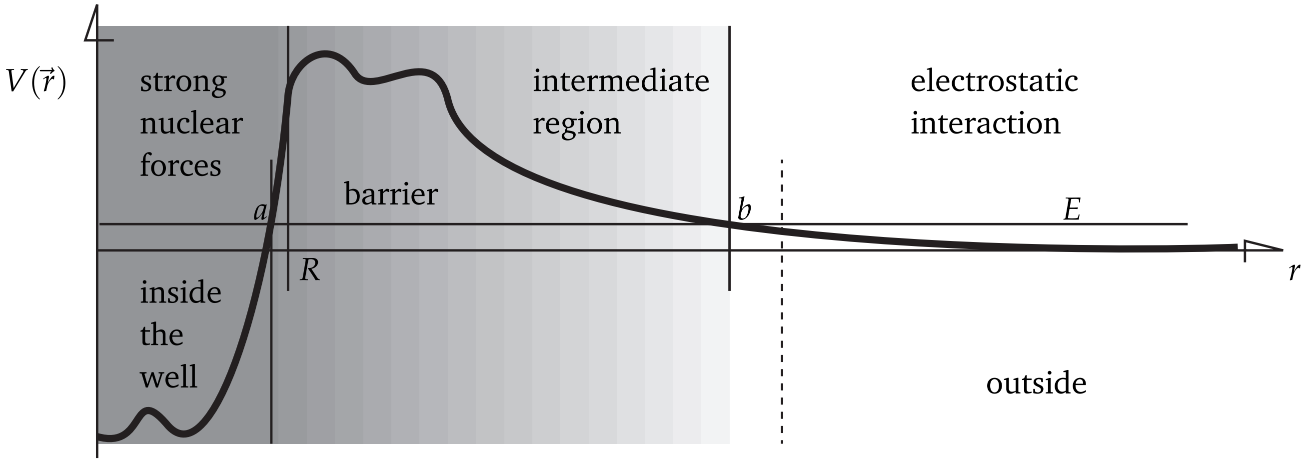

Figure 1: A sketch of the expected radial dependence of $V(\vec{r})$. In the grayed region, the *collective* potential is due to strong nuclear interaction (the shades of gray indicating the dominance) and its exact shape is not known. Outside this range and where the α-particle is by now well defined as a separate entity, the potential stems from the electrostatic repulsion and is well known.

2.1 Gamow's simple model

In 1928, G.A. Gamow provided a (heuristically speaking) first approximation to the α-decay, which was the first application of quantum mechanics to a nuclear physics problem. It very good qualitative agreement with experimental data provided a good measure of confidence that quantum mechanics was really a theory of Nature and not only of Atomic physics[4]. As a first attempt, Gamow made two assumptions about the potential.

1. Within the nucleus (of radius R), the potential is assumed to be simply a spherically symmetric constant potential well: $V(\vec{r}) = -V_0$ for $0 \leq r < R$.
2. The effect of the nuclear interaction is assumed to vanish outside the radius of the nucleus, for $r > R$. Thus, the potential immediately outside and out to infinity is simply the spherically symmetric electrostatic Coulomb potential: $V(\vec{r}) = 2Ze'^2/r$ for $R < r < \infty$.

That is, there is no intermediate transition region, where the strong nuclear interaction potential would continuously change into the Coulomb potential. Moreover, the potential changes discontinuously at $r = R$, so the force that the α-particle experiences at $r = R$ is (the slope of the transition from[5] $\lim_{r\to R-} V = -V_0$ to $\lim_{r\to R+} V = +2Ze'^2/R$ which is) infinite. This of course is unphysical. Nevertheless, we'll follow Gamow and pursue the analysis of this model.

Owing to spherical symmetry of the potential, the 3-dimensional Schrödinger equation

$$\Big[-\frac{\hbar}{2m_\alpha}\vec{\nabla}^2 + V(\vec{r})\Big]\psi(r,\theta,\phi) \;= E\psi(r,\theta,\phi), \tag{2}$$

[4] It was not uncommon for quantum mechanics to be called "atomic theory", even by its founding fathers: Bohr, Born, Dirac, Einstein, Heisenberg, Planck, Schrödinger... While quantum mechanics was developed to explain atomic physics, it was not clear in the beginning whether this theory should apply to larger systems such as molecules and bigger—hence the importance of the work on molecular physics, see chapter 18 [1], esp. §18.4. Similarly, very little was known of subatomic physics (the neutron was discovered only in 1932 by Chadwick, and artificial radioactivity was discovered only in 1934 by Jolio and Curie), so that Gamow's application of quantum mechanics to this uncharted territory was rather ground-breaking.

[5] "$r \to R\pm$" stands for $r = (R\pm\epsilon) \to R$.

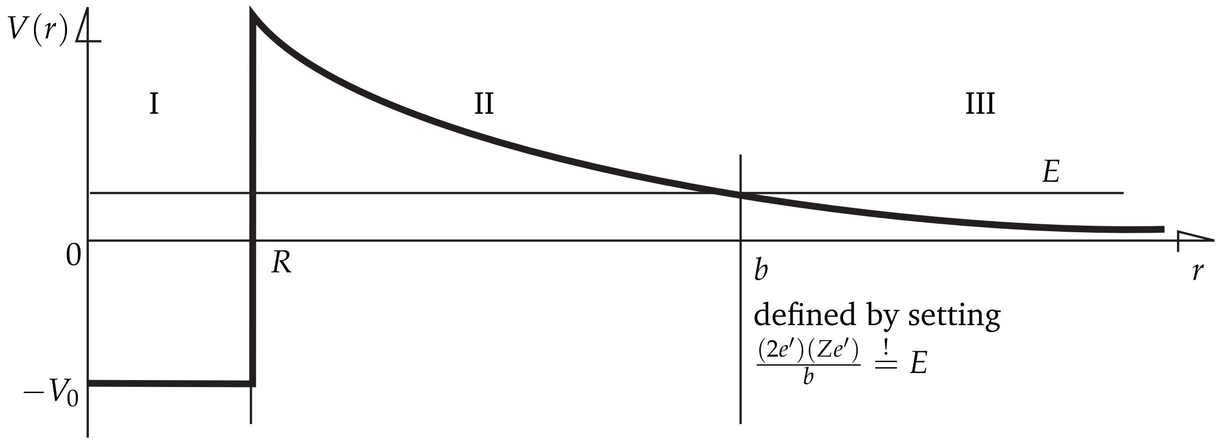

Figure 2: A plot of the simplest approximation of $V(\vec{r})$: spherically symmetric, and defining three regions: two classically allowed ones, the inside and the outside one, and the intermediate classically forbidden region which acts as a barrier.

separates, upon writing[6] $\psi(r,\theta,\phi) = \frac{u(r)}{r} P_\ell^m(\cos\theta) e^{im\phi}$, into

$$\frac{d^2u}{dr^2} + \left[\frac{2m_\alpha}{\hbar^2}[E - V(r)] - \frac{\ell(\ell+1)}{r^2}\right] u \;=\; 0, \tag{3}$$

and the standard associate Legendre equation for $P_\ell^m(\cos\theta)$ and the 'trigonometric' one for $e^{im\phi}$. The latter two being solved in the usual manner, we concentrate on the radial equation (3). Since $\psi(r,\theta,\phi) < \infty$, we must ensure that $[u(r)/r]$ remains finite including the point $r = 0$, which imposes the restriction that $u(r) \sim r^\beta$, $\beta \geq 1$ for $r \to 0$, so that $\lim_{r\to 0}[u(r)/r] < \infty$, which may be regarded as a boundary condition on $u(r)$.

2.2 Spinless decay

Clearly, the case $\ell = 0$, which corresponds to no angular momentum carried by the α-particle[7] is much simpler and we first turn to that.

The region I admits an exact solution (the potential is piece-wise constant),

$$u_I(r) = A\sin(Kr + \delta), \qquad K = \sqrt{\frac{2m_\alpha}{\hbar^2}(E + V_0)}, \tag{4}$$

whereas in the regions II and III we use the WKB solutions:

$$u_{II}(r) = \frac{C}{\sqrt{\kappa(r)}} e^{-\int_R^r dr\,\kappa(r)} + \frac{D}{\sqrt{\kappa(r)}} e^{\int_R^r dr\,\kappa(r)}, \qquad R < r < b, \tag{5a}$$

[6] Please, do not confuse this $u(r)$ with the dependent variable substitution $\psi \to e^{iu}$ we've made in deriving the formulae for the WKB approximation!

[7] As a bound state of four nucleons, we assume that the α-particle is in its own ground state, with no orbital angular momentum around the center of mass of the 4-nucleon system, and that the spins are added up to a total of zero. The total angular momentum of this 4-nucleon system—perceived as the α-particle's spin—is thus zero. In addition, by setting $\ell = 0$ in Eq. (3), we assume also zero orbital angular momentum about the center of the daughter nucleus.

$$u_{III}(r) = \frac{A'}{\sqrt{k(r)}} e^{i\int_b^r \mathrm{d}r\, k(r)} + \frac{B'}{\sqrt{k(r)}} e^{-i\int_b^r \mathrm{d}r\, k(r)}, \qquad b < r, \tag{5b}$$

where

$$\kappa(r) = \sqrt{\frac{2m_\alpha}{\hbar^2}\Big(\frac{2Ze'^2}{r} - E\Big)}, \qquad k(r) = \sqrt{\frac{2m_\alpha}{\hbar^2}\Big(E - \frac{2Ze'^2}{r}\Big)}. \tag{5a$'$, 5b$'$}$$

Now, the $\lim_{r\to 0}[u(r)/r] < \infty$ boundary condition forces $\delta = 0$ (so that $u_I(0) = 0$. Furthermore, the A' term in (5b) represents an outgoing wave, while the B' term represents an incoming wave. Thus, for describing α-decay, we need to set $B' = 0$, while for α-capture we need $B' = 0$; here we set the former.

Next, we need to match $u_I(r)$ and $u_{II}(r)$ across $r = R$, and $u_{II}(r)$ and $u_{III}(r)$ across $r = b$. For the first patching, note that the potential changes *discontinuously* around $r = R$, whence we must impose the standard boundary conditions:

$$\lim_{r\to R-} u_I(r) = \lim_{r\to R+} u_{II}(r), \qquad \lim_{r\to R-} u_I'(r) = \lim_{r\to R+} u_{II}'(r). \tag{6}$$

This gives ($\kappa_R = \kappa(R)$)

$$A\sin(KR) = \frac{C+D}{\sqrt{\kappa_R}}, \qquad AK\cos(KR) = (-C+D)\sqrt{\kappa_R}, \tag{7}$$

or

$$\begin{aligned} C &= \frac{A}{2\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) - K\cos(KR)\big], \\ D &= \frac{A}{2\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) + K\cos(KR)\big]. \end{aligned} \tag{8}$$

In contrast, the potential changes *continuously* around $r = b$, whence here we must impose the WKB boundary conditions, Eq. (4.52b) [1]:

$$C = \vartheta^* e^{\sigma} A', \qquad D = \tfrac{1}{2}\vartheta e^{-\sigma} A', \qquad \vartheta = e^{i\pi/4}, \tag{9}$$

where

$$\sigma = \int_R^b \mathrm{d}r\, \kappa_R = \sqrt{\frac{2m_\alpha}{\hbar^2}} \int_R^b \mathrm{d}r \sqrt{\frac{2Ze'^2}{r} - E}. \tag{10}$$

Together with Eq. (8), these imply

$$\vartheta^* e^{\sigma} A' = \frac{A}{2\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) - K\cos(KR)\big] \tag{11}$$

and

$$\vartheta e^{-\sigma} A' = \frac{A}{\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) + K\cos(KR)\big] \tag{12}$$

Dividing the last one by the former one, we obtain that

$$\vartheta^2 e^{-2\sigma} = 2\frac{\kappa_R \sin(KR) + K\cos(KR)}{\kappa_R \sin(KR) - K\cos(KR)} \tag{13}$$

and realize that something has gone *horribly wrong* here! Recalling that $\vartheta^2 = i$, we see that the left hand side is purely imaginary (κ_R and so σ in Eq. (10) is real), while the right hand side is purely real. Back-tracking, we see that the cause of this inconsistency lies in fixing *both* $\delta = 0$ *and* $B' = 0$—*i.e.*, imposing boundary conditions on both "ends" of the domain $r \in [0, \infty)$. Recall that this was done successfully in the past, but for bound states, and it had the drastic consequence of quantizing the allowed energy into a discrete spectrum. It should be clear that the total energy of the outgoing α-particle should *not* be so quantized.

In other words, using $u(0) = 0$ to set $u_I(r) = A\sin(Kr)$—which we *must* do for $\psi(r, \theta, \phi)$ to be finite near the origin—fixes the far left part of the solution. Then, the two matching conditions at $r = R$ determine the solution in region II (the integration constants C and D) in terms of A *completely*; the two matching conditions at $r = b$ then determine the solution in region III (the integration constants A' and B') in terms of A *completely*. We were therefore not free to choose $B' = 0$ at will. The resulting $\psi(r, \theta, \phi)$ has at this stage only two parameters:

1. A, the overall normalization constant, which then must be determined from a suitable normalization condition,
2. E, the total energy of the sought-for stationary state of the α-particle.

The (careful) phrasing of this last item in fact clarifies the point: the real solution $u_I(r)$ determines the *stationary state* $\psi(r, \theta, \phi)$, and indeed looks like a *standing wave*, composed of both an outgoing and an incoming wave. The former produces the A' part of $u_{III}(r)$ representing α-decay, while the latter one comes from the B' part, representing α-capture.

It is thus *inconsistent* to annihilate the α-capture "partial wave" by hand.

The simple-minded potential above does not admit solutions which describe exclusively (spinless, vanishing angular momentum) α-decay, but requires that there is a B' term (representing α-capture) in $u_{III}(r)$, representing an incoming α-particle. We thus correct:

$$A' = \tfrac{1}{2}\vartheta e^{-\sigma} C + \vartheta^* e^{\sigma} D, \qquad B' = \tfrac{1}{2}\vartheta^* e^{-\sigma} C + \vartheta e^{\sigma} D, \tag{14}$$

whereupon Eqs. (8) imply

$$A' = \frac{\vartheta e^{-\sigma} A}{4\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) - K\cos(KR)\big] \tag{15a}$$

$$\quad + \frac{\vartheta^* e^{\sigma} A}{4\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) + K\cos(KR)\big], \tag{15b}$$

$$= \frac{\vartheta e^{\sigma} A\cos(KR)}{4\sqrt{\kappa_R}}\Big[\kappa_R \tan(KR) - K - 2ie^{2\sigma}\big[\kappa_R \tan(KR) + K\big]\Big], \tag{15c}$$

$$B' = \frac{\vartheta^* e^{-\sigma} A}{4\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) - K\cos(KR)\big] \tag{15d}$$

$$\quad + \frac{\vartheta e^{\sigma} A}{4\sqrt{\kappa_R}}\big[\kappa_R \sin(KR) + K\cos(KR)\big], \tag{15e}$$

$$= \frac{\vartheta^* e^{\sigma} A\cos(KR)}{4\sqrt{\kappa_R}}\Big[\kappa_R \tan(KR) - K + 2ie^{2\sigma}\big[\kappa_R \tan(KR) + K\big]\Big]. \tag{15f}$$

Now it is clear that $|A'|^2 = |B'|^2$. The intermediate results reported in Park's text [1] differ because of the systematic and consistent practice of neglecting terms that relate to α-capture; this is routinely done in research, but hardly ever emphasized in the research literature, and nor it would seem in textbook form. Notice also that the final result for B', (15f), implies that the only way for B' to vanish would be a separate cancellation in the real and imaginary part of (15f), which in fact produces the requirements

$$\kappa_{R} \tan(KR) - K = 0 \quad \text{and} \quad \kappa_{R} \tan(KR) + K = 0, \tag{16}$$

which are clearly over-constraining the system, by setting $K = 0$, *i.e.*, $E = -V_0$.

The above then determine completely the stationary states of energy E. Unlike for true bound states, there is no condition on the energy, and E is not quantized. Compare this with the case $-V_0 < E < 0$, where the solution $u_{II}(r)$ would have been valid for all $r > R$, whereupon we'd need to set $D = 0$ for $u_{II}(r)$ not to diverge as $r \to \infty$. Through Eq. (8), this would impose the condition

$$D = \frac{A}{2\sqrt{\kappa_{R}}}\big[\kappa_{R} \sin(KR) + K\cos(KR)\big] = 0, \tag{17}$$

or

$$\tan(KR) = -K/\kappa_{R}, \tag{18}$$

which is the transcendental equation determining the discrete values of E for which there exist (true) bound states, representing the α-particle forever trapped by the potential of the daughter nucleus—thus implying that the system, identifiable as the parent nucleus, in fact does not decay.

However, consider the shape of the wave-function $u_{II}(r)$ according to Eq. (5a), rewritten here as

$$u_{II}(r) = \frac{C\, e^{-\sigma}}{\sqrt{\kappa(r)}} e^{-\int_b^r \mathrm{d}r\, \kappa(r)} + \frac{D\, e^{\sigma}}{\sqrt{\kappa(r)}} e^{\int_b^r \mathrm{d}r\, \kappa(r)}. \tag{19}$$

The C-term decreases exponentially outward, while the D-term grows. Certainly, the former seems to better conform to a situation in which the α-particle is expected to hover about inside the nucleus for a while and then tunnel through the barrier outward. Thus, one should regard Eq. (18) approximately true also for α-decay, at least so that $|D|\, e^{\sigma} \ll |C|\, e^{-\sigma}$, *i.e.*, $|D| \ll |C|\, e^{-2\sigma}$. In this *approximation*,

$$A' \approx \frac{\vartheta e^{\sigma} A K \cos(KR)}{2\sqrt{\kappa_{R}}}. \tag{20}$$

2.3 Numerical evaluations and predictions

The probability of α-decay per second (decay rate) must be proportional to the probability of finding the state outside the nucleus, *i.e.*, $\lambda \propto |A'|^2$. Integration over angles will give a 4π, and

$$\lambda = \frac{4\pi\hbar|A'|}{m_\alpha} \approx \frac{4\pi\hbar K^2 |A'| e^{-2\sigma} \cos^2(KR)}{m_\alpha \kappa_{R}}, \tag{21}$$

is the only expression depending in addition only on initially given quantities that has the correct dimensions; the second, approximate equality holds upon imposing (20). The proportionality of the decay rate to $e^{-2\sigma}$ is very typical of WKB calculations, and we now turn to calculating it.

The exponential σ was defined in (10) as

$$\begin{aligned}\sigma &= \sqrt{\frac{2m_\alpha}{\hbar^2}}\int_R^b \mathrm{d}r\,\sqrt{\frac{2Ze'^2}{r}-E},\\ &= \sqrt{\frac{2m_\alpha E}{\hbar^2}}\int_R^b \mathrm{d}r\,\sqrt{\frac{b}{r}-1},\\ &= \sqrt{\frac{2m_\alpha E}{\hbar^2}}\,b\left[\arccos\sqrt{\frac{R}{b}}-\sqrt{\frac{R}{b}}\sqrt{1-\frac{R}{b}}\right],\end{aligned} \tag{22}$$

the integral being evaluated through the substitution $r = b\cos^2\rho$. Now, since $b = \frac{2Z(e')^2}{E}$ is typically several times R, we can use small-argument (first terms in the Taylor) expansions: $\arccos(x) \approx \frac{\pi}{2} - x$ and $\sqrt{x(1-x)} \approx \sqrt{x}$, so that

$$\hbar\,\sigma \approx \tfrac{\pi}{2}\sqrt{2m_\alpha E}\,b - \sqrt{8m_\alpha E\,b\,R} + \tfrac{1}{3}\sqrt{2m_\alpha E\,R^3/b} + \frac{1}{10\sqrt{2}}\sqrt{\frac{m_\alpha E\,R^5}{b^3}} + \dots \tag{23}$$

$$= \pi\sqrt{\frac{2m_\alpha}{E}}\,Z(e')^2 - 4\sqrt{m_\alpha Z\,R}\,e' + \frac{1}{3}\sqrt{\frac{m_\alpha}{Z}R^3}\,\frac{E}{e'} + \frac{1}{40}\sqrt{\frac{m_\alpha}{Z}R^5}\,\frac{E^2}{(e')^3} + \dots \tag{24}$$

This produces a very good agreement with experimental data (the first two terms reproduce the so-called Geiger-Nuttall experimental law; see figure 3), verifying the model and the approximations that were made in the process of the above derivations.

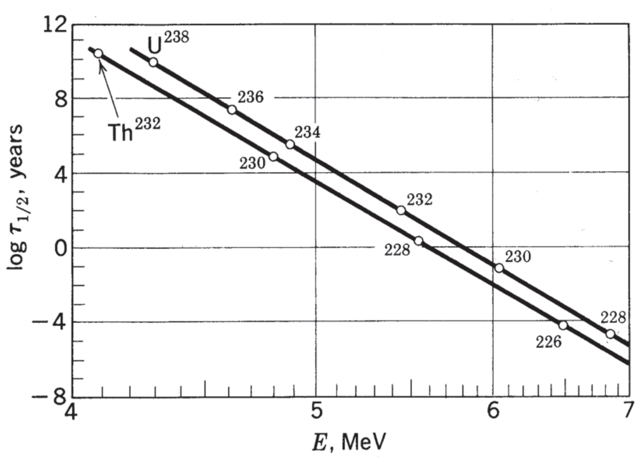

FIGURE 14.4
Geiger-Nuttall plot of $\log\tau_{1/2}$ against $E^{-1/2}$ (plotted toward the left, with the scale given in terms of E) for isotopes U and Th. (*Adapted from I. Perlman and J. O. Rasmussen, "Alpha Radioactivity," in Encyclopedia of Physics, vol.* 42, 1957, *by permission of Springer-Verlag OHG and the authors.*)

Figure 3: The Geiger-Nuttal plot, copied from Ref. [1, p.451].

2.4 Exact solution

While the above analysis does agree very well with experiments, it is not conclusive as it (**1**) limits to $\ell = 0$, "spinless" α-decay, and (**2**) was done in an approximation, the WKB analysis, partly following the historical account and partly because of the intuitiveness of the solution.

Thus, the cases of $\ell \neq 0$ remain to be analyzed; we now turn to this general case and present the general (and in fact exact) solution to the model that is still based on Gamow's simplified potential as in Fig. 2.

Return to the radial equation (3):

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\left[\frac{2m_\alpha}{\hbar^2}\big[E-V(r)\big]-\frac{\ell(\ell+1)}{r^2}\right]u \;=\; 0. \tag{3}$$

For $0<r<R$, we write $R(r)=[u(r)/r]$ and obtain the original

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r}+\left[K^2-\frac{\ell(\ell+1)}{r^2}\right]R \;=\; 0, \tag{25}$$

spherical Bessel equation. Since $\psi(r,\theta,\phi)<\infty$, the von Neumann solution must be excluded, and we have $R_{\text{in}}(r)=Aj_\ell(Kr)$.

For $r>R$, we take hint from the Hydrogen atom, and expect the radial solution in the form $R(r)=r^\ell e^{-\beta r}f(r)$. Substituting this, we obtain (upon multiplying throughout by $r^{\ell-1}e^{\alpha r}$)

$$rf''+\big[2(\ell{+}1)-2\beta r\big]f'-\big[(2\beta(\ell{+}1)+4Ze'^2m_\alpha/\hbar^2)+(\beta^2+2m_\alpha E/\hbar^2)r\big]f \;=\; 0. \tag{26}$$

By setting $\beta^2=-2m_\alpha E/\hbar$, the 'no-derivative term' simplifies, and after rescaling $z=2\beta r$:

$$zf''(z)+\big[2(\ell{+}1)-z\big]f'(z)-\big[(\ell{+}1)+2Ze'^2m_\alpha/\beta\hbar^2)\big]f(z) \;=\; 0. \tag{27}$$

has become the confluent hypergeometric equation. Writing $a=(\ell{+}1){+}2Ze'^2m_\alpha/\beta\hbar^2$ and $c=\ell{+}1$, the general solution is given by [2]:

$$f(z) \;=\; B\,{}_1F_1\big({a \atop c};z\big)+C\,z^{1-c}\,{}_1F_1\big({a+1-c \atop 2-c};z\big). \tag{28}$$

Writing $w=\frac{2Ze'^2m_\alpha}{\beta\hbar^2}$, the radial function may be specified (after redefining C slightly) as

$$R_{\text{out}}(r)=r^\ell e^{-\beta r}\Big[B\,{}_1F_1\big({\ell+1+w \atop 2\ell+2};2\beta r\big)+C\,r^{-2\ell-1}\,{}_1F_1\big({w-\ell \atop -2\ell};2\beta r\big)\Big]. \tag{29}$$

Of course, the confluent hypergeometric functions ${}_1F_1\big({a \atop b};x\big)$ *can* be expressed in terms of Bessel functions, and *vice versa*; see Ref. [2].

Note that β is bound to be imaginary, since $E>0$ and so $\beta^2=-2m_\alpha E/\hbar<0$; choose $0<\arg(\beta)<\pi$. The exponential $e^{-\beta r}$ then describes an incoming wave. To determine the incoming/outgoing nature of the terms in the square brackets, we use the asymptotic behavior of the confluent hypergeometric function [3]:

$${}_1F_1\big({a \atop c};z\big)\sim e^{-i\pi a}\frac{\Gamma(c)}{\Gamma(c-a)}z^{-a}+\frac{\Gamma(c)}{\Gamma(a)}e^z z^{a-c}, \qquad z\to\infty. \tag{30}$$

The $e^{2\beta r}$ factor is outgoing, and upon multiplication through with the $e^{-\beta r}$ prefactor, we see that the first term is an incoming wave (α-capture) and the second term is the outgoing wave (α-decay). That is, the asymptotic form of $R_{\text{out}}(r)$ is

$$R_{\text{out}}(r)\sim Fr^{-(w+1)}\,e^{-\beta r}+Gr^{w-1}\,e^{+\beta r} \tag{31}$$

where (upon using the reflection formula for the Γ-function)

$$F = B\Big(\frac{e^{-i\pi}}{2\beta}\Big)^{w+\ell+1}\frac{\Gamma(2\ell{+}2)}{\Gamma(\ell{+}1{+}w)} + C\Big(\frac{e^{-i\pi}}{2\beta}\Big)^{w-\ell}\frac{\Gamma(\ell{+}1{+}w)\sin[\pi(\ell{+}w)]}{\Gamma(2\ell{+}1)\sin[2\ell\pi]}, \tag{32}$$

$$G = B(2\beta)^{w-\ell-1}\frac{\Gamma(2\ell{+}2)}{\Gamma(\ell{+}1{+}w)} + C(2\beta)^{w-\ell}\frac{\Gamma(\ell{+}1{-}w)\sin[\pi(\ell{-}w)]}{\Gamma(2\ell{+}1)\sin[2\ell\pi]}. \tag{33}$$

So, for a purely outgoing wave (α-decay), we need[8] $F \approx 0$, *i.e.*,

$$C \approx B\frac{(2\ell{+}1)}{(2\beta)^{2\ell+1}}\frac{\Gamma^2(2\ell{+}1)\sin[2\ell\pi]}{\Gamma^2(\ell{+}1{+}w)\sin[\pi(\ell{+}w)]}, \tag{34}$$

where we used that ℓ is integral so the prefactor $-e^{-i\pi(2\ell+1)} = +1$.

Note that $w = \frac{2Ze'^2 m_\alpha}{\beta\hbar^2}$ is non-integral, moreover imaginary, since β is imaginary. The above condition then simply entails that $C = 0$ since $\sin[2\ell\pi] = 0$ for integral ℓ.

This over-constrains the solution (as discussed before), for there is now one undetermined constant in $R_{\text{in}}(r)$ and another undetermined constant in $R_{\text{out}}(r)$, but two matching conditions at $r = R$ (equating both the functions and their derivatives). Finally, an overall constant ought to remain so that the wave-function could be normalized. That means that of the two matching conditions, one should be used to relate A from $R_{\text{in}}(r)$ and B from $R_{\text{out}}(r)$, while the other should be used to restrict the possible values of the energy E—the only parameter left free. Explicitly carrying this out is rather tedious and will not be done here, but is clearly possible.

This follows on noticing that equating $\lim_{\epsilon\to 0} R_{\text{in}}(R{-}\epsilon)$ with $\lim_{\epsilon\to 0} R_{\text{out}}(R{+}\epsilon)$ may clearly be used to relate A from $R_{\text{in}}(r)$ and B from $R_{\text{out}}(r)$. Thereafter, these constants drop out of the second matching condition if that is written as:

$$\lim_{r\to R-}\frac{1}{R_{\text{in}}(r)}\frac{\mathrm{d}R_{\text{in}}(r)}{\mathrm{d}r} = \lim_{r\to R+}\frac{1}{R_{\text{out}}(r)}\frac{\mathrm{d}R_{\text{out}}(r)}{\mathrm{d}r}. \tag{35}$$

(Once the functions have been made to match at the boundary, one can modify the derivative matching condition

$$\lim_{r\to R-} R'_{\text{in}}(r) = \lim_{r\to R+} R'_{\text{out}}(r) \tag{36}$$

by dividing the l.h.s. by $R_{\text{in}}(r)$ and the r.h.s. by $R_{\text{out}}(r)$ and then take the limits.) Therefore, this last condition (admittedly cumbersome and obviously transcendental) appears to be a (quantization) condition on the only free parameter appearing in it—the energy E.

Had we not required that the solution be of the form of an outgoing wave at $r \to \infty$, there would have been one more constant in $R_{\text{out}}(r)$ and thus no condition on the energy. Therefore, the exact stationary states (by definition) describe both an outgoing (α-decay) wave and an incoming (α-capture) wave for $r \gg R$.

[8] As discussed above, it is *inconsistent* to set $F = 0$ for a stationary wave, which by definition must include both traveling waves. The relation "≈" will be used to denote the approximate setting to zero, as discussed above, and we explore the consequences in general.

Indeed, for a purely outgoing wave, there would be a net probability current outward:

$$\begin{aligned}\vec{\jmath} &= \frac{2e}{m_\alpha}\,\Re e\left\{G^* r^{w-1} e^{\beta^* r}\,\hat{\mathbf{e}}_r \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}r} G r^{w-1} e^{\beta} r\right\} \\ &= \frac{2e\hbar}{m_\alpha}\,|G|^2\,\Im m(\beta)\,\hat{\mathbf{e}}_r\, r^{2w-2}\, e^{2\,\Re e(\beta) r},\end{aligned} \tag{37}$$

which results in a "probability leak" through a sphere of radius ρ:

$$\oint \mathrm{d}\omega\,\rho^2 \vec{\jmath}(\rho) \;=\; \frac{8\pi e\hbar\,\Im m(\beta)}{m_\alpha} |G|^2 r^{2w}\, e^{2\,\Re e(\beta) r}. \tag{38}$$

Thus, unless $\Re e(\beta) < 0$, this does not vanish in the $\rho \to \infty$ limit, an probability is not conserved! Since $\beta^2 = -2m_\alpha E/\hbar$, and the quantities on the r.h.s. are real, $\Re e(r) = 0$, and a purely outgoing (α-decay) wave would imply probability non-conservation.

We have thus proved that the stationary states must include both an outgoing (α-decay) and an incoming (α-capture) wave. This does not invalidate the estimates of subsection 2.3 above, since the probability of finding the α-particle in the escaping wave remains proportional to the square of the absolute value of the amplitude of the outgoing wave, regardless of the amplitude of the incoming wave.

3 Better Models

The above presents detailed analysis of the simple spherically symmetric potential as given in Fig. 2.

As noted in the previous section, the discontinuous jump of the potential at $r = R$ is unphysical, as the force there becomes infinite. The simplest modification of this potential is then presented by smoothing the jump from the bottom of the potential well inside the nucleus to the top of the Coulomb barrier at the outer side of $r = R$. The resulting potential is then sketched in Fig. 4 below. Now there is no force on the α-particle while it is within $0 < r < \varrho$, then there is a constant attractive force (equal to the slope of the potential) within $\varepsilon < r < R$. Finally, outside $r = R$, there is again the Coulomb repulsion force. This would seem to be more reasonable physically. The potential function is now

$$V(r) \;=\; \begin{cases} \alpha r - \beta & \varrho \le r \le R, \\ \frac{2Ze'^2}{r} & R \le r < \infty, \end{cases} \tag{39}$$

where

$$\alpha = \frac{2Ze'^2 + V_0 R}{R(R-\varrho)}, \qquad \beta = \frac{2Ze'^2\varrho + V_0 R^2}{R(R-\varrho)}. \tag{40}$$

Now $a = (E-\beta)/\alpha$, and $b = 2Ze'^2/E$ remains.

As to the solutions (stationary states) in this modified model, one can apply WKB for $\varepsilon < r < \infty$, and then match this with $u_I(r) = A\sin(Kr)$ which is still the exact solution for $0 \le r \le \varrho$. The decay rate would again involve $e^{2\sigma}$, where

$$\sigma = \sqrt{\frac{2m_\alpha}{\hbar^2}} \int_a^b \mathrm{d}r\,\sqrt{V(r) - E}, \tag{41}$$

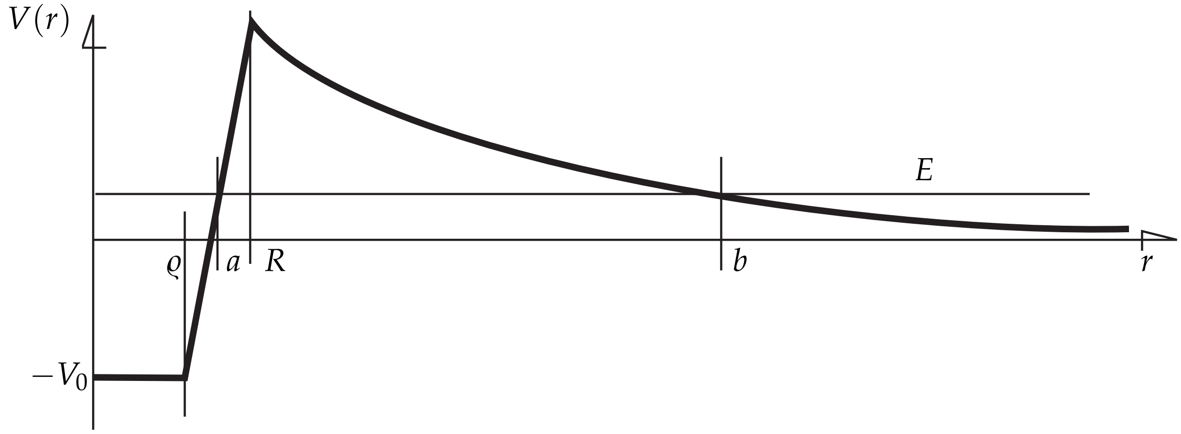

Figure 4: A plot of a next-to-simplest approximation of $V(\vec{r})$. The discontinuous jump from $-V_0$ for $0 < r < R$ to $2Ze'^2/R$ has been replaced by a linear segment.

$$= \sqrt{\frac{2m_\alpha}{\hbar^2}}\Big[\int_a^R \mathrm{d}r\ \sqrt{\alpha r - \beta - E} + \int_R^b \mathrm{d}r\ \sqrt{\frac{2Ze'^2}{r} - E}\Big], \tag{42}$$

$$= \sqrt{\frac{2m_\alpha}{\hbar^2}}\frac{2}{3\alpha}\Big[\sqrt{(\alpha R - \beta - E)^3} - \sqrt{(\alpha a - \beta - E)^3}\Big] \qquad \Leftarrow \text{slope correction} \tag{43}$$

$$+ \sqrt{\frac{2m_\alpha E}{\hbar^2}}\, b\left(\arccos\sqrt{\frac{R}{b}} - \sqrt{\frac{R}{b}}\sqrt{1-\frac{R}{b}}\right), \tag{44}$$

The second line in (44) is of course the same as before, whereas the first line is the correction due to the finite slope in Fig. 3. Note that, unlike the old result, this now does depend on the depth of the potential well, V_0, and of course also on the new parameter, ϱ. This makes an estimation quite harder, but we note that these correction terms will definitely involve higher positive powers of E than what already appeared in Eq. (24). Since the old result (24) already showed a good agreement with the experimental data, it follows that the corrections should be minimal and so $\varrho \lesssim R$, and $R{-}\varrho$ may be used as a small parameter to expand the square-roots in the first line of (44). The result is a power-series in non-negative integral powers of $(R-\varrho)E$, with an over-all $\sqrt{R{-}\varrho}$ pre-factor. The 0^{th} term of this series shifts the E-independent second term in (24), while the remaining terms are all new, as compared to (24). Note that in the limit $\varrho \to R$, these new terms all vanish because of the over-all $\sqrt{R{-}\varrho}$ pre-factor.

While comparison with experimental data, and very little deviation from the power-laws in Eq. (24), indicates that $\varrho \approx R$, we may nevertheless attempt to describe the 'inside' region of the potential by means of a (spherical) harmonic oscillator. The resulting potential is sketched in Fig. 4 This time, the 'inside' radial function would be a solution of the 3-dimensional harmonic oscillator (for $\ell = 0$, this does reduce to the 1-dimensional case). Note that since the solutions are needed only for finite r, it is not necessary to limit the series or discard the solution which would diverge at infinity; both solutions of the resulting radial equation can be used (they can be expressed in terms of hypergeometric functions). Now $a = \sqrt{2E/m_\alpha}/\omega$.

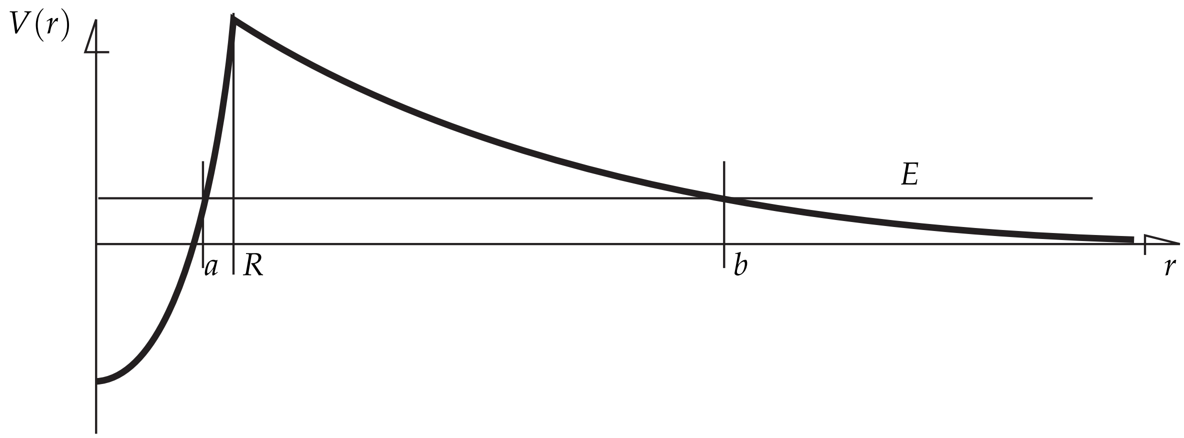

Figure 5: A further elaboration on the shape of $V(\vec{r})$; the inside potential has been assumed to be of the (spherical) harmonic kind.

The decay rate, estimated again by WKB methods, will again involve $e^{-2\sigma}$, where this time

$$\sigma = \sqrt{\frac{2m_\alpha}{\hbar^2}}\left[\int_a^R \mathrm{d}r\,\sqrt{\tfrac{1}{2}m_\alpha\omega r^2 - E} + \int_R^b \mathrm{d}r\,\sqrt{\frac{2Ze'^2}{r} - E}\right], \tag{45}$$

$$\tag{46}$$

$$= \sqrt{\frac{2m_\alpha}{\hbar^2}}\left[\frac{R}{2}\sqrt{\tfrac{1}{2}m_\alpha\omega^2R^2 - E} - \frac{a}{2}\sqrt{\tfrac{1}{2}m_\alpha\omega^2a^2 - E}\right. \tag{47}$$

$$\left. + \frac{E}{\omega\sqrt{2m_\alpha}}\ln\left(\frac{am_\alpha\omega^2 + \omega\sqrt{2m_\alpha}\sqrt{\tfrac{1}{2}m_\alpha\omega^2a^2 - E}}{Rm_\alpha\omega^2 + \omega\sqrt{2m_\alpha}\sqrt{\tfrac{1}{2}m_\alpha\omega^2R^2 - E}}\right)\right] \tag{48}$$

$$+ \sqrt{\frac{2m_\alpha E}{\hbar^2}}\,b\left(\arccos\sqrt{\frac{R}{b}} - \sqrt{\frac{R}{b}}\sqrt{1 - \frac{R}{b}}\right), \tag{49}$$

The correction with respect to the Gamow result (24) now appears in the first two lines of (49). Note especially the logarithmic terms; since $\lambda \propto e^{-2\sigma}$, the decay rate will be proportional to a power of the ratio which appears inside the logarithm, quite unlike all other terms so far. Again, for these corrections to be small, it must be that $a \lesssim R$, whence $\omega \gtrsim \sqrt{2E/m_\alpha}/R$, which gives an estimate on typical frequencies of the α-particle while in the nucleus. This is quite consistent with the estimates based on the Gamow model (see Ref. [1], Eq. (14.17), p. 449),

$$\omega \approx \frac{\hbar K}{m_\alpha R} = \sqrt{2(E + V_0)/m_\alpha}/R. \tag{50}$$

Finally, the models in Fig. 3 and 4 both have the potential $V(r)$ continuous at $r = R$, but in both cases, the derivative is not continuous, and the force field would have discontinuities. This can be remedied by introducing a smoothing intermediate region, and use the ‘upside-down’ harmonic oscillator for $R \leq r \leq R'$:

$$V(r) = \begin{cases} \frac{1}{2} m_\alpha \omega^2 r^2 & 0 \leq r \leq R, \\ W - \frac{1}{2} m_\alpha \nu^2 (r - r_0)^2 & R \leq r \leq R', \\ \frac{2Ze'^2}{r} & R' \leq r \leq \infty, \end{cases} \tag{51}$$

where W, ν, r_0 and R' are chosen so that both $V(r)$ and $V'(r)$ are continuous across $r = R$ and $r = R'$. A sketch is given in Fig. 6.

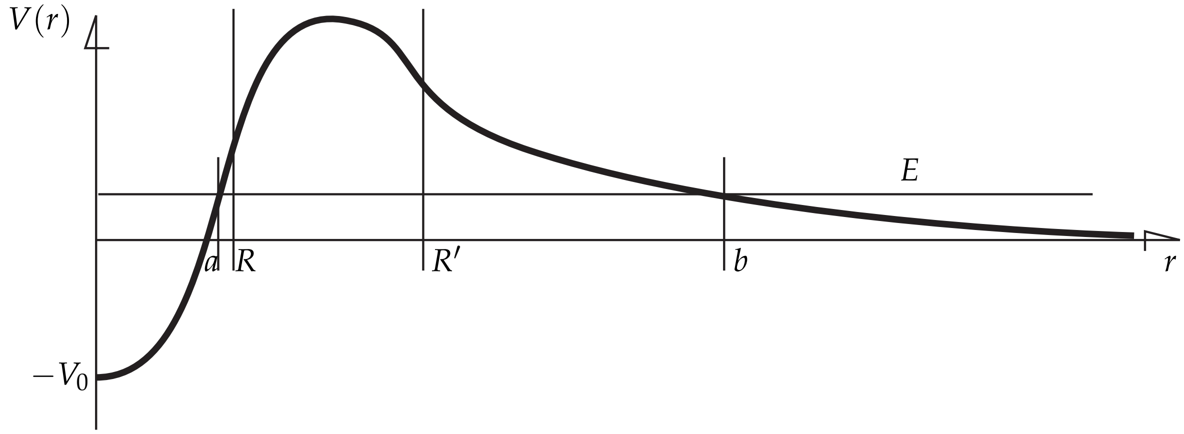

Figure 6: Our last attempt at refining the shape of $V(\vec{r})$. The inside potential has been assumed to be of the (spherical) harmonic kind, just as above. However, instead of the quadratic function simply turning into the $1/r$ curve, there is now an intermediate region, where the potential is the 'upside-down' quadratic curve. At the points, R, R', the shapes are joined smoothly, which ensures that the force is continuous.

Stationary states would be found by using the solutions to the 3-dimensional harmonic oscillator for $0 \leq r \leq R$, their analytic continuation for $R \leq r \leq R'$ and the above analyzed confluent hypergeometric functions for $r \geq R$. The decay rate is again estimated using WKB methods, and the integral for σ breaks into two or three parts, depending whether $a > R$ or $a < R$. The integrals $\int \mathrm{d}r \sqrt{V(r) - E}$ can again be solved exactly, and some useful integrals are collected in the appendix A.

4 Conclusion

Given the longish and somewhat disheartening introductory note, one might have expected a rather poor predictive power of such a simple model as has been analyzed in section 2. Fortunately, it turns out that all the integrals for σ, the exponential of which is the dominant factor in the WKB approximation for the decay rate, could be evaluated exactly. Given also that the WKB approximation is typically rather accurate for decay problems, this combined into a very good agreement with experiments.

The simplest model on purpose ignores many of the details discussed in the introduction, but captures two essential characteristics of the process: 1. the Coulomb repulsion between the daughter nucleus and the α-particle, and 2. the fortuitously negligible dependence on the details of the potential inside the nucleus. Modifications of this model will include more realistic potentials inside the nucleus, and more realistic transition regions between 'inside' and 'well outside' the nucleus. Some of these have been described in the previous section, and many more can be constructed in similar vein.

A Some Useful Integrals

When estimating the decay rate using WKB methods through a potential barrier described in part by some simple functions, the following integrals are useful:

$$\int \mathrm{d}x\ \sqrt{1+x^2} = \tfrac{1}{2}x\sqrt{1+x^2} + \tfrac{1}{2}\sinh^{-1}(x), \tag{52}$$

$$\int \mathrm{d}x\ \sqrt{1-x^2} = \tfrac{1}{2}x\sqrt{1-x^2} + \tfrac{1}{2}\sin^{-1}(x), \tag{53}$$

$$\int \mathrm{d}x\ \sqrt{x^2-1} = \tfrac{1}{2}x\sqrt{x^2-1} - \tfrac{1}{2}\ln\Big(r - \sqrt{x^2-1}\Big), \tag{54}$$

$$\int \mathrm{d}x\ \sqrt{1+\frac{1}{x}} = x\sqrt{1+\frac{1}{x}} + \tfrac{1}{2}\ln\Big(1+2x+2x\sqrt{1+\tfrac{1}{x}}\Big), \tag{55}$$

$$\int \mathrm{d}x\ \sqrt{1-\frac{1}{x}} = x\sqrt{1-\frac{1}{x}} - \tfrac{1}{2}\ln\Big(-1+2x+2x\sqrt{1-\tfrac{1}{x}}\Big), \tag{56}$$

$$\int \mathrm{d}x\ \sqrt{\frac{1}{x}-1} = x\sqrt{\frac{1}{x}-1} - \tfrac{1}{2}\tan^{-1}\Big(\frac{(2x-1)\sqrt{\frac{1}{x}-1}}{2(x-1)}\Big). \tag{57}$$

References

[1] D. Park, *Introduction to Quantum Physics*, 3rd ed., McGraw-Hill, Inc., 1992.

[2] G. B. Arfken and H. Weber, *Mathematical Methods for Physicists*, 6th ed., Academic Press, 2005.

[3] S. Flügge, *Practical Quantum Mechanics*, Vol. I and II, Springer-Verlag, 1971.