Quantum Mechanics I

Quantum Mechanics

Three-Dimensional Space: Rotations, Angular Momenta and Composite Systems

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Pink Floyd: "Another Brick in the Wall (Pt. II)"

Angular Momenta

Algebraic Solution of LHO

- Changed variables: $Q, P \to a, a^\dagger$ so that $[a, a^\dagger] = 1$:
  - Hamiltonian $H = \hbar\omega(a^\dagger a + ½)$ $\quad H\,|n\rangle = E_n\,|n\rangle \quad E_n = \hbar\omega(n+\frac{1}{2})$
  - w/ $N := a^\dagger a$, so $\quad [N, a] = -a \qquad [N, a^\dagger] = +a^\dagger$
  - Then $\quad N\,|n\rangle = n\,|n\rangle \qquad a\,|n\rangle = \sqrt{n}\,|n-1\rangle \quad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\mathscr{H} = \Big\{ |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle : \ \langle n|n'\rangle = \delta_{n,n'}, \ \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1} \Big\}$$

  - and

$$\langle m|R(a, a^\dagger)|n\rangle = \sum_{p,q=0}^{\infty} c_{p,q}\langle m|(a^\dagger)^p(a)^q|n\rangle$$
$$= \sum_{p,q=0}^{\infty} c_{p,q}\sqrt{n(n-1)\cdots(n-q+1)}$$
$$\times\sqrt{(n-q+1)(n-q+2)\cdots(n-q+p)}\,\delta_{m,n-q+p}$$

  - and even

$$\psi_0(x) = N\,e^{-\frac{1}{2}\alpha x^2} \qquad \alpha = \frac{M\omega}{\hbar} \qquad N = \sqrt[4]{\frac{\alpha}{\pi}}$$

Angular Momenta

3D Space & Rotations

- In 3D space, $W(\vec{r}) = W(r)$ $\quad \frac{\partial W}{\partial \theta} = 0 = \frac{\partial W}{\partial \phi}$ $\quad$ 3D rotational symmetry
- Then [see Arfken & Webber, exercises 2.5.13–2.5.17]

$$\vec{\nabla}^2 f(\vec{r}) = \tfrac{1}{r}\big(\tfrac{\partial^2}{\partial r^2}\, r\, f\big) - \tfrac{1}{r^2}\vec{L}^2 f \qquad \vec{L}^2 f := -\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}\big[\sin\theta \tfrac{\partial f}{\partial\theta}\big] - \tfrac{1}{\sin^2\theta}\tfrac{\partial^2 f}{\partial\phi^2}$$

$$\vec{L} = -i(\vec{r}\times\vec{\nabla}) = i\big(\hat{e}_\theta \tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\phi} - \hat{e}_\phi \tfrac{\partial}{\partial\theta}\big)$$

- Use algebra. In Cartesian coordinates [A&W 1.8.7],

$$L_i = -i\varepsilon_{jk}{}^{\ell}\, x^k \tfrac{\partial}{\partial x^\ell} \qquad L_x = -i\big(y\tfrac{\partial}{\partial z} - z\tfrac{\partial}{\partial y}\big),\ \ etc.$$

$$[L_j, L_k] = i\varepsilon_{jk}{}^{\ell}\, L_\ell \qquad (L_j)^\dagger = L_j$$

- No two of L_j commute, no simultaneous eigenvectors
- Instead: $\vec{L}^2 := L_x^2 + L_y^2 + L_z^2 \qquad [\vec{L}^2, L_j] = 0$
  - Pick $L_z = L_3$, so $\quad \vec{L}^2\,|\lambda, m\rangle = \lambda\,|\lambda, m\rangle \qquad L_3\,|\lambda, m\rangle = m\,|\lambda, m\rangle$
  - & figure out everything we can about λ and m.

Angular Momenta

3D Space & Rotations

$$\vec{L}^2 := L_x^2 + L_y^2 + L_z^2$$

- Now, compute

$$\langle\lambda, m|\vec{L}^2|\lambda, m\rangle = \lambda, \quad = \underbrace{\langle L_x^2\rangle + \langle L_y^2\rangle}_{\geqslant 0} + [\langle L_z^2\rangle = m^2]$$

so $\lambda \geq m^2$

- Now, define $L_\pm := L_x \pm i L_y$
- so $[L_3, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_3 \qquad [\vec{L}^2, L_\pm] = 0$

$$L_\pm L_\mp = L_x^2 + L_y^2 \mp i[L_x, L_y] = L_x^2 + L_y^2 \pm L_z$$

- Then $L_3(L_\pm|\lambda, m\rangle) = (L_\pm L_3 \pm L_\pm)|\lambda, m\rangle = (m \pm 1)(L_\pm|\lambda, m\rangle)$

$$L_\pm|\lambda, m\rangle = C_\pm|\lambda, m\pm 1\rangle$$

- so $\big|L_\pm|\lambda, m\rangle\big|^2 = \big|C_\pm|\lambda, m\pm 1\rangle\big|^2 = |C_\pm|^2 \geqslant 0$

$$= \langle\lambda, m|L_\mp L_\pm|\lambda, m\rangle = \langle\lambda, m|[L_x^2 + L_y^2 \mp L_3]|\lambda, m\rangle$$

$$= \langle\lambda, m|[\vec{L}^2 - L_3^2 \mp L_3]|\lambda, m\rangle = \lambda - m^2 \mp m$$

- and $\lambda \geq m(m\pm 1)$. $\quad j := \max(|m|)$, $\lambda = j(j+1)$

$$C_\pm = \sqrt{j(j+1) - m(m\pm 1)}$$

Angular Momenta

3D Space & Rotations

- Redefine $|\lambda,m\rangle \to |j,m\rangle$ $\vec{L}^2\,|j,m\rangle = j(j{+}1)\,|j,m\rangle$ $L_3\,|j,m\rangle = m\,|j,m\rangle$
  - where $L_\pm\,|j,m\rangle = \sqrt{j(j{+}1) - m(m{\pm}1)}\,|j,m{\pm}1\rangle$
  - Notice: $L_+\,|j,j\rangle = 0$ $L_-\,|j,-j\rangle = 0$ (just like $a\,|0\rangle = 0$)
- Thus: $V_j := \{\,|j,m\rangle : -j \leqslant m \leqslant j,\ \ \triangle m \in \mathbb{Z}\}$ $2j \in \mathbb{Z}$

$$U_{\vec{\varphi}}\,V_j := \exp\{-i\vec{\varphi}\cdot\vec{L}\}\,V_j = \exp\{\,-i(\varphi^{\pm}L_\pm + \varphi^3 L_3)\}\,V_j = V_j$$

- Generalize:

  $L_j \to J_j = L_j + S_j$

  - where L_j operate on the positional factor
  - while S_j operate on the directional factor

	dim.	formal ket-notation
V_0	1	$\{\,|0,0\rangle\,\}$
$V_{\frac{1}{2}}$	2	$\left\{\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle, |\tfrac{1}{2},+\tfrac{1}{2}\rangle\,\right\}$
V_1	3	$\{\,|1,-1\rangle, |1,0\rangle, |1,+1\rangle\,\}$

Angular Momenta

3D Space & Rotations

- Can use (spherical) coordinate representation

$$\vec{L}^2 = -\Big[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial}{\partial\theta}\Big) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\Big]$$

$$L_\pm = \pm e^{\pm i\phi}\Big[\frac{\partial}{\partial\theta} \pm i\cot(\theta)\frac{\partial}{\partial\phi}\Big], \qquad L_3 = -i\frac{\partial}{\partial\phi},$$

and define

$$Y_\ell^m(\theta,\phi) := \langle\vec{r}|\ell, m\rangle$$

- Then $|1,\pm 1\rangle \leftrightarrow Y_1^{\pm 1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi},$

$$|1,0\rangle \leftrightarrow \quad Y_1^0(\theta,\phi) = +\sqrt{\frac{3}{4\pi}}\cos\theta,$$

- but also

$$x = r\sin\theta\cos\phi = -r\sqrt{\frac{2\pi}{3}}\Big(Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)\Big) \quad \leftrightarrow \quad |1,+1\rangle + |1,-1\rangle\,,$$

$$y = r\sin\theta\sin\phi = i\,r\sqrt{\frac{2\pi}{3}}\Big(Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)\Big) \quad \leftrightarrow \quad |1,+1\rangle - |1,-1\rangle\,,$$

$$z = r\sin\phi \qquad = r\sqrt{\frac{4\pi}{3}}\,Y_1^0(\theta,\phi) \quad \leftrightarrow \quad |1,0\rangle\,.$$

Angular Momenta

"Addition" of Angular Momenta

- By definition,

$$e^{-i\theta\cdot J}\Psi(\boldsymbol{r},t) = \left\langle \boldsymbol{r} \middle| e^{-i\theta\cdot J} \middle| \Psi(t) \right\rangle = \left\langle \boldsymbol{r} \middle| e^{-i\theta\cdot(L+S)} \middle| \Psi(t) \right\rangle$$

- so L_j (S_j) act in the "real/positional" (Hilbert) space (of states)
- they commute: $[L_j, S_k] = 0$ and $[L_j, L_k] = i\,\varepsilon_{jk}{}^m L_m$ & $[S_j, S_k] = i\,\varepsilon_{jk}{}^m S_m$
- then $J_j := L_j + S_j, \qquad \Rightarrow \qquad [J_j, J_k] = i\varepsilon_{jk}{}^m J_m,$
- and

$$L^2|\ell,m_\ell\rangle = \ell(\ell+1)|\ell,m_\ell\rangle, \qquad L_3|\ell,m_\ell\rangle = m_\ell|\ell,m_\ell\rangle;$$

$$S^2|s,m_s\rangle = s(s+1)|s,m_s\rangle, \qquad S_3|s,m_s\rangle = m_s|s,m_s\rangle;$$

$$J^2|j,m_j\rangle = j(j+1)|j,m_j\rangle, \qquad J_3|j,m_j\rangle = m_j|j,m_j\rangle.$$

- The maximal subset of independent commuting operators:

composite					
composite $\vec{J}^2 \quad J_3$	$\vec{L}^2 \quad \vec{S}^2$	$L_3 \quad S_3$ **product**
$\lvert j,\ell,s;m_j\rangle$		$\lvert\ell,s;m_\ell,m_s\rangle := \lvert\ell,m_\ell\rangle\otimes\lvert s,m_s\rangle$

Angular Momenta

"Addition" of Angular Momenta

- In the product basis, by definition:

$$\begin{aligned} L^2|\ell,s;m_\ell,m_s\rangle &= \ell(\ell+1)|\ell,s;m_\ell,m_s\rangle, \\ L_3|\ell,s;m_\ell,m_s\rangle &= m_\ell|\ell,s;m_\ell,m_s\rangle, \\ S^2|\ell,s;m_\ell,m_s\rangle &= s(s+1)|\ell,s;m_\ell,m_s\rangle, \\ S_3|\ell,s;m_\ell,m_s\rangle &= m_s|\ell,s;m_\ell,m_s\rangle. \end{aligned}$$

$\{L^2, L_3, S^2, S_3\}$

- Also, $J_3|\ell,s;m_\ell,m_s\rangle = (m_\ell+m_s)|\ell,s;m_\ell,m_s\rangle$ ...but $J_3 = L_3 + S_3$, so it is not independent.

- In turn, $[J^2, L_3] = 2i\,\varepsilon^{jk}{}_3 L_j S_k = 2i(L_1 S_2 - L_2 S_1) = -[J^2, S_3]$

- In the composite basis,

$$\begin{aligned} J^2|j,\ell,s;m_j\rangle &= j(j+1)|j,\ell,s;m_j\rangle, \\ J_3|j,\ell,s;m_j\rangle &= m_j|j,\ell,s;m_j\rangle, \\ L^2|j,\ell,s;m_j\rangle &= \ell(\ell+1)|j,\ell,s;m_j\rangle, \\ S^2|j,\ell,s;m_j\rangle &= s(s+1)|j,\ell,s;m_j\rangle. \end{aligned}$$

$\{J^2, L^2, S^2, J_3\}$

Angular Momenta

“Addition” of Angular Momenta

- Since both bases are complete,

$$|\ell,s;m_\ell,m_s\rangle=\sum_{j=|\ell-s|}^{\ell+s}C^{j,m_j}_{\ell,s;m_\ell,m_s}|j,\ell,s;m_j\rangle,$$

$$|j,\ell,s;m_j\rangle=\sum_{\substack{m_\ell=-\ell\\|m_s|=|m_j-m_\ell|\leqslant s}}^{\ell}(C^{j,m_j}_{\ell,s;m_\ell,m_s})^*|\ell,s;m_\ell,m_s\rangle,$$

- where

$$C^{j,m_j}_{\ell,s;m_\ell,m_s}:=\langle j,\ell,s;m_j|\ell,s;m_\ell,m_s\rangle\equiv\langle j,m_j|\ell,s;m_\ell,m_s\rangle$$

- are the Clebsch-Gordan coefficients
- They vanish unless

$$|\ell-s|\leqslant j\leqslant(\ell+s),\qquad|j-\ell|\leqslant s\leqslant(j+\ell),\qquad|j-s|\leqslant\ell\leqslant(j+s),$$

$$m_j=m_\ell+m_s,\qquad|m_j|\leqslant j,\qquad|m_\ell|\leqslant\ell,\qquad|m_s|\leqslant s,$$

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Now, go forth and calculate!!

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