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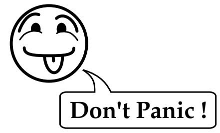


Quantum Mechanics I 8th Oct. '03.

1st Midterm Exam Solutions T. Hübsch

DISCLAIMER: This solution set presents more detail than was required of the Student, and is meant as an additional resource for learning. Please *do* study not just the solutions as presented, but try also to understand the rationale behind the approach.

1. Given the two state vectors, $|u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, and $|u_2\rangle = \begin{bmatrix}0\\1\end{bmatrix}$,

a. prove that they form a complete basis for 2-dimensional vectors.

Solution_____.

For them to form a (complete) basis for 2-dimensional vectors, the *two* $|u_i\rangle$'s must be linearly independent. This we prove by showing that the requirement

$$\sum_{i=1}^{2} c_i\,|u_i\rangle = \frac{c_1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = 0\ , \qquad \Rightarrow \qquad \begin{cases} c_1 \overset{!}{=} 0\ , \\ \frac{c_1}{\sqrt{2}} + c_2 \overset{!}{=} 0\ , \end{cases} \quad .$$

which would hold only if both c_1 and c_2 vanish. [=8pt]

b. Specify the $\langle u_i|$, $i = 1, 2$ and show that this basis it not orthonormal.

Solution_____.

For the $\langle u_i|$ to act on the column-vectors $|u_i\rangle$ and to produce (in general, complex) numbers, they must be hermitian-conjugate row-vectors:

$$\langle u_1| = \tfrac{1}{\sqrt{2}}[1,1]\ , \quad \langle u_2| = [0,1]\ .$$

Non-orthogonality is proven by finding $\langle u_i|u_j\rangle \neq 0$ for any $i \neq j$. In this case:

$$\langle u_1|u_2\rangle = \tfrac{1}{\sqrt{2}}[1,1]\begin{bmatrix}0\\1\end{bmatrix} = \tfrac{1}{\sqrt{2}}(1{\cdot}0 + 1{\cdot}1) = \tfrac{1}{\sqrt{2}} \ \neq\ 0\ .$$

Since $\langle u_i|u_j\rangle \neq 0$ when $i \neq j$, this basis, albeit complete, is not orthogonal. [=7pt]

c. Construct an orthonormal basis $\{|v_1\rangle = |u_1\rangle\ ,\ |v_2\rangle = ?\}$.

Solution_____.

We start with the $|v_1\rangle = |u_1\rangle$, as instructed, write $|v_2\rangle = a\,|u_1\rangle + b\,|u_2\rangle$, and require orthogonality:

$$0 \overset{!}{=} \langle v_1|v_2\rangle = \langle u_1|\,\big[a\,|u_1\rangle + b\,|u_2\rangle\,\big] = a + \frac{b}{\sqrt{2}}\ , \quad \Rightarrow \quad b = -\sqrt{2}a\ ,$$

and so

$$|v_1\rangle = \frac{a}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} - \sqrt{2}a\begin{bmatrix}0\\1\end{bmatrix} = \frac{a}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\ , \quad \text{and} \quad \langle v_2| = \frac{a}{\sqrt{2}}[1,-1]\ .$$

Finally, we normalize: [=15pt]

$$1 \stackrel{!}{=} \langle v_2|v_2\rangle = \frac{|a|^2}{2}[1,-1]\begin{bmatrix}1\\-1\end{bmatrix} = \frac{|a|^2}{2}\cdot 2, \quad \Rightarrow \quad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}.$$

d. Construct the projection operators $\hat{\mathsf{P}}_i = |v_i\rangle\langle v_i|$ and prove that $\sum_{i=1}^{2}\hat{\mathsf{P}}_i = \mathbb{1}$, and that $\hat{\mathsf{P}}_i\hat{\mathsf{P}}_j = \delta_{ij}\hat{\mathsf{P}}_j$, $\forall\, i,j = 1,2$.

Solution

Straightforwardly:

$$\hat{\mathsf{P}}_1 = |v_1\rangle\langle v_1| = \tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}[1,1] = \begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix},$$

and

$$\hat{\mathsf{P}}_2 = |v_2\rangle\langle v_2| = \tfrac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}[1,-1] = \begin{bmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{bmatrix}.$$

And, with these, indeed:

$$\begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix} + \begin{bmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

and, similarly, [=20pt]

$$\begin{bmatrix}1/2 & \pm 1/2\\ \pm 1/2 & 1/2\end{bmatrix}\begin{bmatrix}1/2 & \pm 1/2\\ \pm 1/2 & 1/2\end{bmatrix} = \begin{bmatrix}1/2 & \pm 1/2\\ \pm 1/2 & 1/2\end{bmatrix}, \quad \text{but} \quad \begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

2. An observable of the system in problem 1 is represented by $\hat{F} = \begin{bmatrix}2 & 2\\ 2 & -1\end{bmatrix}$.

a. Determine all possible results of (single attempts of) measuring $\hat{F}$.

Solution

Possible outcomes of single measurements are the eigenvalues of $\hat{F}$. These we obtain by solving the secular equation:

$$0 \stackrel{!}{=} \det[\hat{F} - f\mathbb{1}] = \begin{vmatrix}2-f & 2\\ 2 & -1-f\end{vmatrix} = -(2-f)(1+f) - 4 = (f+2)(f-3),$$

so the possible single measurement results are $f = -2, +3$. [=7pt]

b. Determine all eigenvectors of $\hat{F}$.

Solution

Writing $\hat{F}\,|f\rangle = f\,|f\rangle$ as $[\hat{F} - f\mathbb{1}]\,|f\rangle = 0$, we calculate for each of $f = -2, +3$: [=8pt]

$$0 \stackrel{!}{=} [\hat{F} + 2\mathbb{1}]\,|{-2}\rangle = \begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, \quad \Rightarrow \quad \left.\begin{matrix}4x+2y=0\\ 2x+y=0\end{matrix}\right\} \quad \Rightarrow \quad |{-2}\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}.$$

$$0 \stackrel{!}{=} [\hat{F} - 3\mathbb{1}]\,|{+3}\rangle = \begin{bmatrix}-1 & 2\\ 2 & -4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, \quad \Rightarrow \quad \left.\begin{matrix}-x+2y=0\\ 2x-4y=0\end{matrix}\right\} \quad \Rightarrow \quad |{+3}\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}.$$

c. Calculate the expectation value of $\hat{F}$ in the pure state v_1.

Solution

This *expectation value* equals $\mathrm{Tr}[\hat{\rho}_{u_1}\hat{F}] = \langle u_1|\hat{F}|u_1\rangle$, and we can calculate it directly:

$$\langle\hat{F}\rangle_{u_1} = \mathrm{Tr}\left[\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right] = \mathrm{Tr}\begin{bmatrix} 2 & 1/2 \\ 2 & 1/2 \end{bmatrix} = \frac{5}{2} ,$$

or [=5pt]

$$\langle\hat{F}\rangle_{u_1} = \tfrac{1}{2}[1,1]\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{2}[1,1]\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{5}{2} ,$$

d. Calculate the expectation value of $\hat{F}$ in the impure state $\hat{\rho} = \frac{1}{4}\hat{\mathsf{P}}_1 + \frac{3}{4}\hat{\mathsf{P}}_2$.

Solution

Doing the straightforward matrix algebra:

$$\hat{\rho} = \tfrac{1}{4}\hat{\mathsf{P}}_1 + \tfrac{3}{4}\hat{\mathsf{P}}_2 = \tfrac{1}{4}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \tfrac{3}{4}\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix} ,$$

we have that

$$\langle\hat{F}\rangle_\rho = \mathrm{Tr}\left[\begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\right] = \mathrm{Tr}\begin{bmatrix} 1/2 & 5/4 \\ 1/2 & -1 \end{bmatrix} = -\frac{1}{2} .$$

Alternatively, using the linearity of Tr[ ], we have that

$$\langle\hat{F}\rangle_\rho = \mathrm{Tr}[(\tfrac{1}{4}\hat{\mathsf{P}}_1 + \tfrac{3}{4}\hat{\mathsf{P}}_2)\hat{F}] = \tfrac{1}{4}\,\mathrm{Tr}[\hat{\mathsf{P}}_1\hat{F}] + \tfrac{3}{4}\,\mathrm{Tr}[\hat{\mathsf{P}}_2\hat{F}] = \tfrac{1}{4}\langle v_1|\hat{F}|v_1\rangle + \tfrac{3}{4}\langle v_2|\hat{F}|v_2\rangle ,$$

$$= \tfrac{1}{4}\left(\tfrac{1}{2}[1,1]\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \tfrac{3}{4}\left(\tfrac{1}{2}[1,-1]\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \tfrac{1}{4}\big(\tfrac{5}{2}\big) + \tfrac{3}{4}\big(-\tfrac{3}{2}\big) = -\tfrac{1}{2} .$$

Notice that this is not a pure state, so that there exists no $|\psi\rangle$ such that $\langle\hat{F}\rangle_\rho$ would be calculable as $\langle\psi|\hat{F}|\psi\rangle$ for all hermitian $\hat{F}$'s. [=10pt]

e. Calculate the probability that the measurement of the observable $\hat{F}$ in the system in state pure state v_1 would turn out to be 3.

Solution

As given in the text, and used in the homework solution, $\mathrm{Prob}(\hat{F}{=}f|\hat{\rho}) = \mathrm{Tr}[\hat{\rho}\,\hat{\mathsf{P}}_f]$. Since $\hat{\mathsf{P}}_f \stackrel{\text{def}}{=} |f\rangle\langle f|$, and for the pure state $\hat{\rho}_{v_1} = |v_1\rangle\langle v_1|$, we have that

$$\mathrm{Prob}(\hat{F}{=}f|\hat{\rho}) = \mathrm{Tr}\big[\hat{\rho}\,|f\rangle\langle f|\big] = \langle f|\hat{\rho}|f\rangle , \quad = \mathrm{Tr}\big[\,|v_1\rangle\langle v_1|\,\hat{\mathsf{P}}_f\big] = \langle v_1|\hat{\mathsf{P}}_f|v_1\rangle$$

that is,

$$\mathrm{Prob}(\hat{F}{=}f|\hat{\rho}) = \mathrm{Tr}\big[\,|v_1\rangle\langle v_1|\,|f\rangle\langle f|\,\big] = \langle f|v_1\rangle\langle v_1|f\rangle = \big|\langle v_1|f\rangle\big|^2 .$$

Alternatively (as done in other texts focusing on the wave-functions), we can expand $|v_1\rangle = c_{-2}\,|{-2}\rangle + c_{+3}\,|{+3}\rangle$, use the orthonormalization of the $|f\rangle$ to obtain that $\langle +3|v_1\rangle = c_{+3}$ is the probability amplitude, and deduce that $\text{Prob}(\hat{F}{=}f|\hat{\rho}) = |c_f|^2 = \big|\langle f|v_1\rangle\big|^2$, in agreement with the above, more direct calculation.

For our case at hand, [=10pt]

$$\text{Prob}(\hat{F}{=}3|\hat{\rho}_{v_1}) = \big|\langle v_1|{+}3\rangle\big|^2 = \left|\tfrac{1}{\sqrt{2}}[1,1]\cdot\tfrac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}\right|^2 = \left|\tfrac{3}{\sqrt{10}}\right|^2 = \frac{9}{10}\ .$$

f. Calculate the probability that the measurement of the observable $\hat{F}$ in the system in impure state with $\hat{\rho} = \frac{1}{4}\hat{\mathsf{P}}_1 + \frac{3}{4}\hat{\mathsf{P}}_2$ would turn out to be 3.

Solution____.

Using above results, we have that $\text{Prob}(\hat{F}{=}f|\hat{\rho}) = \langle f|\hat{\rho}|f\rangle$ now is:

$$\text{Prob}(\hat{F}{=}3|\hat{\rho}) = \tfrac{1}{5}[2,1]\begin{bmatrix}{}^1\!/_2 & -{}^1\!/_4\\ -{}^1\!/_4 & {}^1\!/_2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \frac{3}{10}\ .$$

We could have also calculated this using the linearity of Tr[ ]:

$$\begin{aligned}\text{Prob}(\hat{F}{=}f|\hat{\rho}) &= \text{Tr}[(\tfrac{1}{4}\hat{\mathsf{P}}_1 + \tfrac{3}{4}\hat{\mathsf{P}}_2)\hat{\mathsf{P}}_f] = \tfrac{1}{4}\,\text{Tr}[\hat{\mathsf{P}}_1\hat{\mathsf{P}}_f] + \tfrac{3}{4}\,\text{Tr}[\hat{\mathsf{P}}_2\hat{\mathsf{P}}_f]\ ,\\ &= \tfrac{1}{4}\big|\langle v_1|f\rangle\big|^2 + \tfrac{3}{4}\big|\langle v_2|f\rangle\big|^2\end{aligned}$$

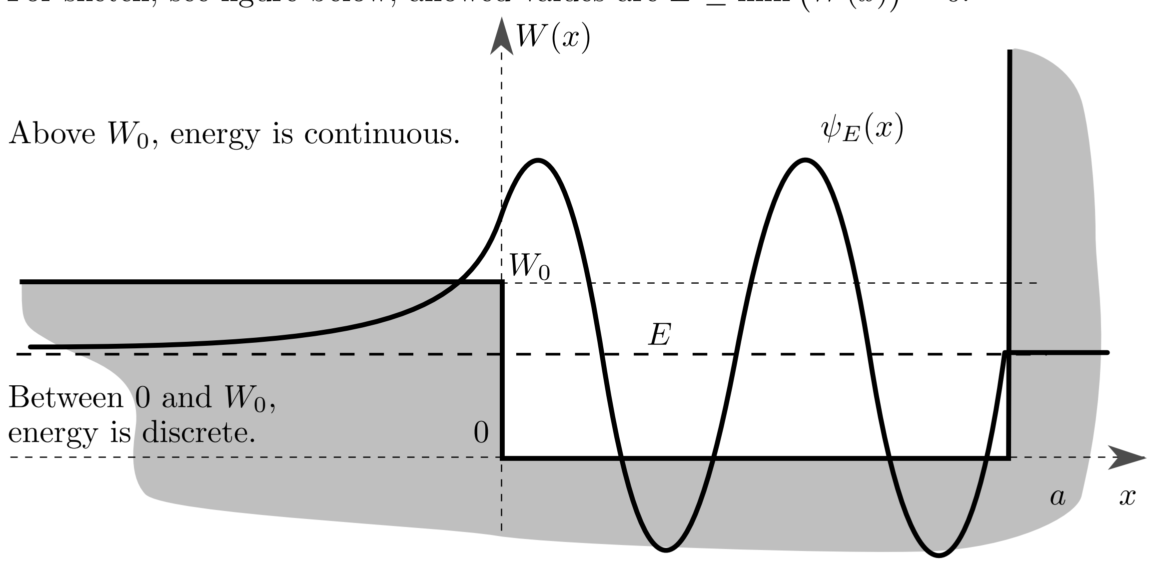
Thus, $\text{Prob}(\hat{F}{=}3|\hat{\rho}) = \frac{1}{4}\big|\langle v_1|{+}3\rangle\big|^2 + \frac{3}{4}\big|\langle v_2|{+}3\rangle\big|^2 = \frac{1}{4}\big|\frac{1\cdot2+1\cdot1}{\sqrt{2\cdot5}}\big|^2 + \frac{3}{4}\big|\frac{1\cdot2-1\cdot1}{\sqrt{2\cdot5}}\big|^2 = \frac{3}{10}$. [=20pt]

3. Consider a particle under the influence of the potential: $W(x) = W_0$ for $x < 0$, $W(x) = 0$ for $0 < x < a$, and $W(x) = +\infty$ for $a < x$, with $W_0, a > 0$.

a. Sketch potential and determine the allowed values of the total energy.

Solution____.

For sketch, see figure below; allowed values are $E \geq \min\big(W(x)\big) = 0$. [=5pt]



b. State/specify all boundary conditions for stationary states with $0 < E < W_0$.

Solution____.

Besides the mathcing condition at $x = 0$:

$$\lim_{\epsilon\to0}\psi(-\epsilon) = \lim_{\epsilon\to0}\psi(+\epsilon)\ ,\quad \text{and}\quad \lim_{\epsilon\to0}\psi'(-\epsilon) = \lim_{\epsilon\to0}\psi'(+\epsilon)\ ,$$

we also have that $\psi(a) = 0$ and $\lim_{x\to-\infty}\psi(x) = 0$. [=10pt]

c. Sketch the wave-function of a stationary state with $0 < E < W_0$.

Solution____.

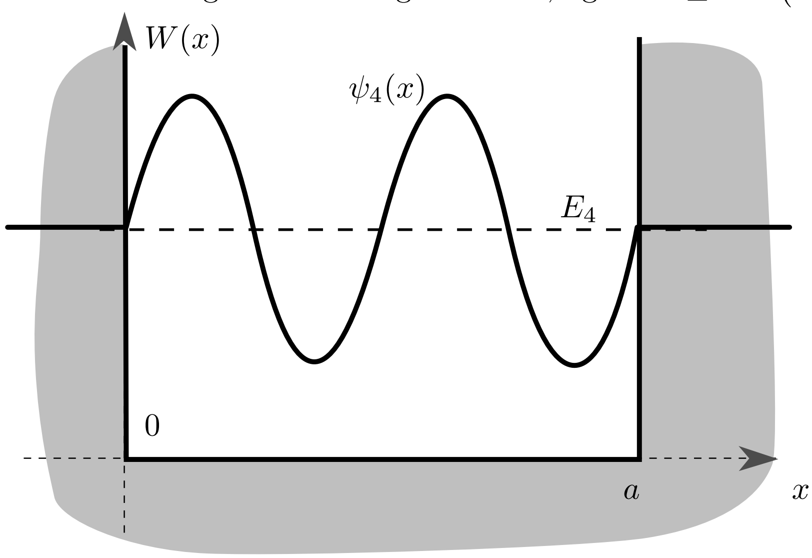
A wave-function is sketched in the figure on the previous page. Note that it satisfies the boundary conditions spelled out in the previous part. Furthermore, it identically vanishes in the region $x{>}a$, where $W(x) = +\infty$. Finally, note that it is oscillatory for $0{<}x{<}a$, where $E > W(x)$, but that it decays (to the left) for $x < 0$ where $E{<}W(x)$. [=10pt]

Now let $W_0 \to \infty$.

d. Sketch potential and determine the allowed values of the total energy.

Solution____.

The sketch is given in the figure below; again $E \geq \min\big(W(x)\big) = 0$. [=5pt]



Note that all $\psi(x) \equiv 0$ for $x < 0$ and $x > a$.

e. State all boundary conditions and sketch the wave-function of a stationary state.

Solution____.

This time, $W(x) = +\infty$ both when $x{<}0$ and when $x{>}a$: there $\psi(x) \equiv 0$. For $0{<}x{<}a$, allowed, positive values of E are always bigger than $W(x)$, and the solution is oscillatory, as sketched in the figure above. Thus, $\psi(0) = 0 = \psi(a)$ are the only boundary conditions.[=10pt]

f. Calculate the wave-number, k, and thus the energy of all stationary states.

Solution____.

Writing the solution as $\psi(x) = A\sin(kx{+}\delta)$, we find that:

$$@\,x{=}0: \quad \psi(0) = A\sin(\delta) \stackrel{!}{=} 0~, \quad \Rightarrow \quad \delta = 0~,$$

and then:

$$@\,x{=}a: \quad \psi(a)\Big|_{\delta=0} = A\sin(ka) \stackrel{!}{=} 0~, \quad \Rightarrow \quad ka = n\pi~, \quad \text{for} \quad n = 1, 2, 3, \ldots$$

so that $k = n\pi/a$. Since $k^2 = \frac{2m}{\hbar^2}E$, we have that

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}~, \quad n = 1, 2, 3, \ldots$$

are all the allowed values of the energy, corresponding to the stationary states with the wave-functions $\psi_n(x) = A\sin\left(\frac{n\pi\hbar}{a}x\right)$. [=10pt]