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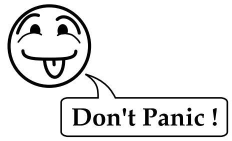

Quantum Mechanics II 4th Dec. '02.

Final Exam

Instructor: T. Hübsch ______________________ (Student name and ID)

This is an "open Textbook (Ballentine), open lecture notes" take-home exam, **due 5 PM of Wednesday, 11th Dec. 2002**. For full credit, show all your work. If you cannot complete one part of a calculation, a clear description of the procedure/method will still earn you partial credit. No collaboration or consultation is allowed, but you may quote (full reference, please) *any published source* for *intermediate* results that you may use.

1. Consider a quantum particle of mass M, moving under the influence of the potential $V(x) = \frac{1}{2}M\omega^2(x^2 - a^2) + \lambda$ for $|x| < a$, and $V(x) = 0$, for $|x| > a$; $a, \lambda > 0$.

a. Specify which energies are permissible, and for each energy (band) *all* the corresponding boundary conditions. [=10pt]

b. Solve the problem exactly*: calculate the exact wave-functions and corresponding energy levels, or at least . [=20pt]

c. For a fixed value of a, estimate the smallest value of λ (call it λ_0) for which there is least one bound state. [=10pt]

d. Describe the Hilbert space of the system when $0 < \lambda_0 < \lambda$ and when $0 < \lambda < \lambda_0$. [=10pt]

e. Assume that λ is slowly changed from $\lambda > \lambda_0 > 0$ to $0 < \lambda < \lambda_0$. Describe what happens to the bound state(s). [=15pt]

* It is convenient to divide the x-space of this problem into three regions, $x \in (-\infty, -a, +a, +\infty)$, so that for $x \in [-a, +a]$ this would reduce to the linear harmonic oscillator. However, note that this x-space domain is *finite*, so the boundary conditions at $|x|{=}a$ do not rule out the "divergent" solution of LHO.

2. Consider a particle of mass M moving freely within an infinitely deep spherical potential well, with $V = 0$ for $r < R$ and $V = \infty$ for $r > R$,

a. Expand the wave-functions in spherical harminics and determine the radial equation; [=10pt]

b. Specify the boundary conditions for the differential equation satisfied by the radial factor, and then solve it. [=10pt]

c. Determine the complete energy spectrum (= list of allowed values). [=15pt]

d. Calculate the degeneracy of *all* stationary states, or at least those with the *three* lowest energies. [=10pt]

e. Explain which part of the degeneracy is caused by the spherical symmetry of the potential. [=10pt]

f. Shift the potential by adding $\hat{H}' = \alpha\sin(\theta)$, where $0 < \alpha = const.$ may be regarded as small, and calculate the correction to the energy of the next-to-lowest lying state(s). Is the degeneracy of this energy level lifted? Fully or only partially? Explain. [=20pt]

3. Consider the 2-dimensional harmonic oscillator, $\hat{H} = \hbar\omega\big[\hat{a}^\dagger{}_1\hat{a}_1 + \hat{a}^\dagger{}_2\hat{a}_2 + 1\big]$, with its Hilbert space of stationary states $\big\{\,|n_1,n_2\rangle = \frac{1}{n_1!n_2!}(\hat{a}^\dagger{}_1)^{n_1}(\hat{a}^\dagger{}_2)^{n_2}\,|0,0\rangle\,,\, n_1,n_2{=}0,1\ldots\big\}$, and the four quadratic operators $\hat{K}_{ij} \stackrel{\text{def}}{=} [(n_i{+}1)n_j]^{-1/2}\hat{a}^\dagger{}_i\hat{a}_j$ $(i,j=1,2)$.

a. List the allowed values of energy, and determine their degeneracy (or the lowest 10 levels, if you cannot determine a general formula). [=10pt]

b. Determine the action of each of the four $\hat{K}_{ij}$ on the stationary states $|n_1,n_2\rangle$. [=10pt]

c. Prove by explicit computation that all $\hat{K}_{ij}$ commute with $\hat{H}$. [=15pt]

d. Prove that all $\hat{K}_{ij}$ commute with $\hat{H}$ by only using their action on the $|n_1,n_2\rangle$. [=15pt]

Write $\hat{L}_0 \stackrel{\text{def}}{=} (\hat{K}_{11} + \hat{K}_{22})$, $\hat{L}_\pm \stackrel{\text{def}}{=} C_\pm(\hat{K}_{12} \pm \beta_\pm\hat{K}_{21})$, and $\hat{L}_3 \stackrel{\text{def}}{=} C_3(\hat{K}_{11} - \hat{K}_{22})$.

e. Specify the constants $C_\pm, \beta_\pm, C_3$ so that the operators $\hat{L}_\pm, \hat{L}_3$ would satisfy the angular momentum commutation relations. [=20pt]

f. What possibly physical meaning can be given to $\hat{L}_0$? [=10pt]