Howard University
Washington, DC 20059

Department of Physics and Astronomy
(202)-806-6245 (Main Office)
(202)-806-5830 (fax)

2355 Sixth St., NW, TKH Rm.215
thubsch@howard.edu
(202)-806-6257

Don't Panic !

Physical Mechanics II
Case Studies

Spring '04.
Handout (T. Hübsch)

1. Small oscillations

The solutions to the examples considered here are organized in a way that outlines the general Lagrangian method.

1.1. Two beads between springs

Consider two beads, with masses m_1, m_2, moving in one dimension and connected by springs: the left wall to the 1st bead and with spring constant k_1, the 1st to the 2nd bead at k_2, and the 2nd bead to the right wall at k_3. Find normal modes and their frequencies.

1. There are two beads, the motion of which we need to determine. For each bead, we'll need it's 1-dimensional displacement about the equilibrium position, and hence one generalized coordinate for each bead: two in all.

2. Now that we have chosen the generalized coordinates, we can write down the kinetic and potential energies:

a. Kinetic energy is the ability of each bead to do work owing to the energy stored in its motion. Furthermore, the kinetic energies of each bead add, so we have:

$$T = T_1 + T_2 = \tfrac{1}{2}m_1\dot{x}_1^{\,2} + \tfrac{1}{2}m_2\dot{x}_2^{\,2} \ .$$

b. Now, the potential energy is the ability to do work owing to the energy stored in the stretching or contraction of each spring. And again, these energies add:

$$V = V_1 + V_2 + V_3 = \tfrac{1}{2}k_1 x_1^{\,2} + \tfrac{1}{2}k_2(x_1 - x_2)^2 + \tfrac{1}{2}k_3 x_2^{\,2} \ .$$

Notice here that the amount of stretching or contracting that befalls the 1st (3rd) spring depends solely on the displacements of the 1st (2nd) bead; however, the amount of stretching or contracting of the 2nd spring depends on *both*: it depends on the *relative* displacement of the 1st and the 2nd bead, $x_1 - x_2$!

3. The Lagrangian for the system now reads:

$$L = \tfrac{1}{2}m_1\dot{x}_1^{\,2} + \tfrac{1}{2}m_2\dot{x}_2^{2} - \tfrac{1}{2}k_1 x_1^{\,2} - \tfrac{1}{2}k_2(x_1 - x_2)^2 - \tfrac{1}{2}k_3 x_2^{\,2} \ ,$$

and we are ready to write out the concrete form of the general equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{x}_i}\Big) = \frac{\partial L}{\partial x_i} \ ,$$

for $i=1,2$ (each of the beads). Straightforwardly, we get:

$$\begin{aligned} m\ddot{x}_1 = -k_1x_1 - k_2(x_1-x_2) &= -(k_1+k_2)x_1 + k_2x_2 \ , \\ m\ddot{x}_2 = -k_2(x_2-x_1) - k_3x_2 &= -(k_2+k_3)x_3 + k_2x_1 \ , \end{aligned} \tag{1.1}$$

We notice that this is a *coupled* system of linear, homogeneous, ordinary differential equations of second order: solving either one would require the solution of the other.

4. Recall the fact that for a single bead of mass m, attached to a single spring of constant k, the displacement oscillates as $x(t) = a\,\cos(\omega t - \delta)$, where a, δ are the *amplitude* and *phase*, respectively, and $\omega = \sqrt{k/m}$ is the frequency of oscillation. Now, it should be clear that there will exist ways for the two beads to oscillate in a coherent way, with the same frequency and phase. This leads us to look for solutions in the form

$$x_i = a_i\,\cos(\omega t - \delta) \ , \qquad i = 1,2 \ ; \tag{1.2}$$

note that $\ddot{x}_i(t) = -\omega^2 a_i \cos(\omega t - \delta) = -\omega^2 x_i(t)$. By inserting this into Eq. (1.1), we obtain:

$$\begin{aligned} -m_1x_1\omega^2 &= -(k_1+k_2)x_1 + k_2x_2 \ , \\ -m_2x_2\omega^2 &= -(k_2+k_3)x_3 + k_2x_1 \ . \end{aligned} \tag{1.3}$$

Since x_1, x_2 both have the $\cos(\omega t - \delta)$ factor in common, we can cancel them out and obtain the system of equations

$$\begin{aligned} 0 &= m_1a_1\omega^2 - (k_1+k_2)a_1 + k_2a_2 \ , \\ 0 &= m_2a_2\omega^2 - (k_2+k_3)a_3 + k_2a_1 \ , \end{aligned} \tag{1.4}$$

which may be written, in matrix notation, as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_1\omega^2 - (k_1{+}k_2) & k_2 \\ k_2 & m_2\omega^2 - (k_2{+}k_3) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \ . \tag{1.5}$$

For this homogeneous system of equations to have nontrivial solutions in a_1, a_2, the determinant of the system must vanish:

$$0 \ = \det \begin{bmatrix} m_1\omega^2 - (k_1{+}k_2) & k_2 \\ k_2 & m_2\omega^2 - (k_2{+}k_3) \end{bmatrix} \ , \tag{1.6a}$$

$$= [m_1\omega^2 - (k_1{+}k_2)][m_2\omega^2 - (k_2{+}k_3)] - k_2^2 \ ,$$

$$= m_1m_2\omega^4 - [m_1(k_2{+}k_3) + m_2(k_1{+}k_2)]\omega^2 + (k_1k_2{+}k_2k_3{+}k_1k_3) \ . \tag{1.6b}$$

5. This is a bi-quadratic equation in ω (*i.e.*, it is quadratic in ω). This implies that half of the solutions will be simply the negative of the other half. Owing to the form of the solutions (1.2), however, ω and $-\omega$ represent the same type of motion, and we expect two (physically) distinct solutions, with:

$$\omega_{\pm}^2 = \frac{A{+}B}{2m_1m_2} \pm \sqrt{\frac{(A{+}B)^2 - 4m_1m_2C}{4m_1^2m_2^2}} \ , \tag{1.7a}$$

where

$$A = m_1(k_2{+}k_3)\ , \quad B = m_2(k_1{+}k_2)\ , \quad \text{and} \quad C = (k_1k_2{+}k_2k_3{+}k_1k_3)\ . \tag{1.7b}$$

being the two distinct solutions for ω^2.

There are four increasingly more and more special (simpler) cases:

a. When $k_1 = k_3 = k$, in which case

$$\omega_\pm{}^2 = \frac{(m_1{+}m_2)(k{+}k_2)}{2m_1m_2} \pm \sqrt{\frac{(m_1{+}m_2)^2(k{+}k_2)^2 - 4m_1m_2k(k{+}2k_2)}{4m_1^2m_2^2}}\ , \tag{1.8}$$

b. When $m_1 = m_2 = m$, in which case

$$\omega_\pm{}^2 = \frac{k_1{+}2k_2{+}k_3}{2m} \pm \sqrt{\frac{k_1^2{+}4k_2^2 - 2k_1k_3{+}k_3^2}{4m^2}}\ , \tag{1.9}$$

c. When $k_1 = k_3 = k$ and $m_1 = m_2 = m$, in which case

$$\omega_\pm{}^2 = \frac{k{+}k_2}{m} \pm \frac{k_2}{m}\ , \quad \textit{i.e.}, \quad \begin{cases} \omega_+^2 = \frac{k+2k_2}{m}\ , \\ \omega_-^2 = \frac{k}{m}\ . \end{cases} \tag{1.10}$$

d. When $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, in which case

$$\omega_\pm{}^2 = \frac{3k}{m} \pm \frac{k}{m}\ , \quad \textit{i.e.}, \quad \omega_+^2 = \frac{3k}{m}\ , \quad \text{and} \quad \omega_-^2 = \frac{k}{m}\ . \tag{1.11}$$

6. Substitution of one or the other solution, ω_+^2, ω_-^2 (1.7), into the matrix equation (1.5), permits solving for a_1, a_2. However, as the determinant of the system now vanishes (1.6), the system (1.5) will be underdetermined, so that we can only solve for one of the two amplitudes, a_1, a_2, in terms of the other:

$$a_{2\pm} = \frac{a_{1\pm}}{2k_2m_1}\Big(A - B \mp \sqrt{(A+B)^2 - 4m_1m_2C}\Big)\ . \tag{1.12}$$

These are the (relative) amplitudes of the displacements of the 1st and 2nd bead, respectively, for the "+" and the "−" *normal mode.*

The diligent Reader will have no difficulty obtaining the corresponding results for the four above special cases; in particular, both cases (*c.*) and (*d.*) yield $a_{2\pm} = \mp a_{1\pm}$.[1]

7. The normal modes can be used to provide a complete expression for both generalized coordinates:

$$\begin{aligned} q_1(t) &= a_{1+}\cos(\omega_+t - \delta_+) + a_{1-}\cos(\omega_-t - \delta_-)\ , \\ q_2(t) &= a_{2+}\cos(\omega_+t - \delta_+) + a_{2-}\cos(\omega_-t - \delta_-)\ , \end{aligned} \tag{1.13}$$

which are clearly given as linear combinations (*i.e.*, superpositions) of (or, expansions *over*) the (two in this simple case) normal modes, and this clearly generalizes for n generalized coordinates and n normal modes. The constants $\delta_\pm$ and $a_{i\pm}$ may be determined from an initial (final, intermediate…) condition, where the instantaneous values of $q_i(t)$ and the phases are specified at some particular time.