1. The differential equation (a body falling under the influence of gravity and resisted by air drag)

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = mg - bv(t)\ , \quad \text{or} \quad \frac{\mathrm{d}v(t)}{\mathrm{d}t} = g - \frac{b}{m}v(t)\ , \tag{1}$$

may easily be solved by the 'trial-and-adjust' method. First, we note that the equation is linear in $v(t)$. This allows us to solve the equation by means of a superposition (addition) of the solutions of the two simpler equations:

$$\frac{\mathrm{d}v_1(t)}{\mathrm{d}t} = g\ , \qquad \text{and} \qquad \frac{\mathrm{d}v_2(t)}{\mathrm{d}t} = -\frac{b}{m}v(t)\ . \tag{2\,a,b}$$

The first of these, Eq. (2a), is solved by $v_1(t) = v_{10} + gt$ (as seen before), while Eq. (2b) is solved by remembering that a derivative of the exponential function is proportional to the exponential function itself, so that $v_2(t) = v_{20}e^{-bt/m}$. This indeed satisfies Eq. (2b), as is easy to verify by substituting.

Then, we look for the general solution to the original equation (1) in the form

$$v(t) = A + Bt + Ce^{Dt}\ . \tag{3}$$

From this, we calculate the derivative to be

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = B + CDe^{Dt}\ . \tag{4}$$

Substituting (4) in the left-hand-side and (3) in the right-hand-side of Eq. (1), we obtain:

$$B + CDe^{Dt} = g - \frac{b}{m}\big(A + Bt + Ce^{Dt}\big) = \big(g - \frac{b}{m}A\big) - \frac{b}{m}Bt - \frac{b}{m}Ce^{Dt}\ .$$

Equating the coefficients of $t^0 = 1 = \textit{const.}$, $t^1 = t$ and e^{Dt}, we have that

$$B = g - \frac{b}{m}A\ , \qquad 0 = -\frac{b}{m}B\ , \qquad \text{and} \qquad CD = -\frac{b}{m}C\ . \tag{5\,a,b,c}$$

Of these, Eq. (5b) forces $B = 0$, after which Eq. (5a) sets $A = \frac{mg}{b}$, while Eq. (5c) sets $D = -\frac{b}{m}$. This then produces

$$v(t) = \frac{mg}{b} + Ce^{-bt/m}\ ,$$

which indeed has one integration constant: C, as should be the case for a *first* order differential equation. To determine this last constant, we can use the *initial* condition, noticing that

$$v(0) = \frac{mg}{b} + C \stackrel{\text{def}}{=} v_0\ ,$$

is the initial (at $t = 0$) speed and so can write $C = v_0 - \frac{mg}{b}$, whereupon

$$v(t) = \frac{mg}{b} + \big(v_0 - \frac{mg}{b}\big)e^{-bt/m} = \frac{mg}{b}\big(1 - e^{-bt/m}\big) + v_0\,e^{-bt/m}\ .$$

Finally, we know that the body has reached the terminal speed, v_T, if it no longer accelerates, *i.e.*, when the gravitational force is cancelled by the air drag, $mg = bv_T$, so that $v_T = \frac{mg}{b}$. Indeed, when $t \to +\infty$, then $e^{-bt/m} \to 0$, and so $v(+\infty) = \frac{mg}{b} = v_T$. This provides an auxiliary check for our general solution. Notice that the value of the terminal speed, $v_T = \frac{mg}{b}$, is independent of the initial speed, v_0. Also, note that $v(t)$ does not in fact reach v_T at any *finite* time, but merely approaches it asymptotically.