# Non-Convex Mirror Models of Ricci-Flat Spaces

**Tristan Hübsch** w/ Per Berglund

*@ Conference on Elementary Particle Physics, Astrophysics, and Cosmology*
*Department of Physics, University of Miami, Coral Gables, Florida; 2021.12.19*

*Departments of Physics & Astronomy and Mathematics, Howard University, Washington DC*
*Department of Physics, Faculty of Natural Sciences, Novi Sad University, Serbia*
*Department of Mathematics, University of Maryland, College Part, MD*

# Non-Convex Mirror-Models

Prehistoric Prelude

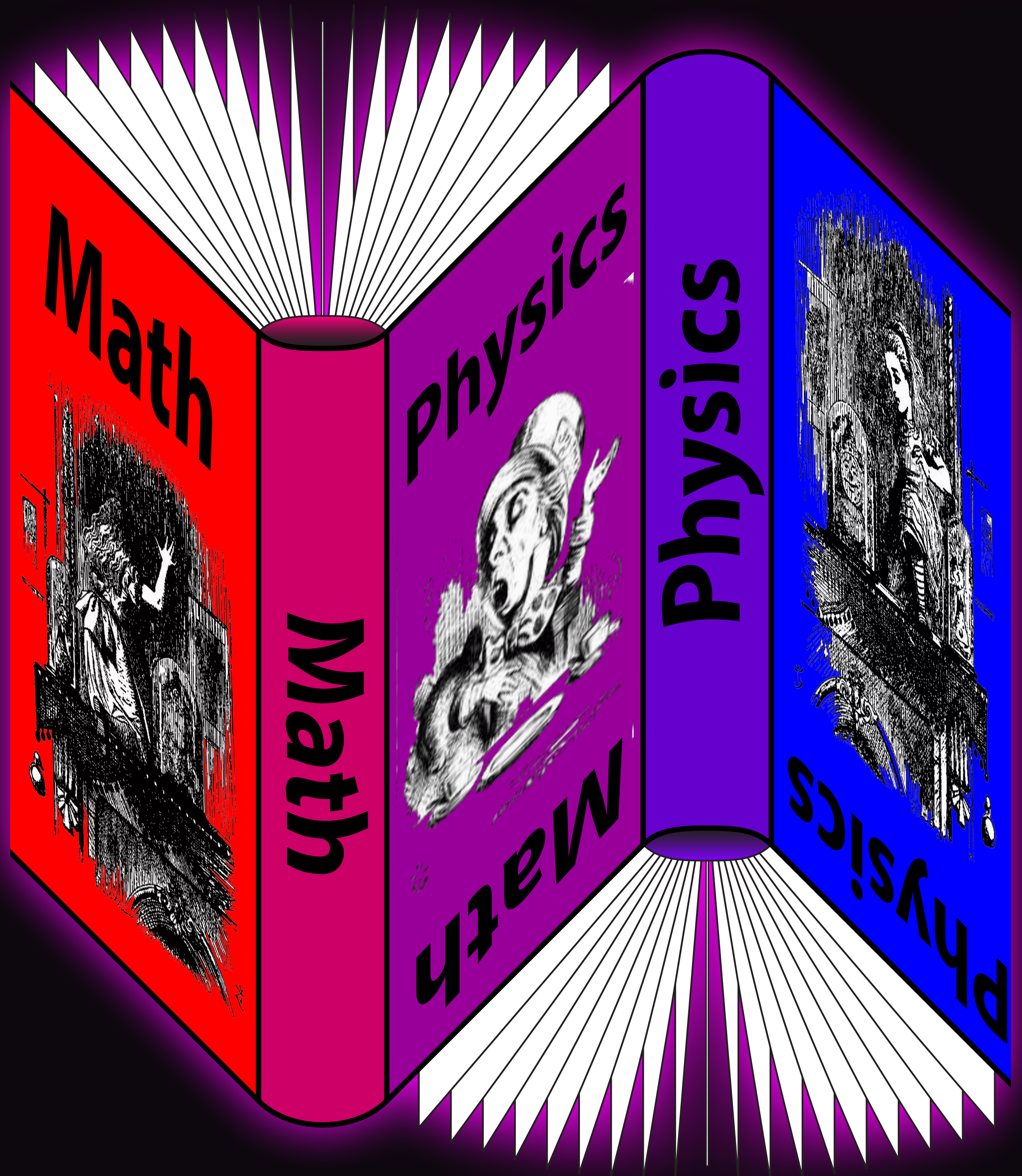
Meromorphic Minuet

*Laurent-Toric Fugue**

Discriminant Divertimento

Mirror Motets

* "It doesn't matter what it's called,
...if it has substance."
S.-T. Yau

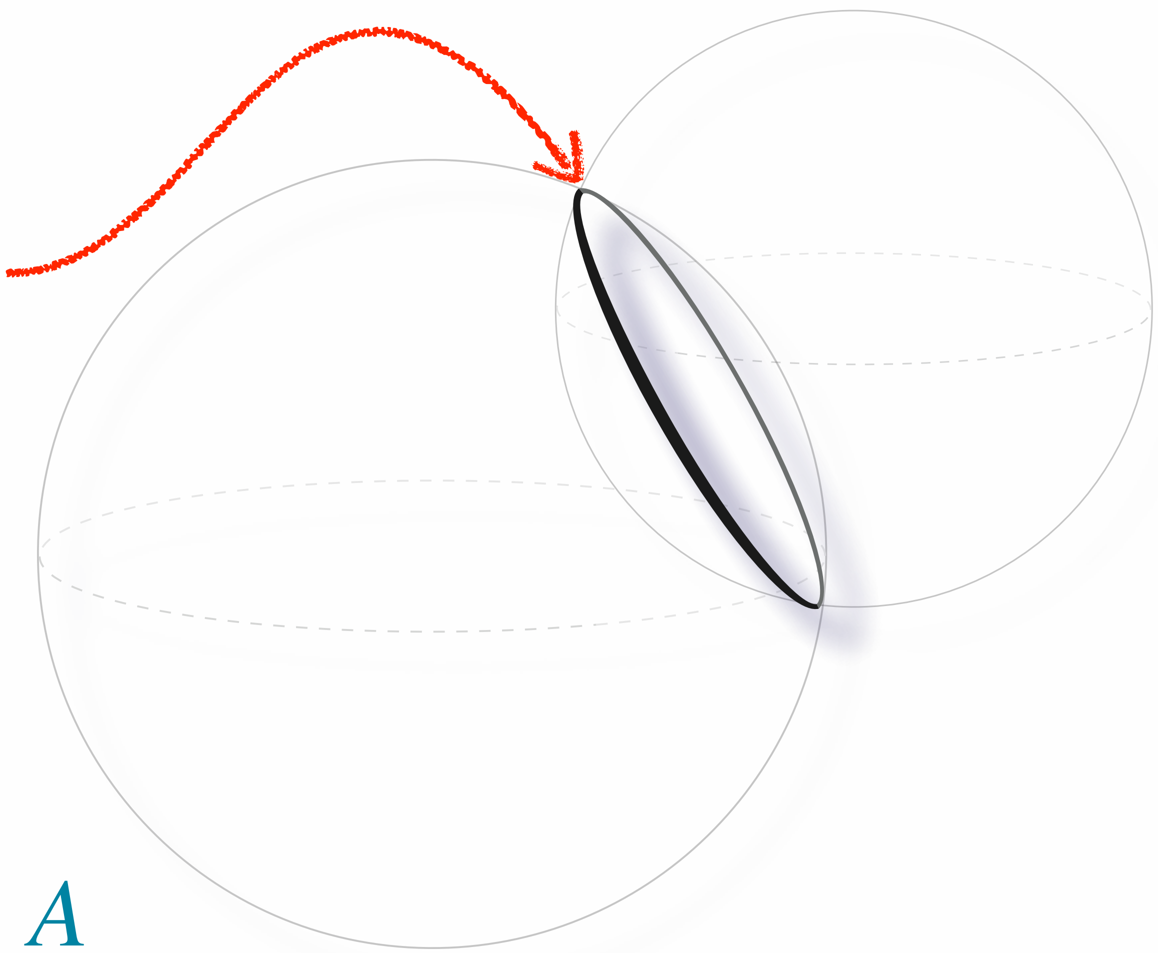


# Pre-Historic Prelude
## (Where are We Coming From?)

# Pre-Historic Prelude


## *Classical Constructions*

- Complete Intersections
  - Ex. $\left.\begin{aligned}(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 &= R_1^{\,2}\\ (x-x_2)^2+(y-y_2)^2+(z-z_2)^2 &= R_2^{\,2}\end{aligned}\right\}$
    - Algebraic (constraint) equations
    - …in a well-understood "ambient" ($A$)



- Work over complex numbers
  - …& incl. "infinity" (e.g., $\mathbb{CP}^n$'s)
- For *hypersurfaces*: $X=\{p(x)=0\}\subset A$
  - "Functions": $[f(x)]_X=[f(x)\simeq f(x)+\lambda\cdot p(x)]_A$
  - Differentials: $[\mathrm{d}x]_X=[\mathrm{d}x\simeq \mathrm{d}x+\tilde{\lambda}\cdot \mathrm{d}p(x)]_A$
  - Homogeneity: $\mathbb{CP}^n=U(n+1)\big/[U(n)\times U(1)]$
    - $i$'th cohomology on $\mathbb{CP}^n\to U(n+1)$-tensors

*Just like gauge transformations*

*D, BV, BFV constraints in the nLSM ⇝ GLSM*

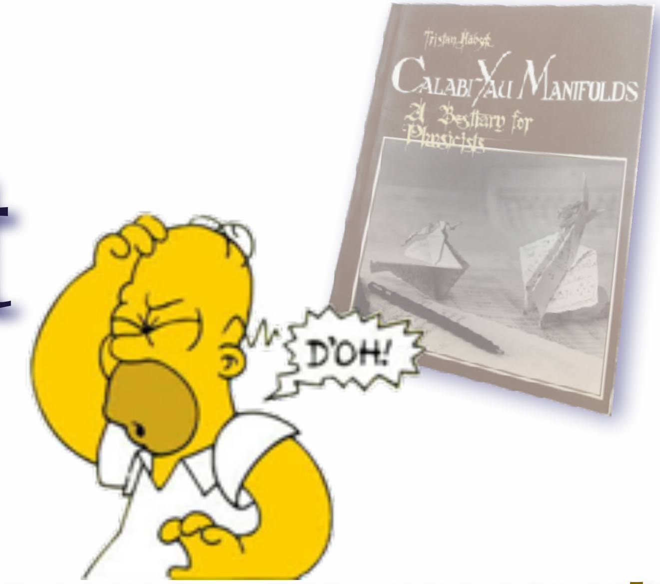
*…with $U(n+1)$ tensors*

# Pre-Historic Prelude


## *Classical Constructions*

- E.g: $X_m \in \boxed{\begin{bmatrix} \mathbb{P}^4 \\ \mathbb{P}^1 \end{bmatrix}\!\!\begin{Vmatrix} 1 \\ m \end{Vmatrix}}\!\begin{matrix} 4 \\ 2-m \end{matrix}\Bigg]^{(2,86)}_{-168}$ $\quad 2 = h^{1,1}$ dim. space of Kähler classes; $86 = h^{2,1}$ dim. space of complex structures; $-168 = \chi = 2(h^{1,1} - h^{2,1})$ the Euler #
  - Zero-set of $p(x,y) = 0$, $\deg[p] = \binom{1}{m}$, & $q(x,y) = 0$, $\deg[q] = \binom{4}{2-m}$
  - Generic $\{p=0\} \cap \{q=0\}$ smooth; $\deg_{\mathbb{P}^n}[p] + \deg_{\mathbb{P}^n}[q] = n+1 \Rightarrow \boxed{R_{\mu\nu} = 0}$
- Sequentially: $X_m \xrightarrow{q=0} \big(F_m \xrightarrow{p=0} \mathbb{P}^4 \times \mathbb{P}^1\big)$ $\quad q(x,y) \sim \dfrac{q_0(x)}{y_0} + \dfrac{q_1(x)}{y_1}$
  - Chern: $c = \dfrac{(1+J_1)^5(1+J_2)^2}{(1+J_1+mJ_2)(1+4J_1+(2-m)J_2)} = 1 + [6J_1^2 + (8-3m)J_1J_2] - [20J_1^3 - (32+15mJ_1^2J_2)]$.
  - Wall: $\underbrace{\kappa_{111} = 2+3m,\ \kappa_{112} = 4}_{\infty\text{-vol. Yukawa}}$, so $(aJ_1 + bJ_2)^3 = [2a + 3(\underline{4b+ma})]a^2$.
  - $p_1[aJ_1 + bJ_2] = -88a - 12(\underline{4b+ma})$… the same "$4b+ma$"
  - So, $F_m \approx F_{m\,(\mathrm{mod}\,4)}$ & $X_m \approx X_{m\,(\mathrm{mod}\,4)}$: 4 <u>diffeomorphism types</u>
- …but, $m = 0, 1, 2, \cdots 3 \rightarrow \deg[q] = \binom{4}{-1}$ ?!

# Meromorphic Minuet




## *Why Haven't We Thought of This Before?*

[AAGGL:1507.03235 + BH:1606.07420]
[+ GvG:1708.00517]

- $\deg[q] = \binom{4}{-1}$ $\mathbb{C}$-*analytic* sections?!
  - Not *everywhere* on $\mathbb{P}^4 \times \mathbb{P}^1$ — (simple poles)
  - but yes on $F_3$ — in fact, 105 of 'em!

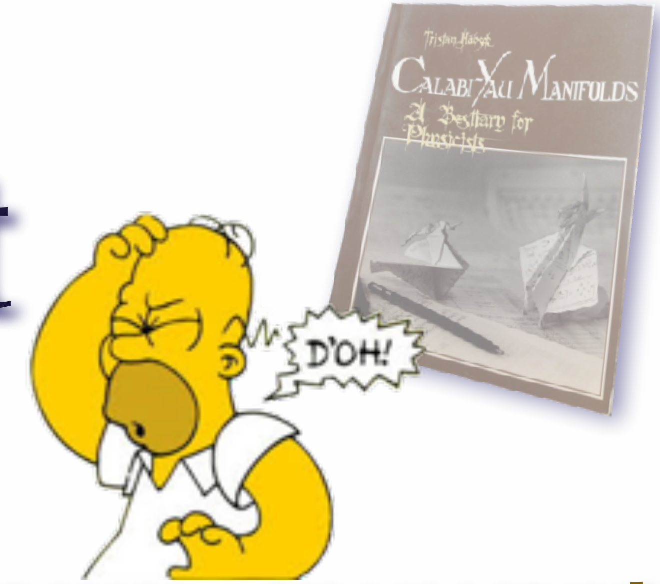
$$X_m \in \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}$$

for $m = 3$

So, $\{q(x,y) = \infty\}$ can avoid $\{p(x,y) = 0\}$?!

How can two codimension-1 subspaces possibly avoid each other ??

# Meromorphic Minuet




## *Why Haven't We Thought of This Before?*

- $\deg[q] = \binom{4}{-1}$ $\mathbb{C}$-*analytic* sections?!

[AAGGL:1507.03235 + BH:1606.07420]
[+ GvG:1708.00517]

$$X_m \in \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}$$

for $m = 3$

  - Not *everywhere* on $\mathbb{P}^4 \times \mathbb{P}^1$ — (simple poles)
  - but yes on $F_3$ — in fact, 105 of 'em!
- How so? On $F_3$, $q(x,y) \simeq q(x,y) + \lambda \cdot p(x,y)$ ← equivalence class!
  - ~[Hirzebruch, 1951]: $p = x_0 y_0{}^3 + x_1 y_1{}^3$ & $q = c(x)\Big(\frac{x_0 y_0}{y_1{}^2} - \frac{x_1 y_1}{y_0{}^2}\Big)$ $\deg[c] = \binom{3}{0}$
  - So, $q_0 = q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y) \overset{\lambda\to-1}{=\!=\!=} c(x)\Big(-2\frac{x_1 y_1}{y_0{}^2}\Big)$ where $y_0 \neq 0$
  - & $q_1 = q(x,y) + \frac{\lambda\, c(x)}{(y_0 y_1)^2} p(x,y) \overset{\lambda\to1}{=\!=\!=} c(x)\Big(2\frac{x_0 y_0}{y_1{}^2}\Big)$ where $y_1 \neq 0$
  - & $q_1(x,y) - q_0(x,y) = 2\frac{c(x)}{(y_0 y_1)^2}\, p(x,y) = 0$, on $F_3 := \{p(x,y) = 0\}$
  - *Just as the Wu-Yang monopole avoids the "Dirac string"…*
  - *… → D, BV, BFV etc. treatment of constraints (in the nLSM → GLSM)*

# Meromorphic Minuet


## …in well-tempered counterpoint

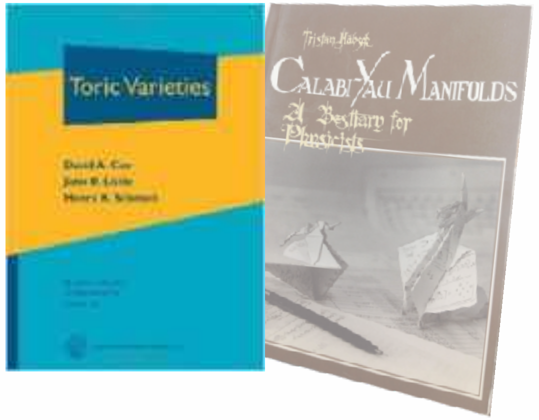
[AAGGL:1507.03235 + BH:1606.07420]

- For $\Big\{\underbrace{x_0\,y_0{}^m + x_1\,y_1{}^m}_{:=\,\mathring{p}(x,y)} = -\sum_\alpha \cancel{\epsilon_\alpha}^{\,0}\,\delta p_\alpha(x,y)\Big\} = F^{(n)}_{m;\cancel{\epsilon}} \in \left[\begin{array}{c\|c} \mathbb{P}^n & 1 \\ \mathbb{P}^1 & m \end{array}\right]$
  - The central ( $\epsilon = 0$ ) member of the family is the "true" $F_m$:
  - Directrix: $S := \{\mathfrak{s}(x,y) = 0\}$, $[S] = [H_1] - m[H_2]$ & $[S]^n = -(n-1)m$;
    - where $\mathfrak{s}(x,y) := \Big(\frac{x_0}{y_1{}^m} - \frac{x_1}{y_0{}^m}\Big) + \frac{\lambda}{(y_0 y_1)^m}[x_0 y_0{}^m + x_1 y_1{}^m]$ degree $\binom{1}{-m}$
  - & $h^0(K^*) = 3\binom{2n-1}{n} + \delta_{\epsilon,0}\,\vartheta_3^m\binom{2n-2}{2}(m-3)$, $h^0(T) = n^2 + 2 + \delta_{\epsilon,0}\,\vartheta_1^m(n-1)(m-1)$
  - All these "extras" *reduce* (decompose) for $(\epsilon_\alpha \neq 0)$ deformations resulting in *discrete deformations* $F^{(n)}_m \to F^{(n)}_{m_1,m_2,\cdots}, \cdots, F^{(n)}_{[m\,(\mathrm{mod}\,n)]}$
- Also, explicit tensorial (residue) representatives
  → can compute coupling ratios

"Linear algebra"

*straightforward & (extremely) tedious*

# Meromorphic Minuet



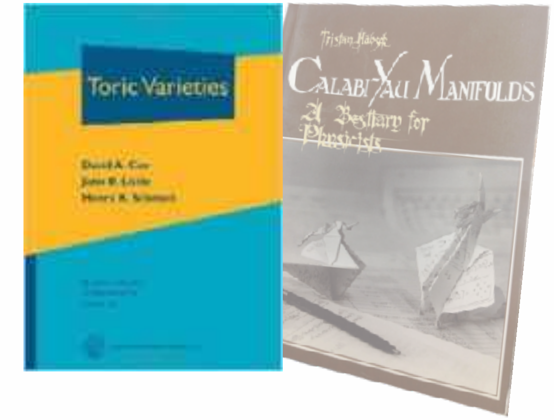

## …in well-tempered counterpoint

- So, on $F_m^{(n)}$: $p(x,y)=0 \;\Rightarrow\; x_0=-x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2)=(x_2,\cdots,x_n;y_0,y_1)$
  - $\mathbb{P}^4\times\mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:
  - Add Lagrange multiplier, $X_0\, f(X)$
  - Need $[f(X)]=\binom{4}{2-m}$, with $\deg[X_1X_{5,6}^m]=\binom{1}{0}=\deg[X_{2,3,4}]$

| $X_0$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|---|
| $-4$ | $1$ | $1$ | $1$ | $1$ | $0$ | $0$ | $\leftarrow\mathbb{P}^4$ |
| $m-2$ | $-m$ | $0$ | $0$ | $0$ | $1$ | $1$ | $\leftarrow\mathbb{P}^1$ |

- $f(X)=X_1^{\,4}X_{5,6}^{2+3m}\oplus X_1^{\,3}X_{2,3,4}X_{5,6}^{2+2m}\cdots\oplus X_1X_{2,3,4}X_{5,6}^2\oplus X_{2,3,4}X_{5,6}^{\boxed{2-m}}$
  - $m>2$,

# Meromorphic Minuet



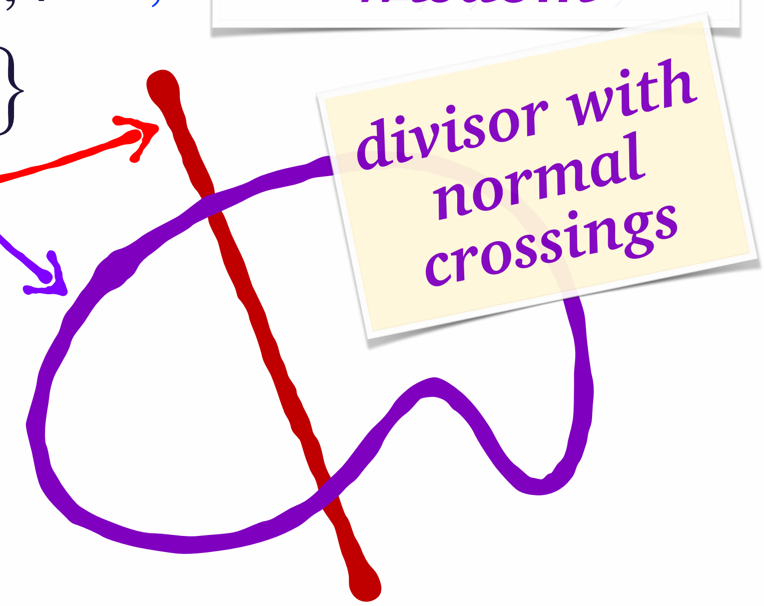

## …in well-tempered counterpoint

- So, on $F_m^{(n)}$: $p(x,y)=0 \;\Rightarrow\; x_0 = -x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:
  - Add Lagrange multiplier, $X_0\, f(X)$
  - Need $[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

| $X_0$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|---|
| $-4$ | $1$ | $1$ | $1$ | $1$ | $0$ | $0$ | $\leftarrow\mathbb{P}^4$ |
| $m-2$ | $-m$ | $0$ | $0$ | $0$ | $1$ | $1$ | $\leftarrow\mathbb{P}^1$ |

- $f(X) = X_1^{\,4} X_{5,6}^{2+3m} \oplus X_1^{\,3} X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4} X_{5,6}^2$

  *standard wisdom*
  - $m>2$, $\{f(X)=0\} = \{X_1=0\} \cup \{\oplus_k X_1^{\,k} X_{2,3,4}^2 X_{5,6}^{2+km} = 0\}$



# Meromorphic Minuet


## …in well-tempered counterpoint

- So, on $F_m^{(n)}$: $p(x,y)=0 \;\Rightarrow\; x_0 = -x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n{+}2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4\times\mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:
  - Add Lagrange multiplier, $X_0\, f(X)$
  - Need $[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

| $X_0$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|---|
| $-4$ | $1$ | $1$ | $1$ | $1$ | $0$ | $0$ | $\leftarrow\mathbb{P}^4$ |
| $m-2$ | $-m$ | $0$ | $0$ | $0$ | $1$ | $1$ | $\leftarrow\mathbb{P}^1$ |

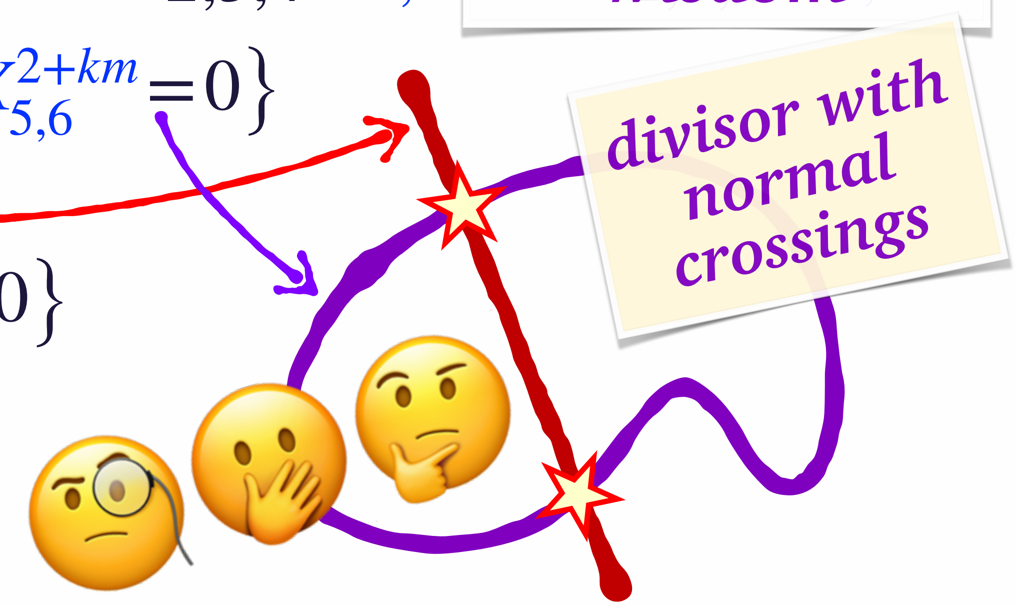
- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4} X_{5,6}^2$

  *standard wisdom*

  - $m>2$, $\{f(X)=0\} = \{X_1=0\} \cup \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km}=0\}$
  - $\{f(X)=0\}^\sharp = \{X_1=0\} \cap \{\oplus_k X_1{}^k X_{2,3,4}^2 X_{5,6}^{2+km}=0\}$

    also $R_{\mu\nu}=0$ (codim-2, K3 in $\mathbb{P}^4\times\mathbb{P}^1$)

$$\left[\begin{array}{c||cc|c} \mathbb{P}^n & 1 & n{-}1 & 1 \\ \mathbb{P}^1 & m & 2 & -m \end{array}\right] = \left[\begin{array}{c||c|cc} \mathbb{P}^n & 1 & 1 & n{-}1 \\ \mathbb{P}^1 & m & -m & 2 \end{array}\right] \overset{\simeq}{\Rightarrow} \left[\begin{array}{c||c} \mathbb{P}^{n-2} & n{-}1 \\ \mathbb{P}^1 & 2 \end{array}\right]$$





# Laurent-Toric Fugue

## (a *not-so-new* Toric Geometry)

A Generalized Construction of
Calabi-Yau Mirror Models
arXiv:1611.10300
+ any day now…

# Meromorphic Minuet

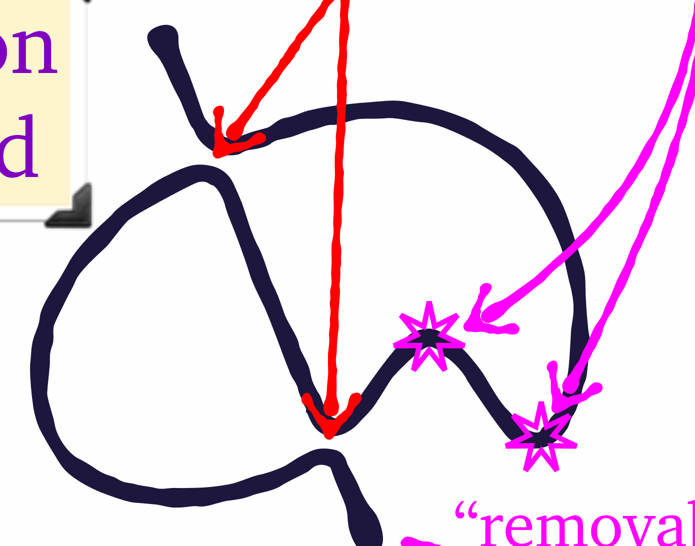

## …in well-tempered counterpoint

- So, on $F_m^{(n)}$: $p(x,y)=0 \;\Rightarrow\; x_0 = -x_1(y_1/y_0)^m$ & $x_1 \to X_1 = \mathfrak{s}$
  - & $(X_i,\ i=2,\cdots,n+2) = (x_2,\cdots,x_n; y_0, y_1)$
  - $\mathbb{P}^4\times\mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:
  - Add Lagrange multiplier, $X_0\, f(X)$
  - Need $[f(X)] = \binom{4}{2-m}$, with $\deg[X_1 X_{5,6}^m] = \binom{1}{0} = \deg[X_{2,3,4}]$

| $X_0$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|---|
| $-4$ | 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow\mathbb{P}^4$ |
| $m-2$ | $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow\mathbb{P}^1$ |

- $f(X) = X_1{}^4 X_{5,6}^{2+3m} \oplus X_1{}^3 X_{2,3,4} X_{5,6}^{2+2m} \cdots \oplus X_1 X_{2,3,4} X_{5,6}^2 \boxed{\oplus X_{2,3,4} X_{5,6}^{\boxed{2-m}}}$
  - $\boxed{m>2,}$ $\{f(X)=0\}$
  - ~~$\{f(X)=0\}^\sharp$ … $\{$~~
  - Embrace Laurent terms
  - "Intrinsic limit" (L'Hôpital-"repaired") smooth (*pre?complex*) <u>*spaces*</u>

SQFT/GLSM as an expansion about "classical" background



"removable" singularity

# Laurent-Toric Fugue

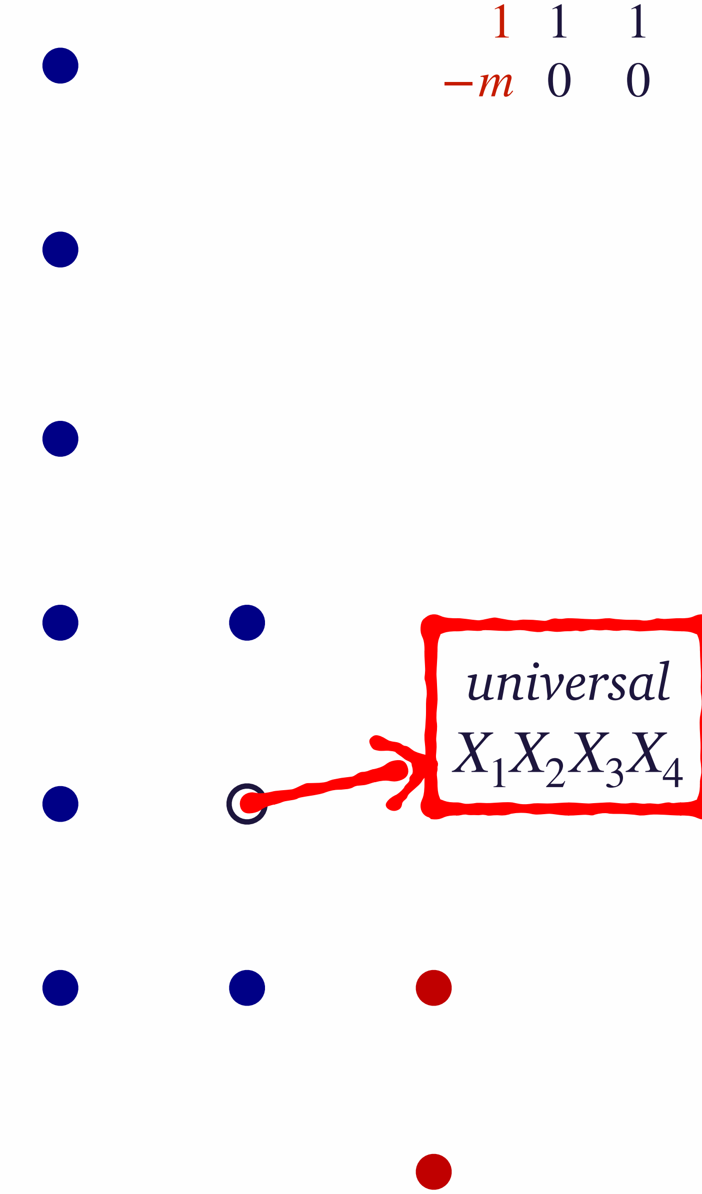

arXiv:1611.10300

## & Non-Convex Mirrors

—2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow \mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow \mathbb{P}^1$ |



# Laurent-Toric Fugue

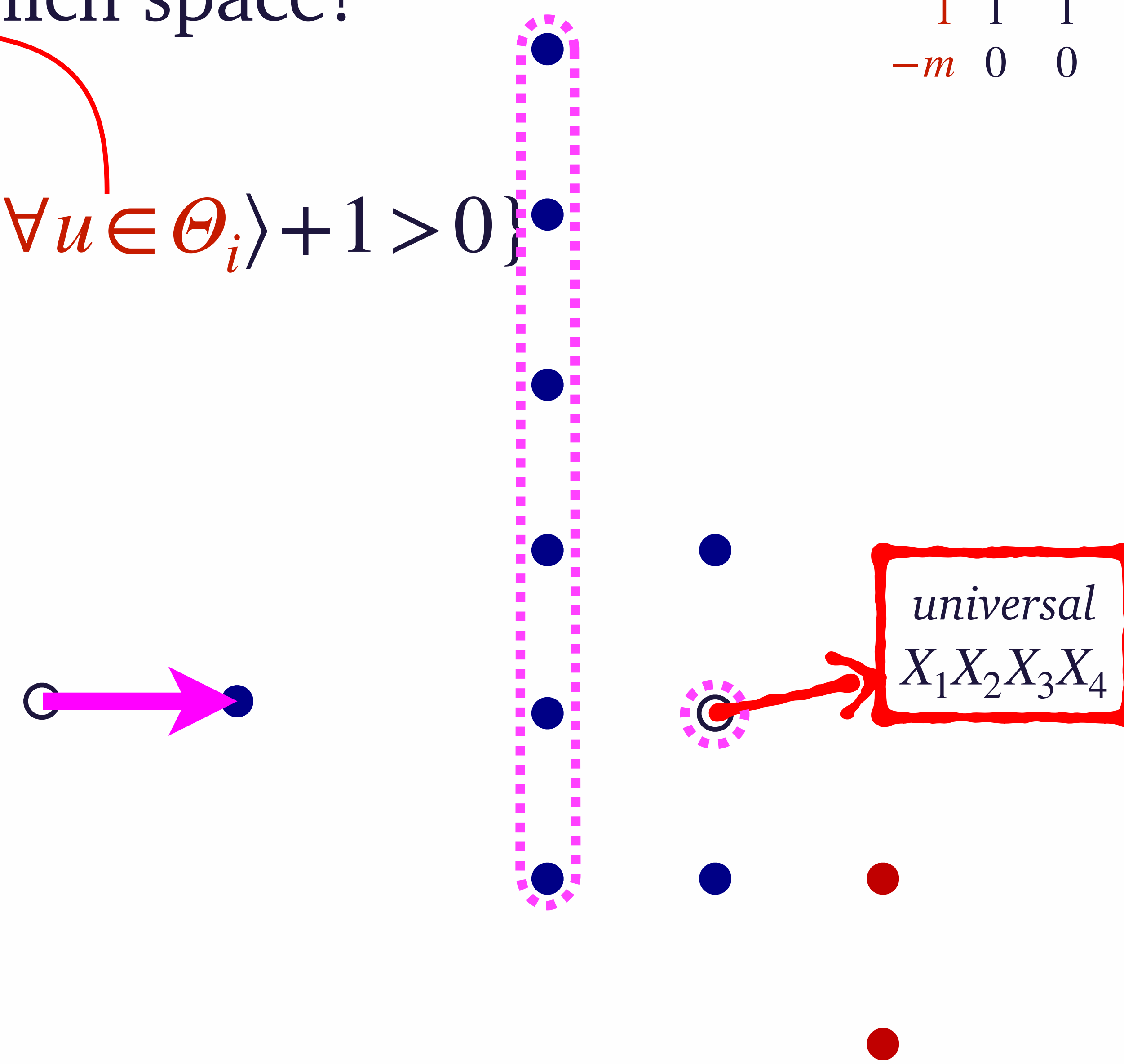

## & Non-Convex Mirrors

—2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow \mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow \mathbb{P}^1$ |

- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$ ;
  - Compute $\Theta_i \to \Theta_i^\circ := \{v\colon \langle v | \forall u \in \Theta_i \rangle + 1 > 0\}$




# Laurent-Toric Fugue

## & Non-Convex Mirrors

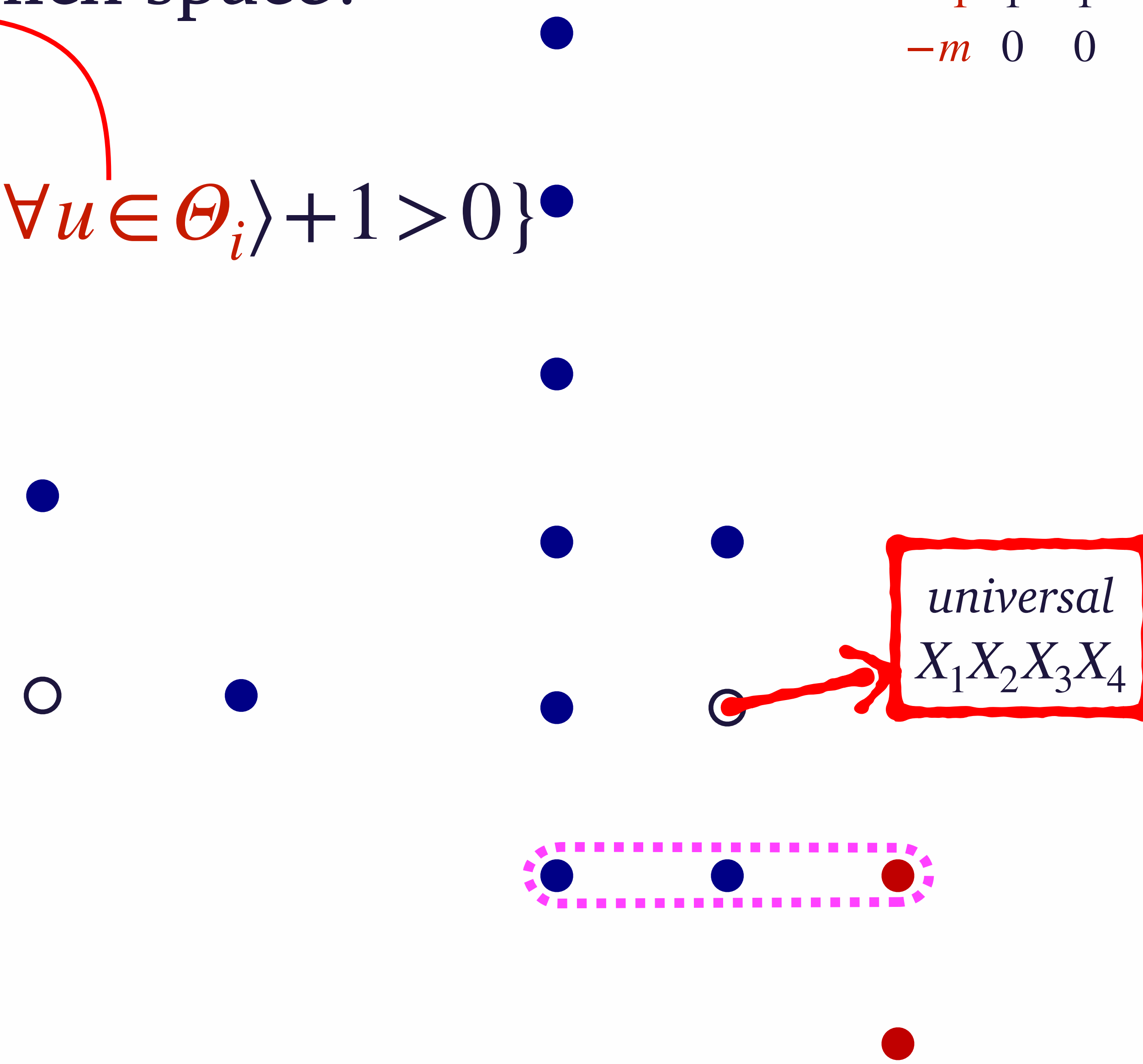
—2D *Proof-of-Concept*—

arXiv:1611.10300

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow \mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow \mathbb{P}^1$ |

- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$ ;
  - Compute $\Theta_i \to \Theta_i^\circ := \{v\!: \langle v | \forall u \in \Theta_i \rangle + 1 > 0\}$



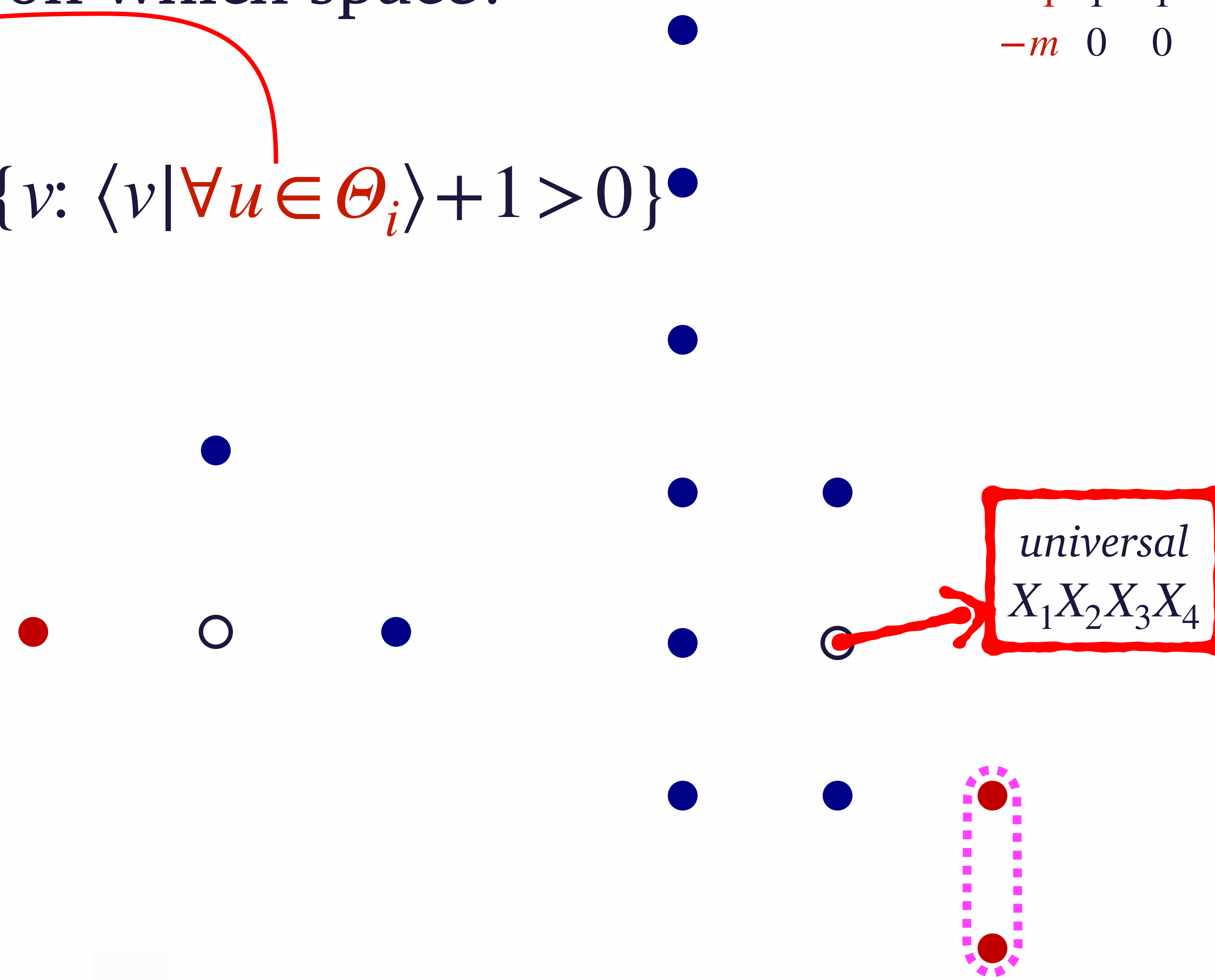
# Laurent-Toric Fugue


## & Non-Convex Mirrors

—2D *Proof-of-Concept*—

- $X_1{}^2X_2{}^0(X_3\oplus X_4)^{2+1m}\oplus X_1{}^1X_2{}^1(X_3\oplus X_4)^{2+0m}\oplus X_1{}^0X_2{}^2(X_3\oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$ ;
  - Compute $\Theta_i\to\Theta_i^\circ:=\{v\colon\ \langle v|\forall u\in\Theta_i\rangle+1>0\}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow\mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow\mathbb{P}^1$ |



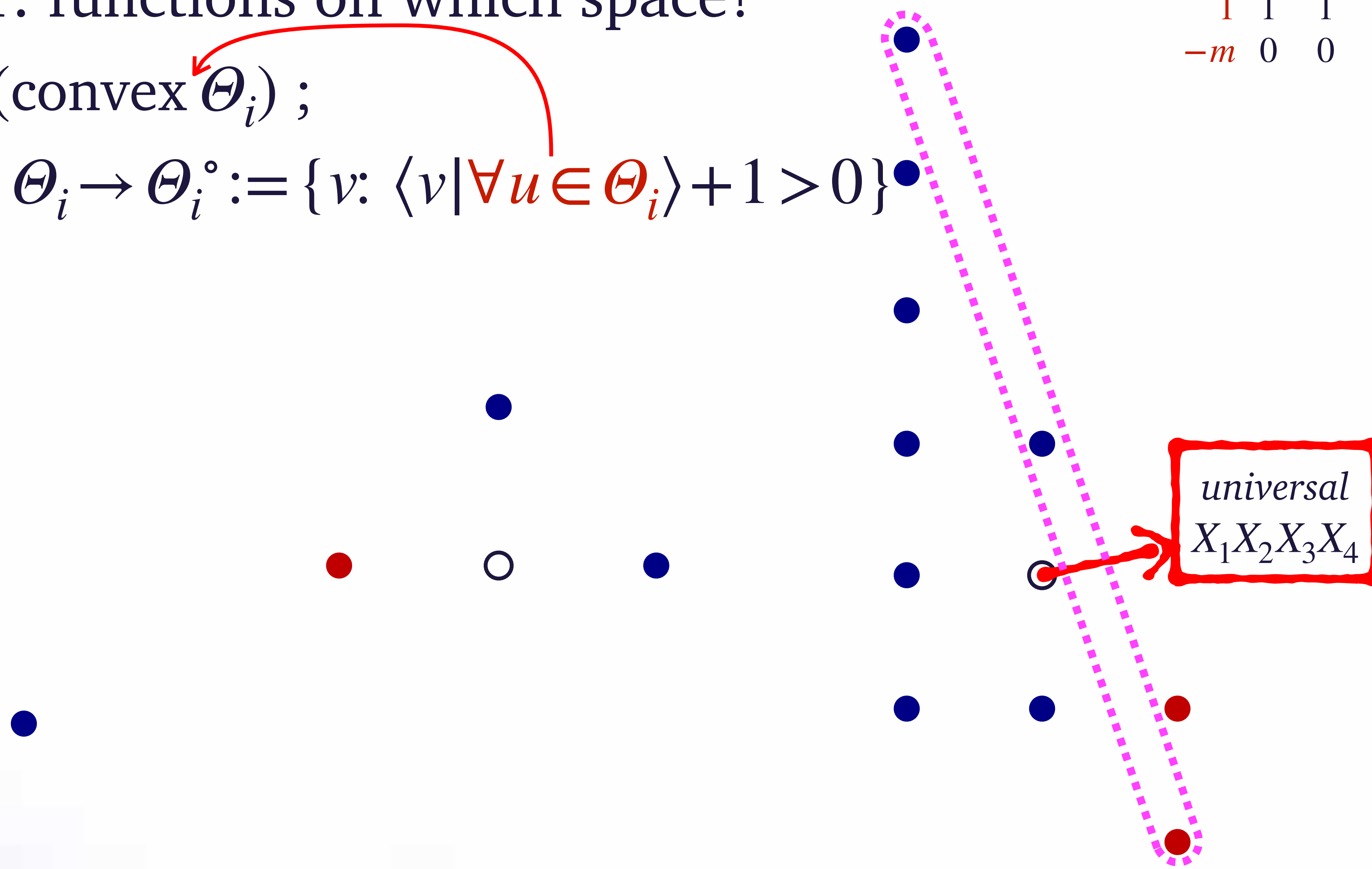
# Laurent-Toric Fugue

## & Non-Convex Mirrors

—2D *Proof-of-Concept*—


- $X_1^{\,2} X_2^{\,0} (X_3 \oplus X_4)^{2+1m} \oplus X_1^{\,1} X_2^{\,1} (X_3 \oplus X_4)^{2+0m} \oplus X_1^{\,0} X_2^{\,2} (X_3 \oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$ ;
  - Compute $\Theta_i \to \Theta_i^\circ := \{ v \colon \langle v | \forall u \in \Theta_i \rangle + 1 > 0 \}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow \mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow \mathbb{P}^1$ |



# Laurent-Toric Fugue


arXiv:1611.10300

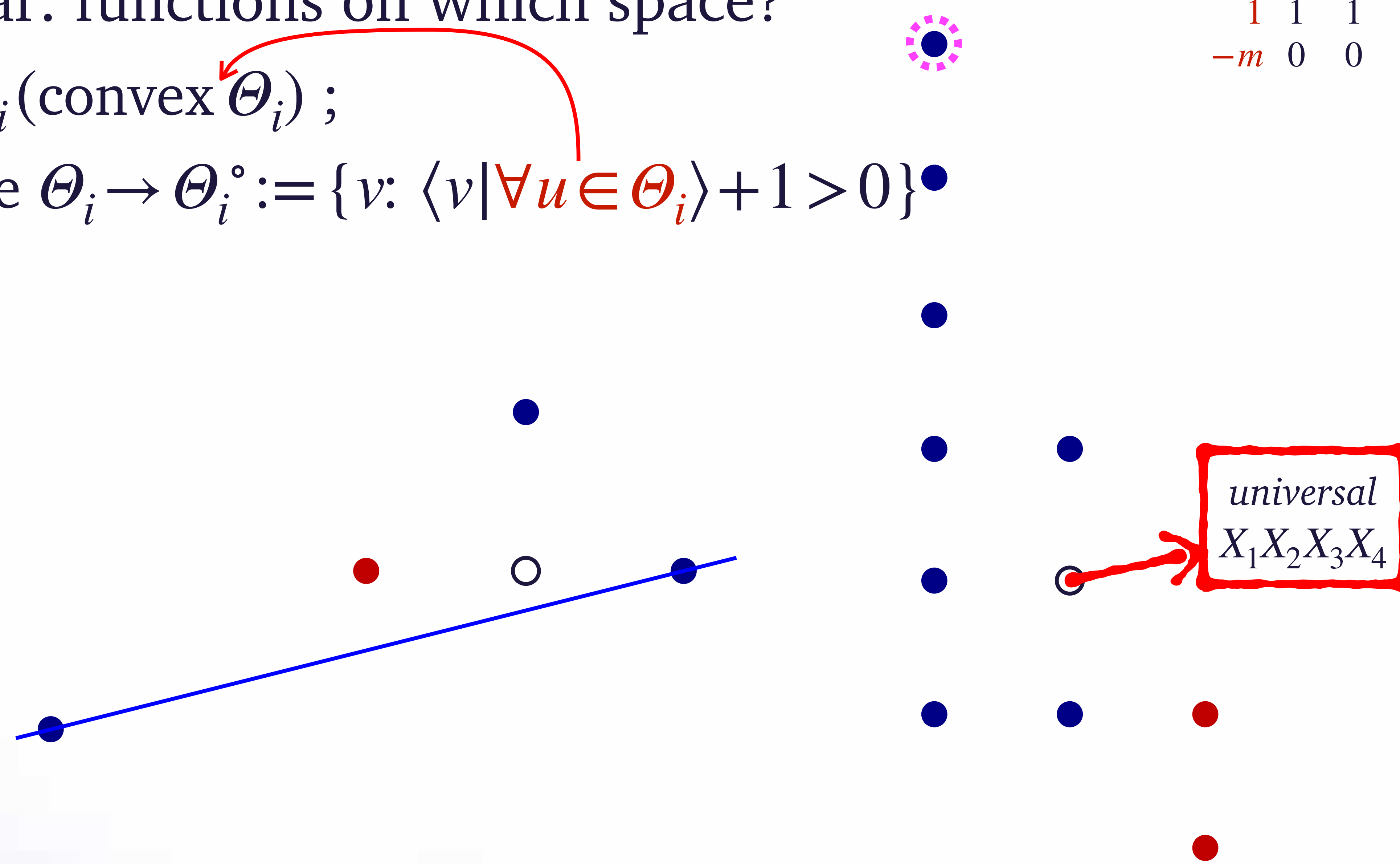
## & Non-Convex Mirrors

—2D *Proof-of-Concept*—

- $X_1{}^2 X_2{}^0 (X_3 \oplus X_4)^{2+1m} \oplus X_1{}^1 X_2{}^1 (X_3 \oplus X_4)^{2+0m} \oplus X_1{}^0 X_2{}^2 (X_3 \oplus X_4)^{2-1m}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow \mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow \mathbb{P}^1$ |

- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$ ;
  - Compute $\Theta_i \to \Theta_i^\circ := \{v\colon \langle v | \forall u \in \Theta_i \rangle + 1 > 0\}$




# Laurent-Toric Fugue

## & Non-Convex Mirrors

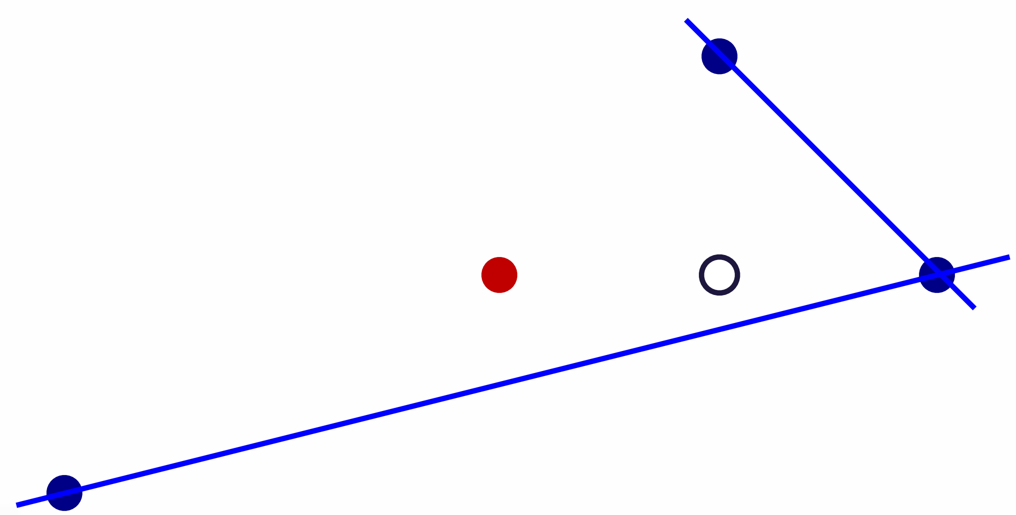
*—2D Proof-of-Concept—*

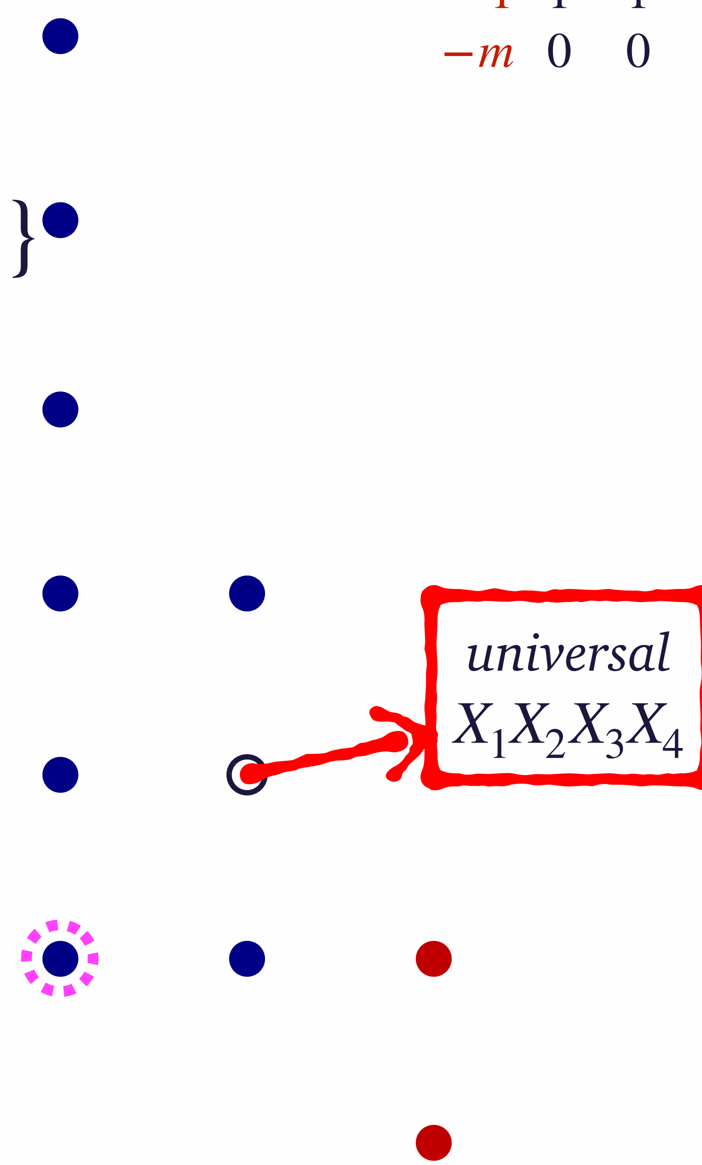
arXiv:1611.10300

- $X_1^{\,2} X_2^{\,0} (X_3 \oplus X_4)^{2+1m} \oplus X_1^{\,1} X_2^{\,1} (X_3 \oplus X_4)^{2+0m} \oplus X_1^{\,0} X_2^{\,2} (X_3 \oplus X_4)^{2-1m}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow \mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow \mathbb{P}^1$ |

- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$ ;
  - Compute $\Theta_i \to \Theta_i^\circ := \{ v \colon \langle v | \forall u \in \Theta_i \rangle + 1 > 0 \}$





# Laurent-Toric Fugue

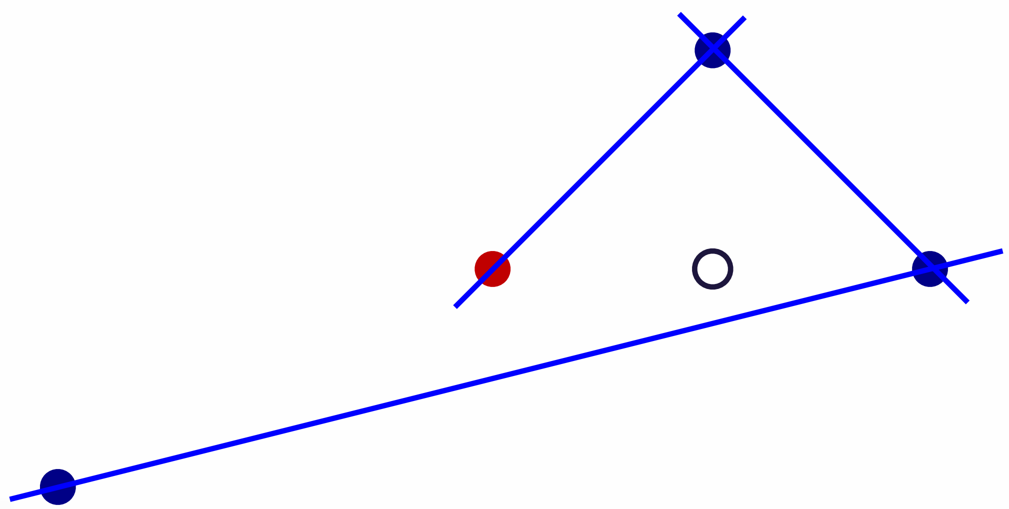
## & Non-Convex Mirrors

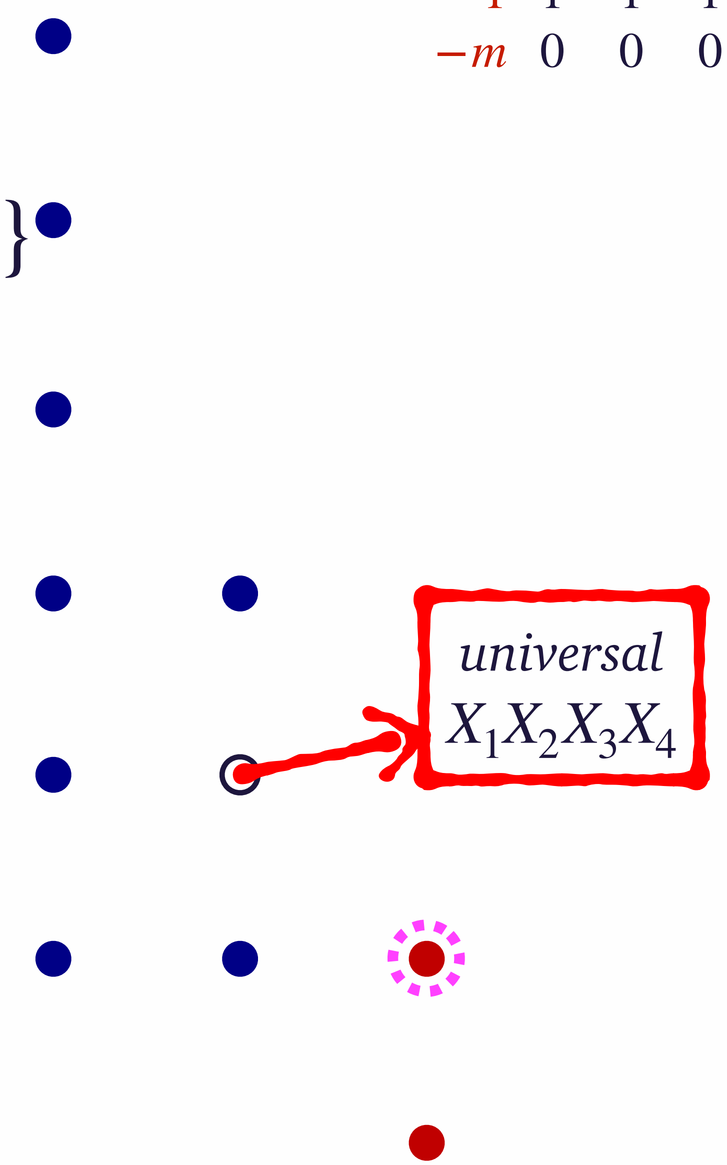
*—2D Proof-of-Concept—*


- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow\mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow\mathbb{P}^1$ |

- *Trans*polar: functions on which space?
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$ ;
  - Compute $\Theta_i\to\Theta_i^{\circ}:=\{v\colon\langle v|\forall u\in\Theta_i\rangle+1>0\}$





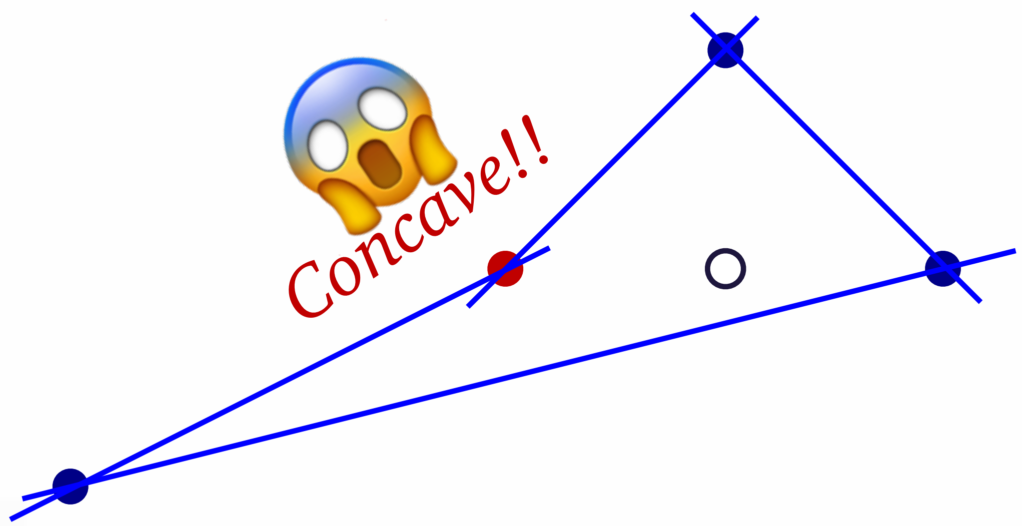
# Laurent-Toric Fugue

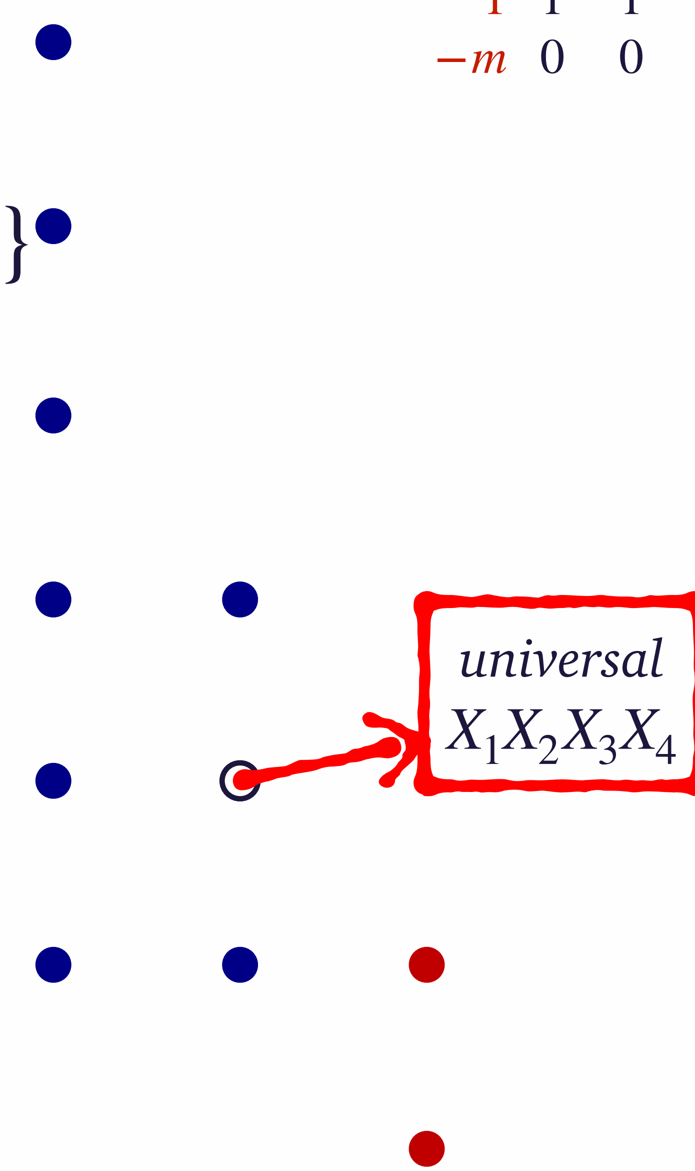

## & Non-Convex Mirrors

—2D *Proof-of-Concept*—

- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$ ;
  - Compute $\Theta_i\to\Theta_i^{\circ}:=\{v\colon\langle v|\forall u\in\Theta_i\rangle+1>0\}$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow\mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow\mathbb{P}^1$ |




# Laurent-Toric Fugue

## & Non-Convex Mirrors

*—2D Proof-of-Concept—*

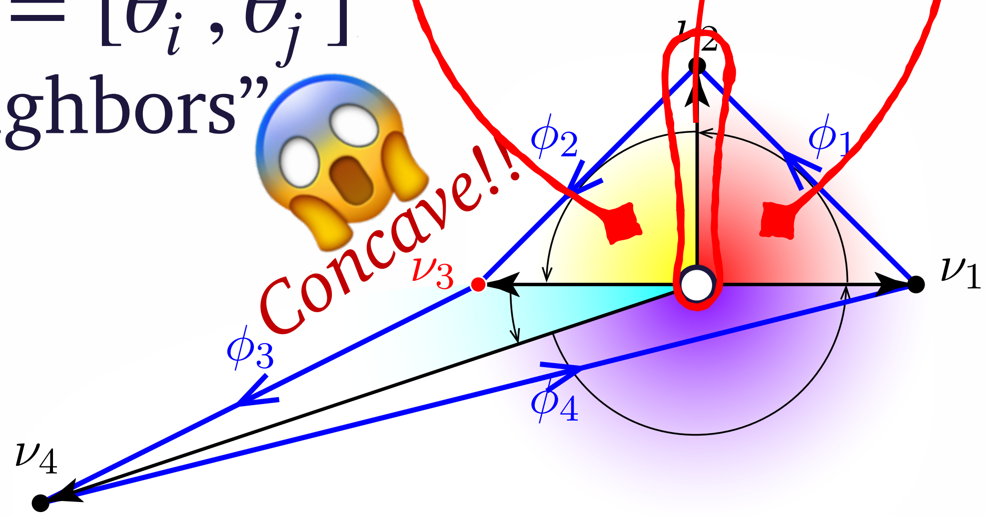
- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$

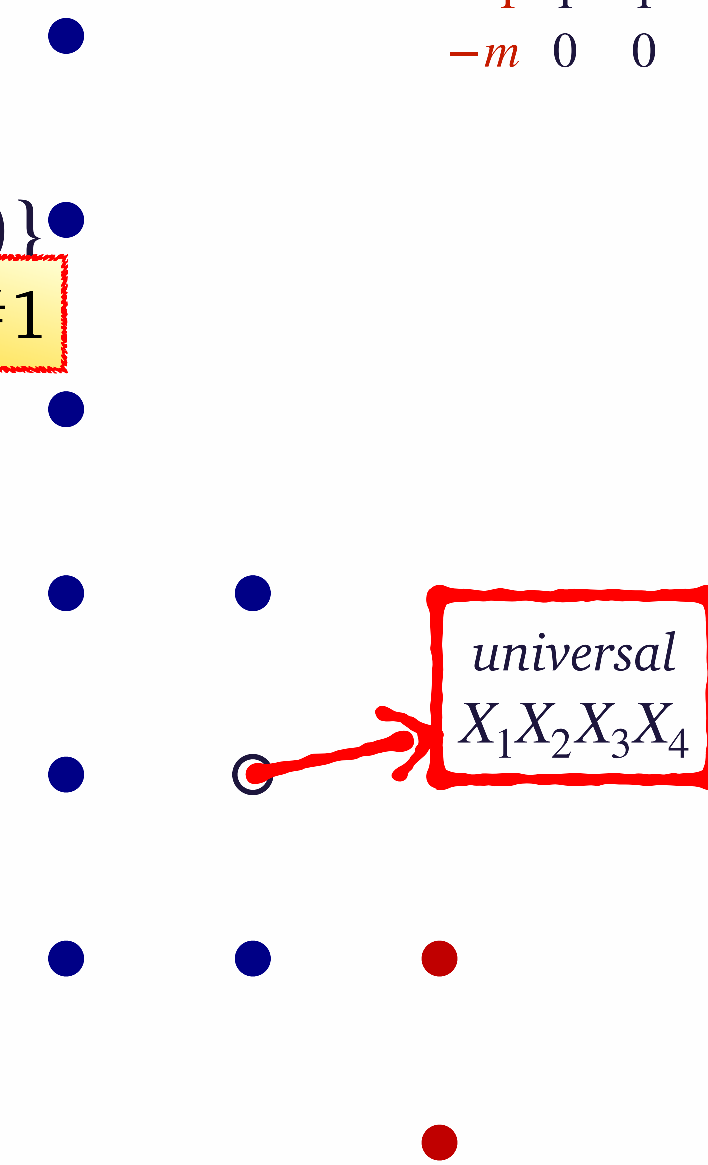
| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow\mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow\mathbb{P}^1$ |

- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$ ;
  - Compute $\Theta_i \to \Theta_i^\circ := \{\nu$ overlap gluing $+1>0\}$
  - (Re)assemble dually $(\theta_i \cap \theta_j)^\circ = [\theta_i^\circ, \theta_j^\circ]$ with “neighbors”

local chart #2

local chart #1



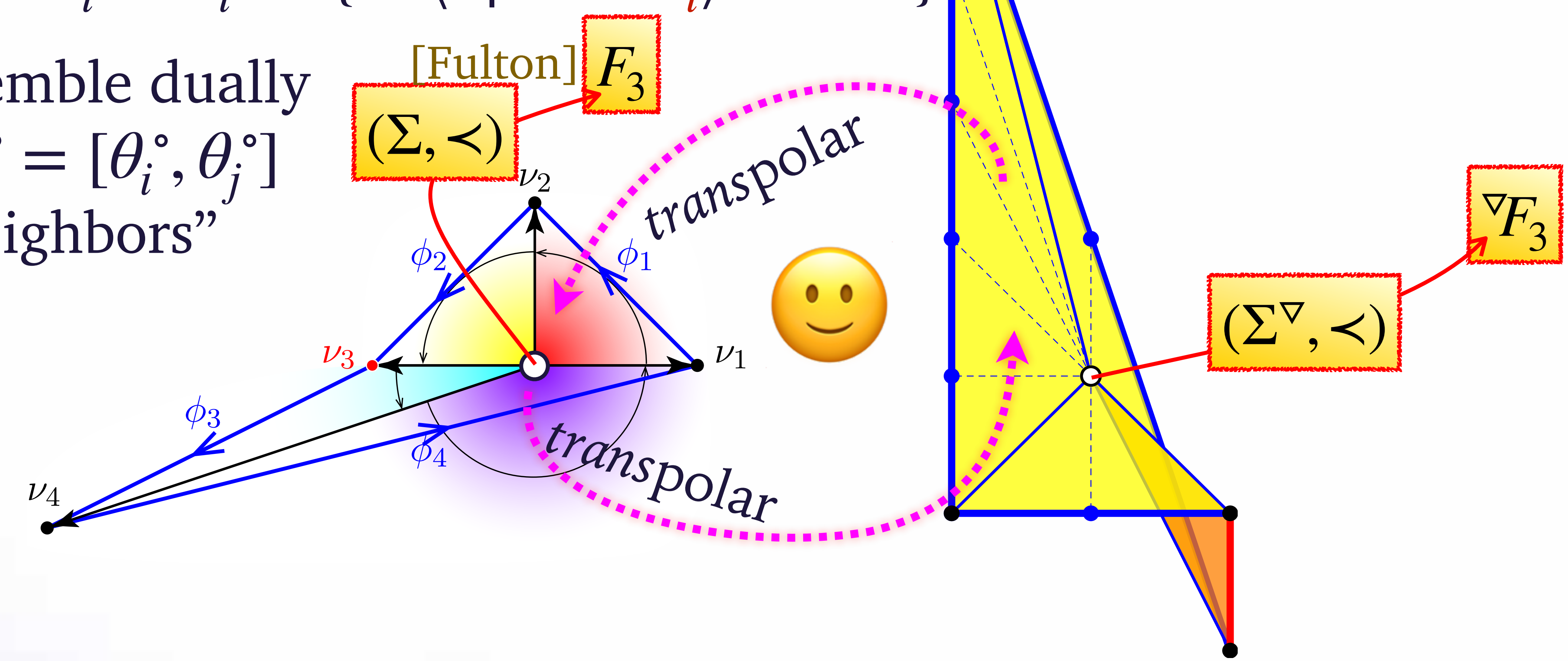
# Laurent-Toric Fugue

## & Non-Convex Mirrors

—2D *Proof-of-Concept*—


- $X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m}\oplus X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m}\oplus X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta\to\bigcup_i(\text{convex }\Theta_i)$ ;
  - Compute $\Theta_i\to\Theta_i^\circ:=\{v\colon\langle v|\forall u\in\Theta_i\rangle+1>0\}$
  - (Re)assemble dually $(\theta_i\cap\theta_j)^\circ=[\theta_i^\circ,\theta_j^\circ]$ with "neighbors"

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | $\leftarrow\mathbb{P}^4$ |
| $-m$ | 0 | 0 | 0 | 1 | 1 | $\leftarrow\mathbb{P}^1$ |



# Laurent-Toric Fugue


## & Non-Convex Mirrors

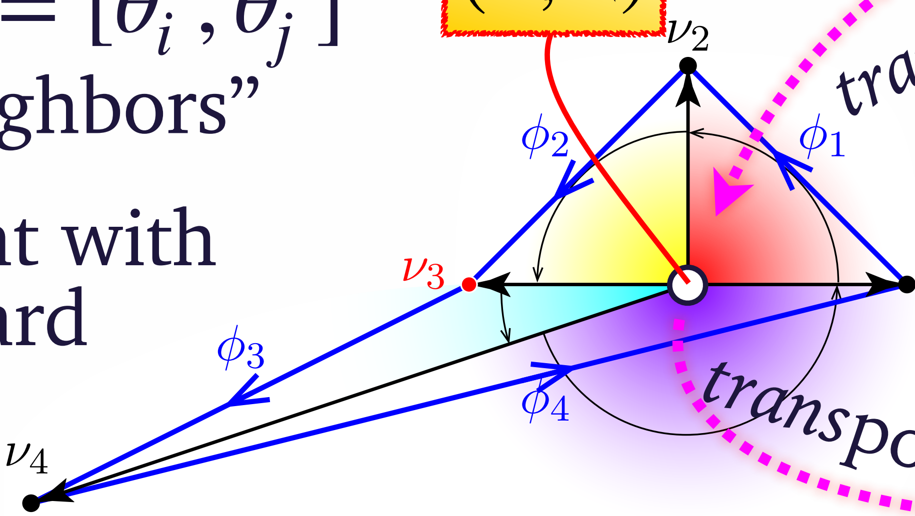
—2D *Proof-of-Concept*—

- $X_1^{\,2} X_2^{\,0} (X_3 \oplus X_4)^{2+1m} \oplus X_1^{\,1} X_2^{\,1} (X_3 \oplus X_4)^{2+0m} \oplus X_1^{\,0} X_2^{\,2} (X_3 \oplus X_4)^{2-1m}$
- *Trans*polar: functions on which space?
  - $\Delta \to \bigcup_i (\text{convex } \Theta_i)$ ;
  - Compute $\Theta_i \to \Theta_i^\circ := \{ v\!: \langle v | \forall u \in \Theta_i \rangle + 1 > 0 \}$
  - (Re)assemble dually $(\theta_i \cap \theta_j)^\circ = [\theta_i^\circ, \theta_j^\circ]$ with "neighbors"
  - Consistent with all standard methods

*(pre)complex algebraic geometry*

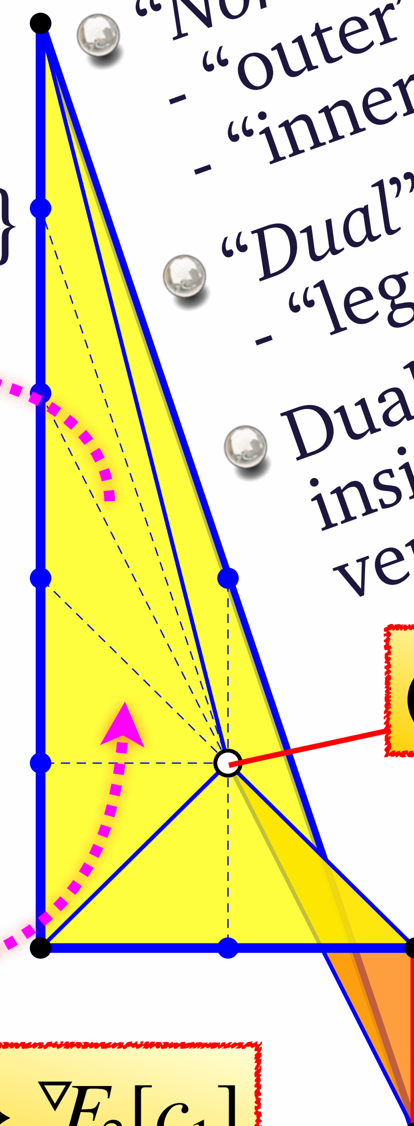
[Fulton] $F_3$

$(\Sigma, \prec)$




*transpolar*

*transpolar*



- "Normal fan"
  - "outer" [GE]
  - "inner/local" [C,L&S]
- "Dual"
  - "legal loops" [P&RV]
- Dual cones ↦ inside opening vertex-cones [?BH]

${}^\nabla F_3$

$(\Sigma^\nabla, \prec)$

'92: Khovanskii +Pukhlikov
'93: Karshon +Tolman
'99: Hattori +Masuda
+lots of

**poset** $(\Sigma, \prec)$ $\leftrightarrow$ **dual poset** $(\Sigma^\nabla, \prec)$

$F_3[c_1] \overset{\text{MM}}{\longleftrightarrow} {}^\nabla F_3[c_1]$

**SYZ**

*(pre)symplectic geometry*

# Laurent-Toric Fugue

## & Non-Convex Mirrors

—*Proof-of-Concept*—


- K3 in Hirzebruch 3-folds, one of *two* “cornerstone” mirror pairs:

$$a_1\,x_4^8 + a_2\,x_3^8 + a_3\,\frac{x_1^3}{x_3} + a_5\,\frac{x_2^3}{x_3}\;:\quad \exp\left\{2i\pi\left[\begin{array}{cccc}\frac13 & \frac23 & 0 & 0\\ \frac1{24} & \frac1{24} & \frac18 & 0\\ \hline \frac38 & \frac38 & \frac18 & \frac18\end{array}\right]\right\}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\;:\quad \begin{cases}G=\mathbb{Z}_3\times\mathbb{Z}_{24},\\ Q=\mathbb{Z}_8.\end{cases}$$

$$\begin{bmatrix}0&0&0&8\\0&0&8&0\\3&0&-1&0\\0&3&-1&0\end{bmatrix}\qquad \begin{array}{c}\mathbb{P}^3_{(3:3:1:1)}[8]\\[1ex] \mathbb{P}^3_{(3:5:8:8)}[24]/\mathbb{Z}_3\end{array}$$

$B^3H^2K$

$$b_1\,y_3^3 + b_2\,y_5^3 + b_3\,\frac{y_2^8}{y_3\,y_5} + b_4\,y_1^8\;:\quad \exp\left\{2i\pi\left[\begin{array}{cccc}\frac18 & 0 & 0 & 0\\ 0 & 0 & \frac13 & \frac23\\ \hline \frac3{24} & \frac5{24} & \frac13 & \frac13\end{array}\right]\right\}\begin{bmatrix}y_1\\y_2\\y_3\\y_5\end{bmatrix}\;:\quad \begin{cases}G^\nabla=\mathbb{Z}_8\\ Q^\nabla=\mathbb{Z}_{24}\times\mathbb{Z}_3\end{cases}$$

$$\frac{|G|}{|Q|}=\frac{3{\cdot}24}{8}=9=\frac{d(\Delta_{\mathcal{F}_3})}{d(\Delta^\star_{\mathcal{F}_3})}=\frac{54}{6}$$

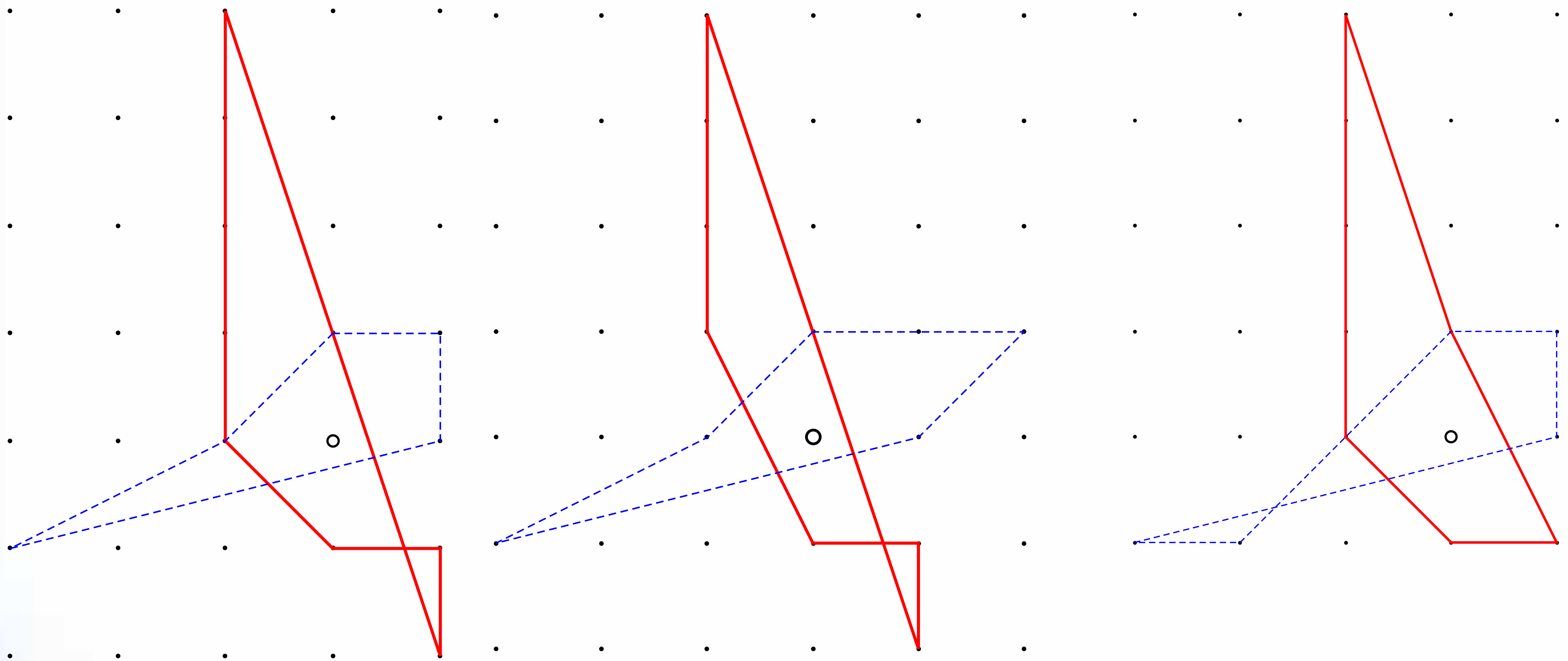
- The Hilbert space & interactions restricted by the symmetries
  - Analysis: classical, semi-classical, quantum corrections…
  - …in spite of the manifest singularity in the (super)potential

# Laurent-Toric Fugue

## & Non-Convex Mirrors

—*Proof-of-Concept*—


- Not just Hirzebruch $n$-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla:\ \Delta^\star \xleftrightarrow{1\text{–}1} \Delta$

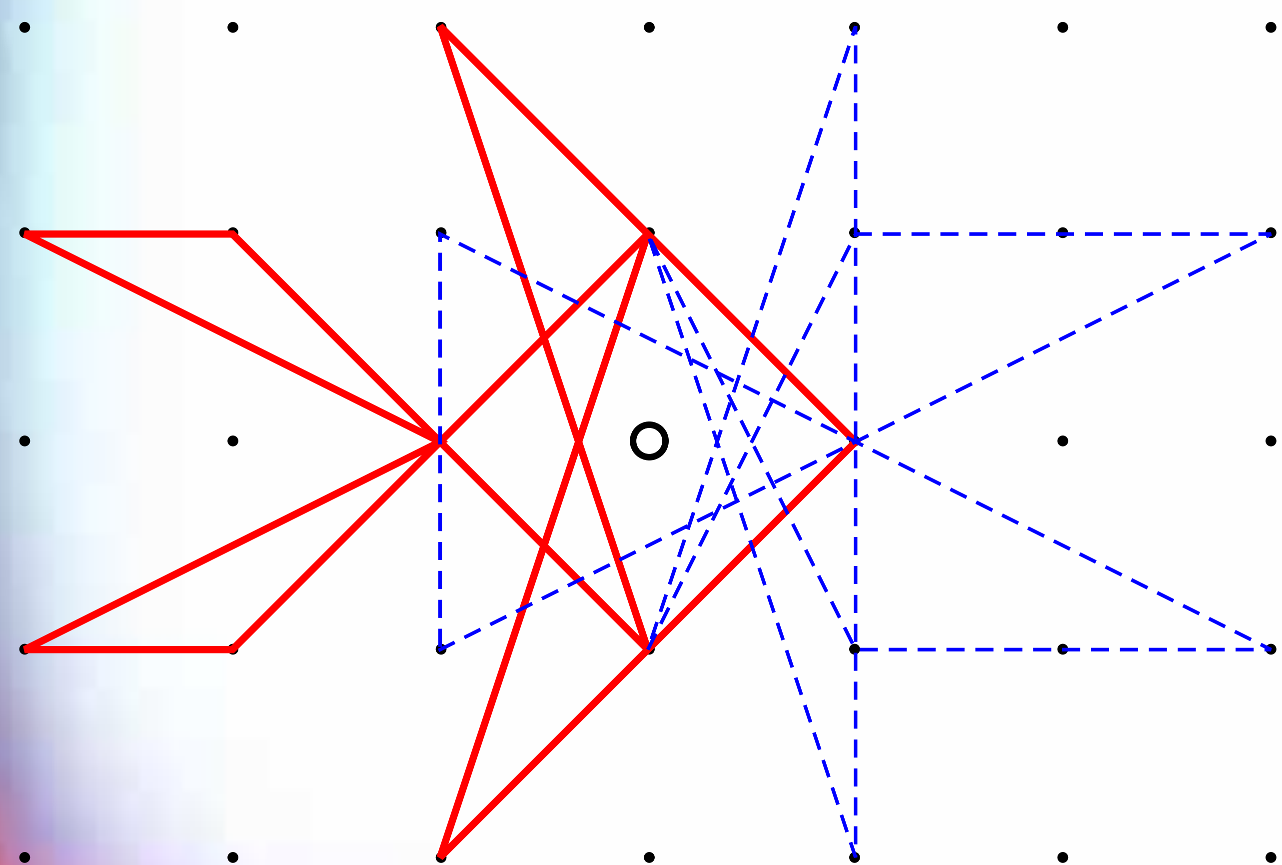


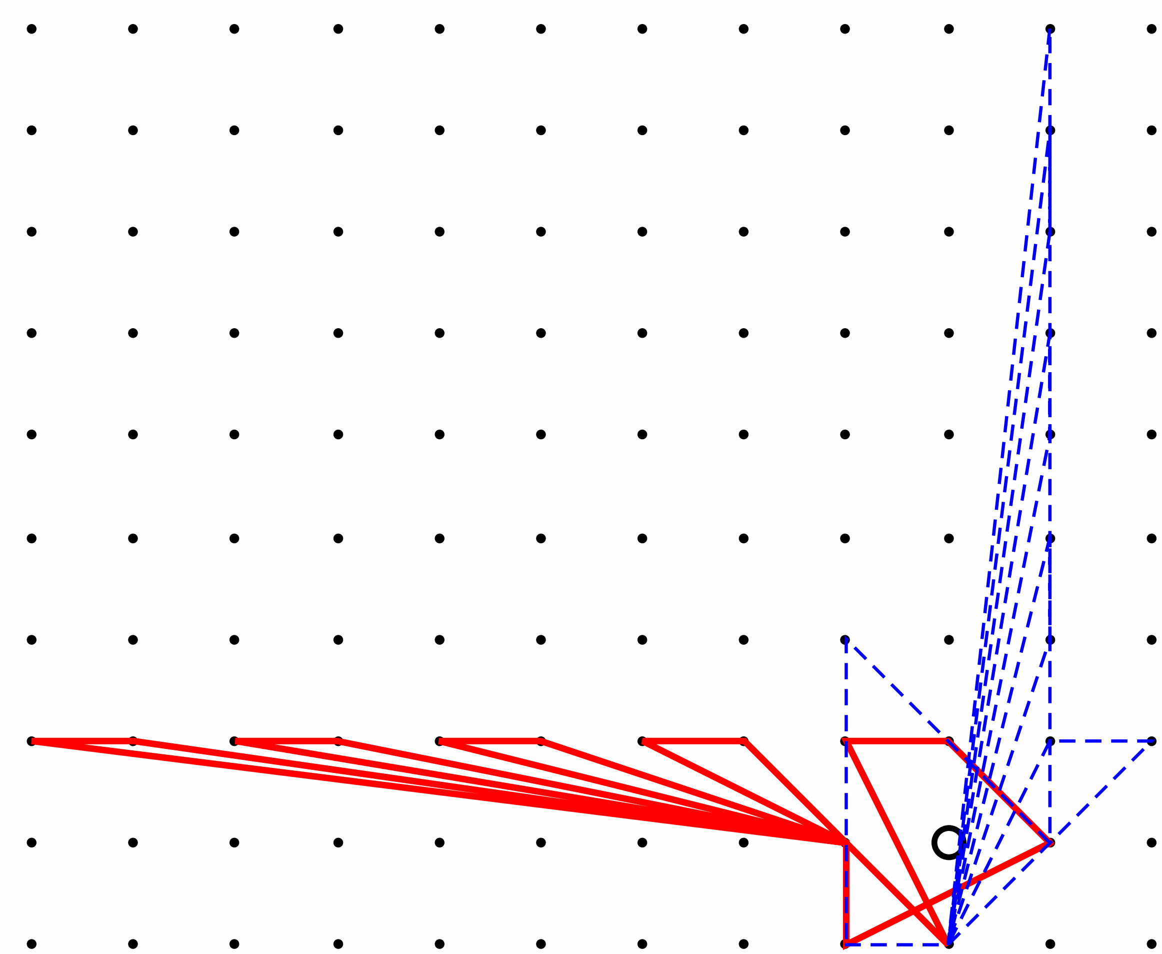
# Laurent-Toric Fugue

## & Non-Convex Mirrors

—*Proof-of-Concept*—


- Not just Hirzebruch $n$-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla\colon \Delta^\star \xleftrightarrow{1\text{–}1} \Delta$





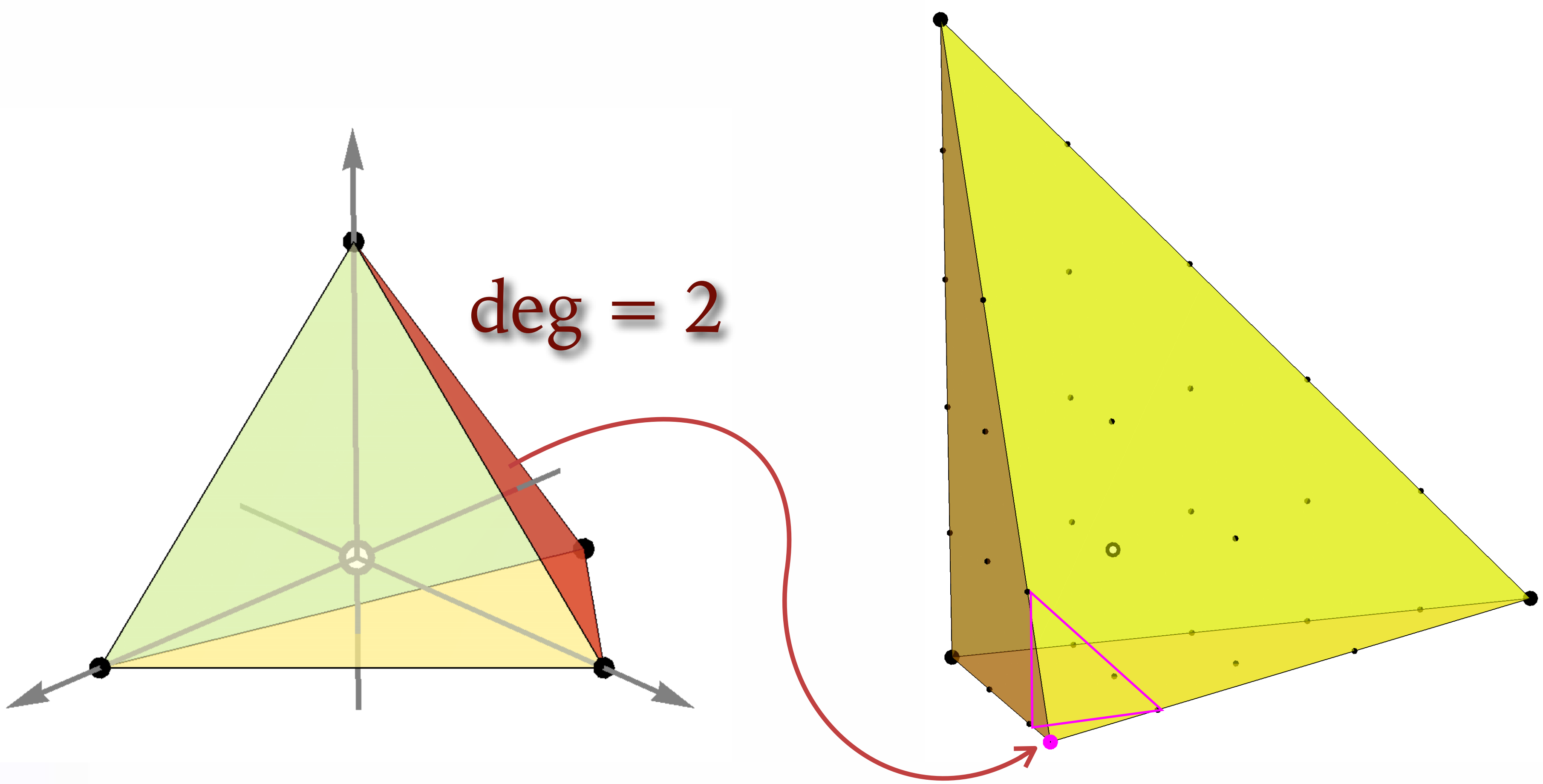
# Laurent-Toric Fugue

## & Non-Convex Mirrors

*—Proof-of-Concept—*

arXiv:1611.10300

- Not just Hirzebruch $n$-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla:\ \Delta^{\star} \xleftrightarrow{1\text{–}1} \Delta$
  - And, plenty of 3-dimensional polyhedra:

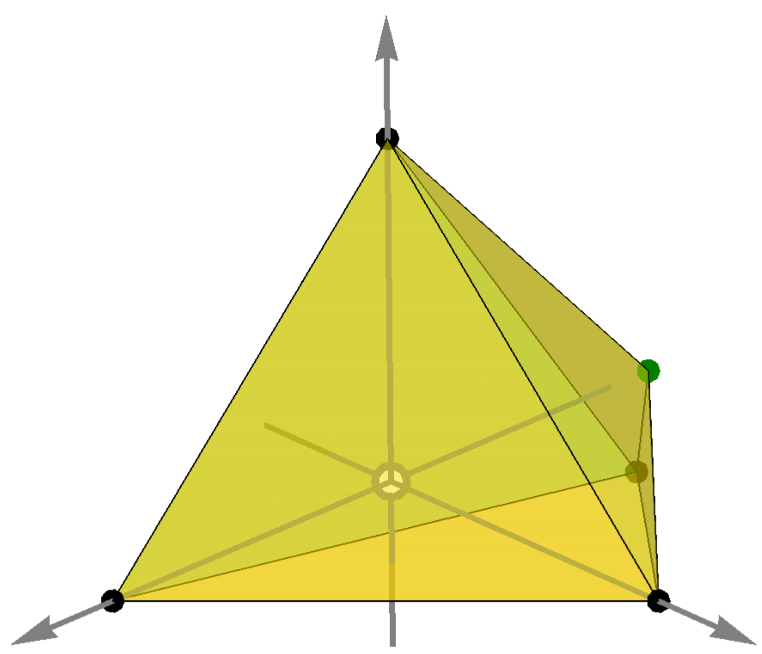


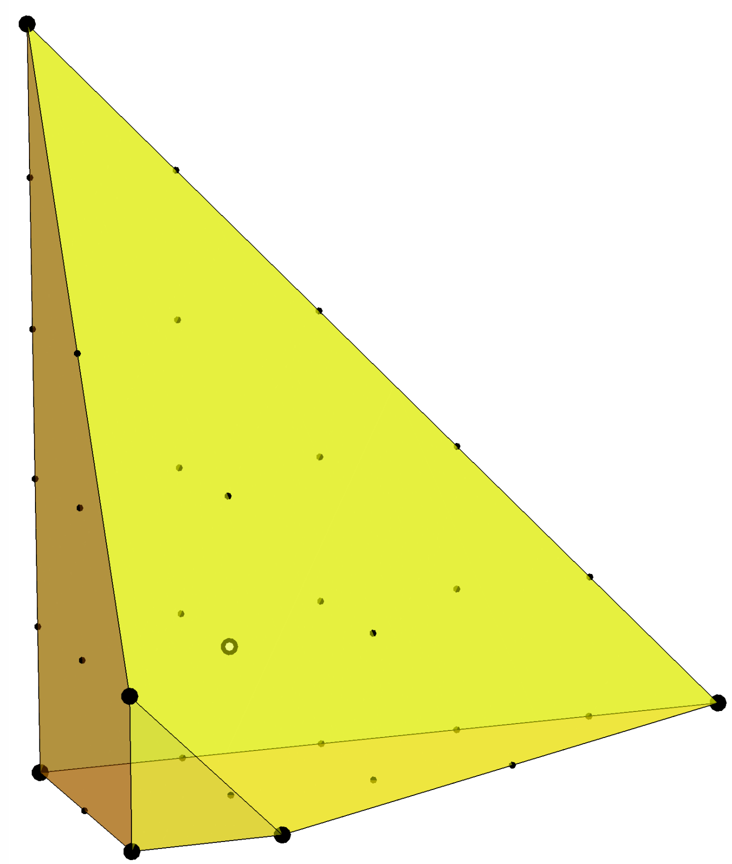
# Laurent-Toric Fugue

## & Non-Convex Mirrors

—*Proof-of-Concept*—


- Not just Hirzebruch $n$-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla : \Delta^{\star} \xleftrightarrow{1\text{–}1} \Delta$
  - And, plenty of 3-dimensional polyhedra:



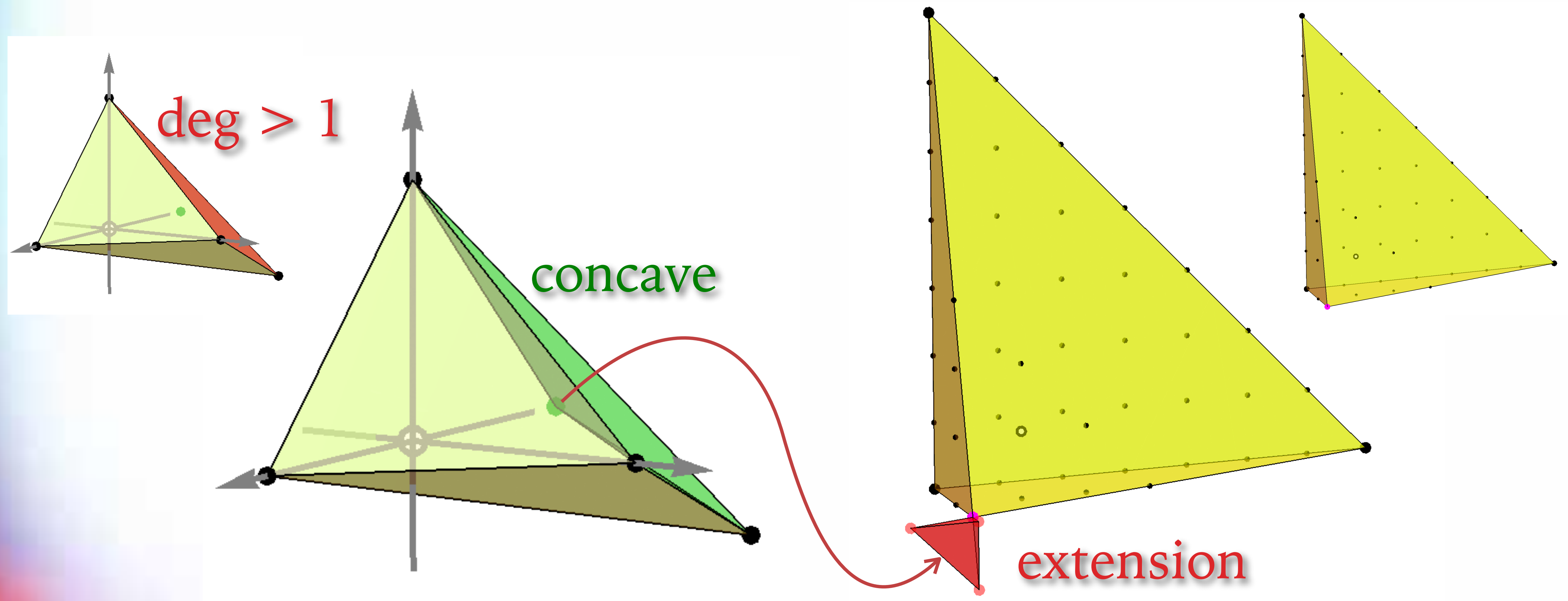


# Laurent-Toric Fugue

## & Non-Convex Mirrors

—*Proof-of-Concept*—


- Not just Hirzebruch *n*-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla: \Delta^\star \xleftrightarrow{1\text{–}1} \Delta$
  - And, plenty of 3-dimensional polyhedra:



# Laurent-Toric Fugue

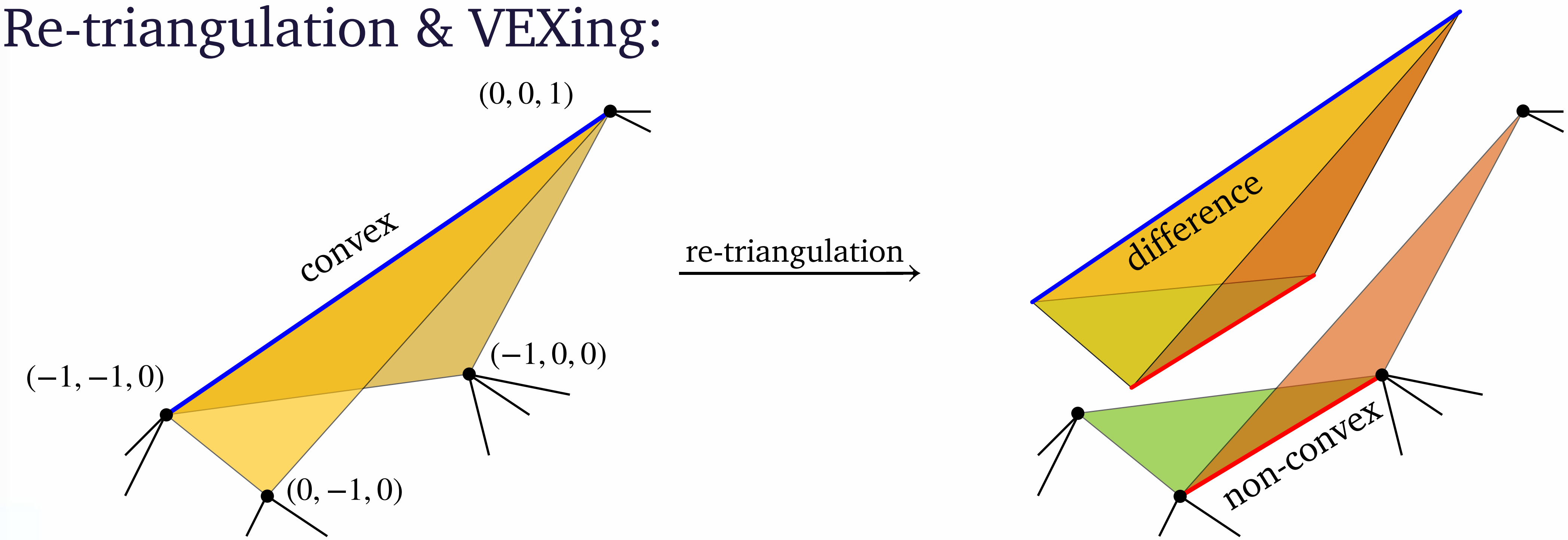
## & Non-Convex Mirrors

—*Proof-of-Concept*—

BH

arXiv:1611.10300

- Not just Hirzebruch $n$-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla : \Delta^\star \xleftrightarrow{1\text{–}1} \Delta$
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & VEXing:



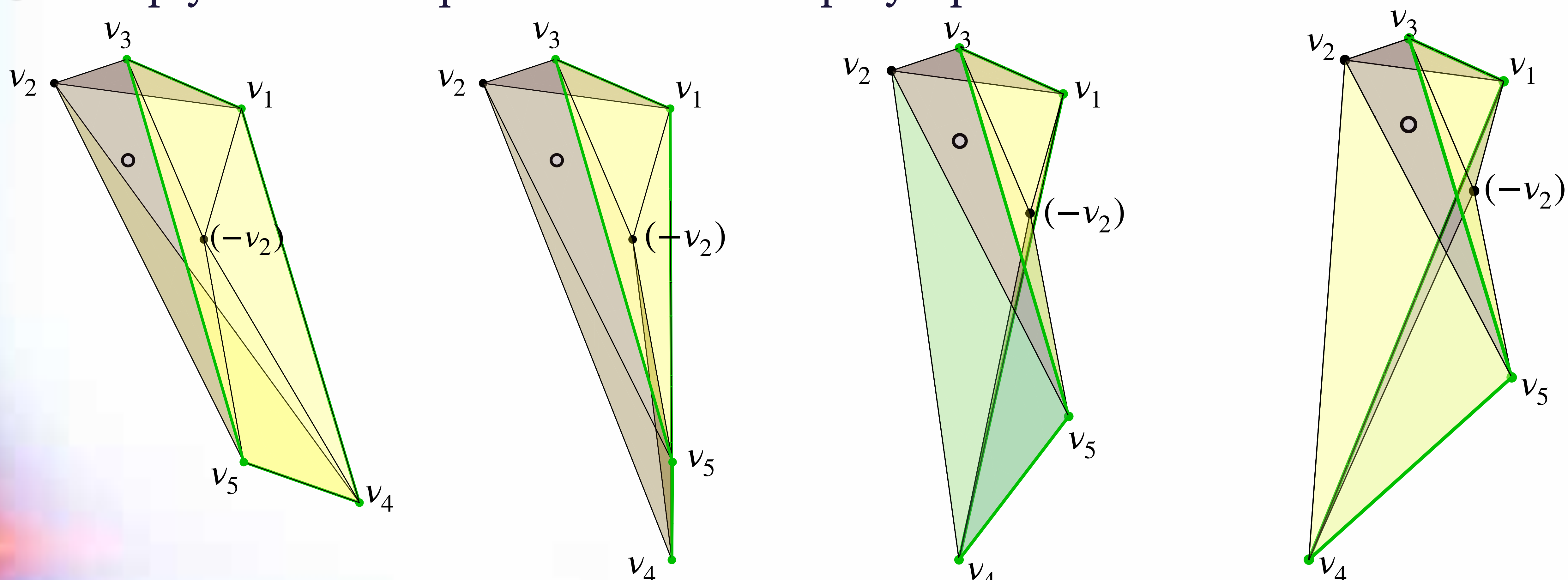
# Laurent-Toric Fugue

## & Non-Convex Mirrors

—*Proof-of-Concept*—

arXiv:1611.10300

- Not just Hirzebruch *n*-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla : \Delta^\star \xleftrightarrow{1\text{–}1} \Delta$
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & VEXing:
  - Multiply infinite sequences of twisted polytopes:

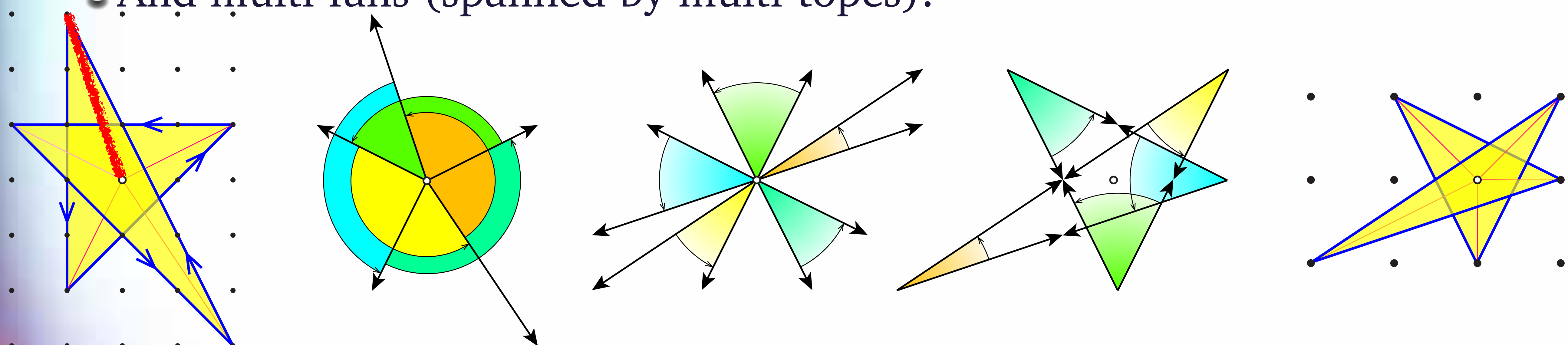


# Laurent-Toric Fugue

## & Non-Convex Mirrors

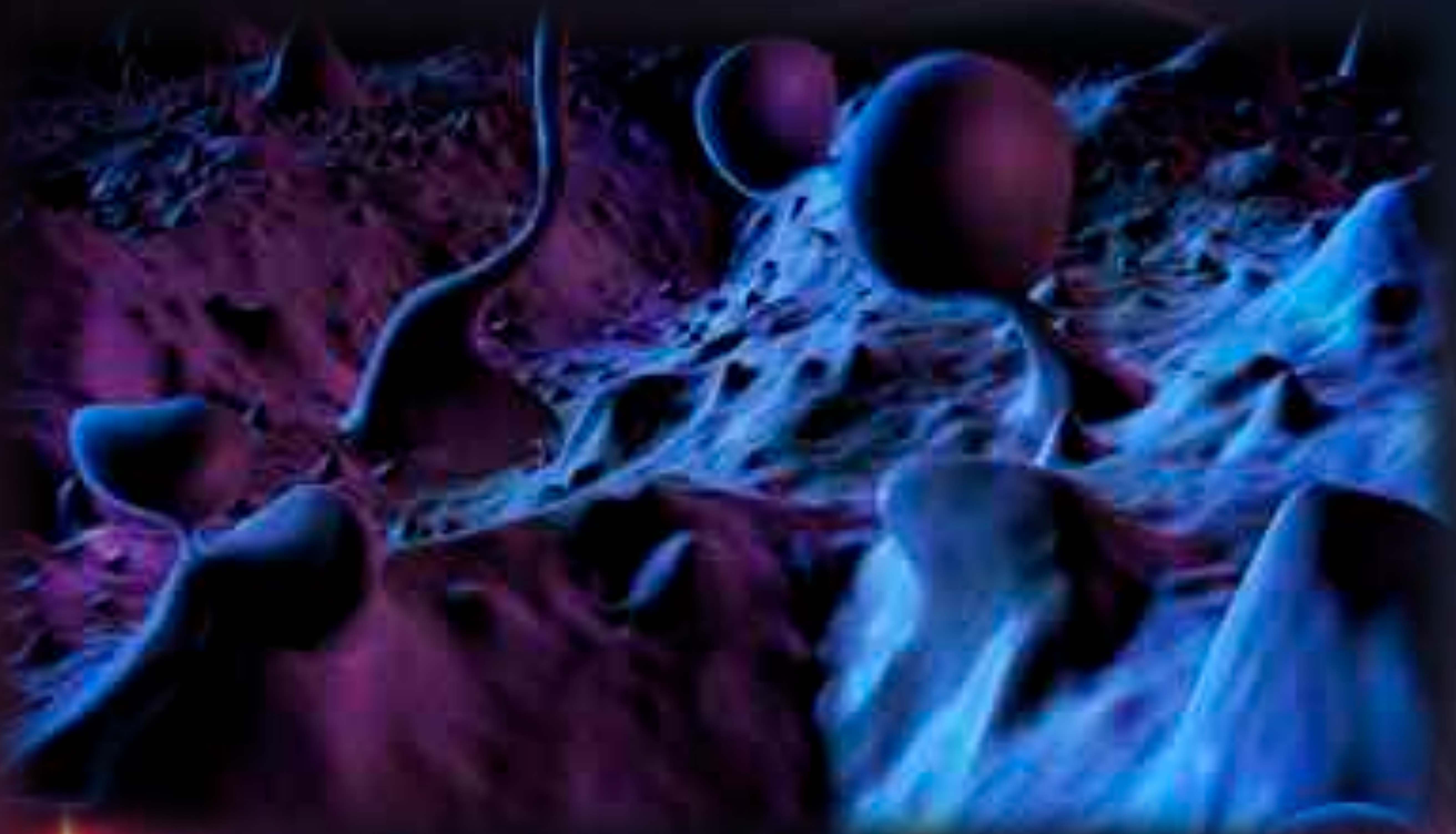
—*Proof-of-Concept*—


- Not just Hirzebruch *n*-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla:\ \Delta^{\star} \xleftrightarrow{1\text{–}1} \Delta$
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & VEXing:
  - Multiply infinite sequences of twisted polytopes:
  - And multi-fans (spanned by multi-topes):



winding number (multiplicity, Duistermaat-Heckman fn.) = 2

[A. Hattori+M. Masuda" *Theory of Multi-Fans*, Osaka J. Math. **40** (2003) 1–68]



# Discriminant Divertimento
## (How Small Can We Go?)

# Discriminant Divertimento

arXiv:1611.10300

## The Phase-Space

—*Proof-of-Concept*—

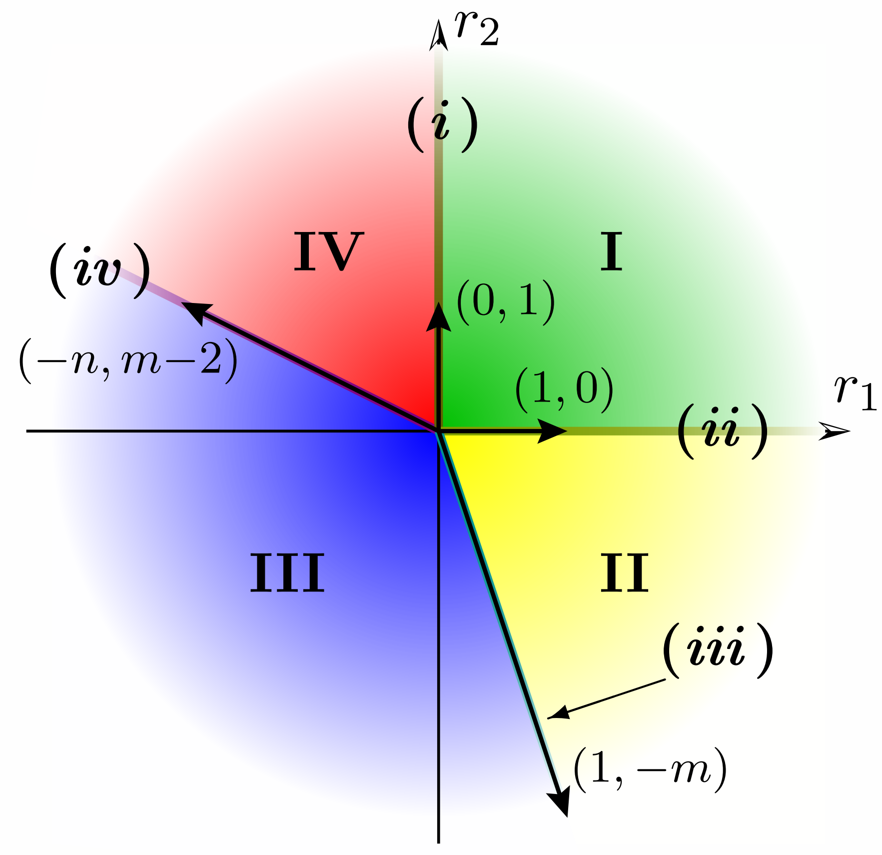
- The (super)potential: $W(X) := X_0 \cdot f(X)$,

$$f(X) := \sum_{j=1}^{2} \Big( \sum_{i=2}^{n} (a_{ij}\, X_i^n) X_{n+j}^{2-m} + a_j\, X_1^n X_{n+j}^{(n-1)m+2} \Big)$$

| | $X_0$ | $X_1$ | $X_2$ | $\cdots$ | $X_n$ | $X_{n+1}$ | $X_{n+2}$ |
|---|---|---|---|---|---|---|---|
| $Q^1$ | $-n$ | 1 | 1 | $\cdots$ | 1 | 0 | 0 |
| $Q^2$ | $m-2$ | $-m$ | 0 | $\cdots$ | 0 | 1 | 1 |

- The possible vevs

| | $\lvert x_0\rvert$ | $\lvert x_1\rvert$ | $\lvert x_2\rvert$ | $\cdots$ | $\lvert x_n\rvert$ | $\lvert x_{n+1}\rvert$ | $\lvert x_{n+2}\rvert$ |
|---|---|---|---|---|---|---|---|
| ***i*** | 0 | 0 | 0 | $\cdots$ | 0 | $*$ | $*$ |
| **I** | 0 | $*$ | $*$ | $\cdots$ | $*$ | $*$ | $*$ |
| ***ii*** | 0 | 0 | $*$ | $\cdots$ | $*$ | 0 | 0 |
| **II** | 0 | $\lvert x_1\rvert = \sqrt{\frac{\sum_j \lvert x_{n+j}\rvert^2 - r_2}{m}} = \sqrt{r_1 - \sum_{i=2}^{n} \lvert x_i\rvert^2} > 0$ | $*$ | $\cdots$ | $*$ | $*$ | $*$ |
| ***iii*** | 0 | $\sqrt{r_1}$ | 0 | $\cdots$ | 0 | 0 | 0 |
| **III** | $\sqrt{\frac{mr_1+r_2}{(n-1)m+2}}$ | $\sqrt{\frac{(m-2)r_1+nr_2}{(n-1)m+2}}$ | 0 | $\cdots$ | 0 | 0 | 0 |
| ***iv*** | $\sqrt{-r_1/n}$ | 0 | 0 | $\cdots$ | 0 | 0 | 0 |
| **IV** | $\sqrt{-r_1/n}$ | 0 | 0 | $\cdots$ | 0 | $*$ | $*$ |



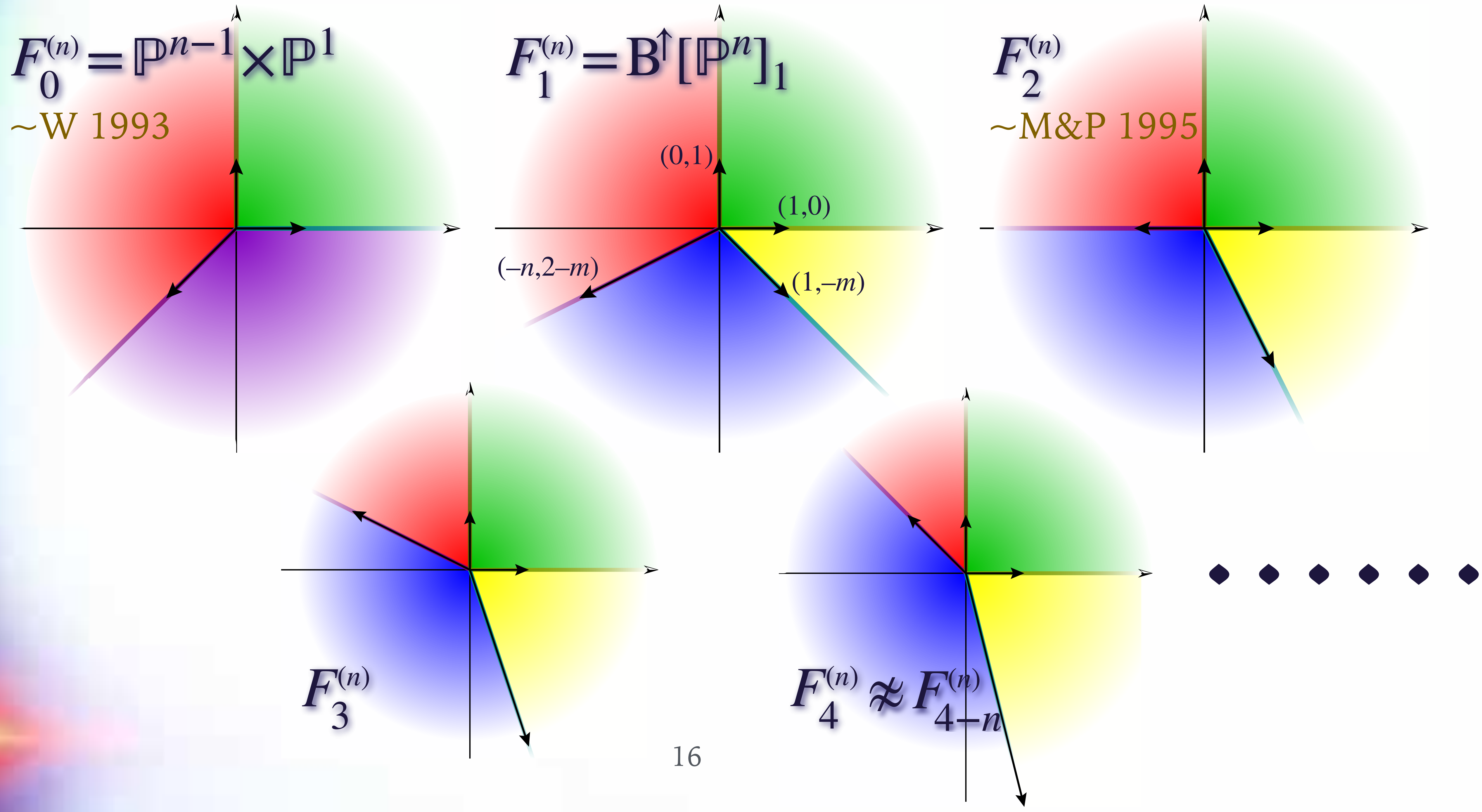
# Discriminant Divertimento

arXiv:1611.10300+

## The Phase-Space

—*Proof-of-Concept*—

- Varying $m$ in $F_m^{(n)}$:



# Discriminant Divertimento


arXiv: "real soon, now"

## The Discriminant

*—Proof-of-Concept—*

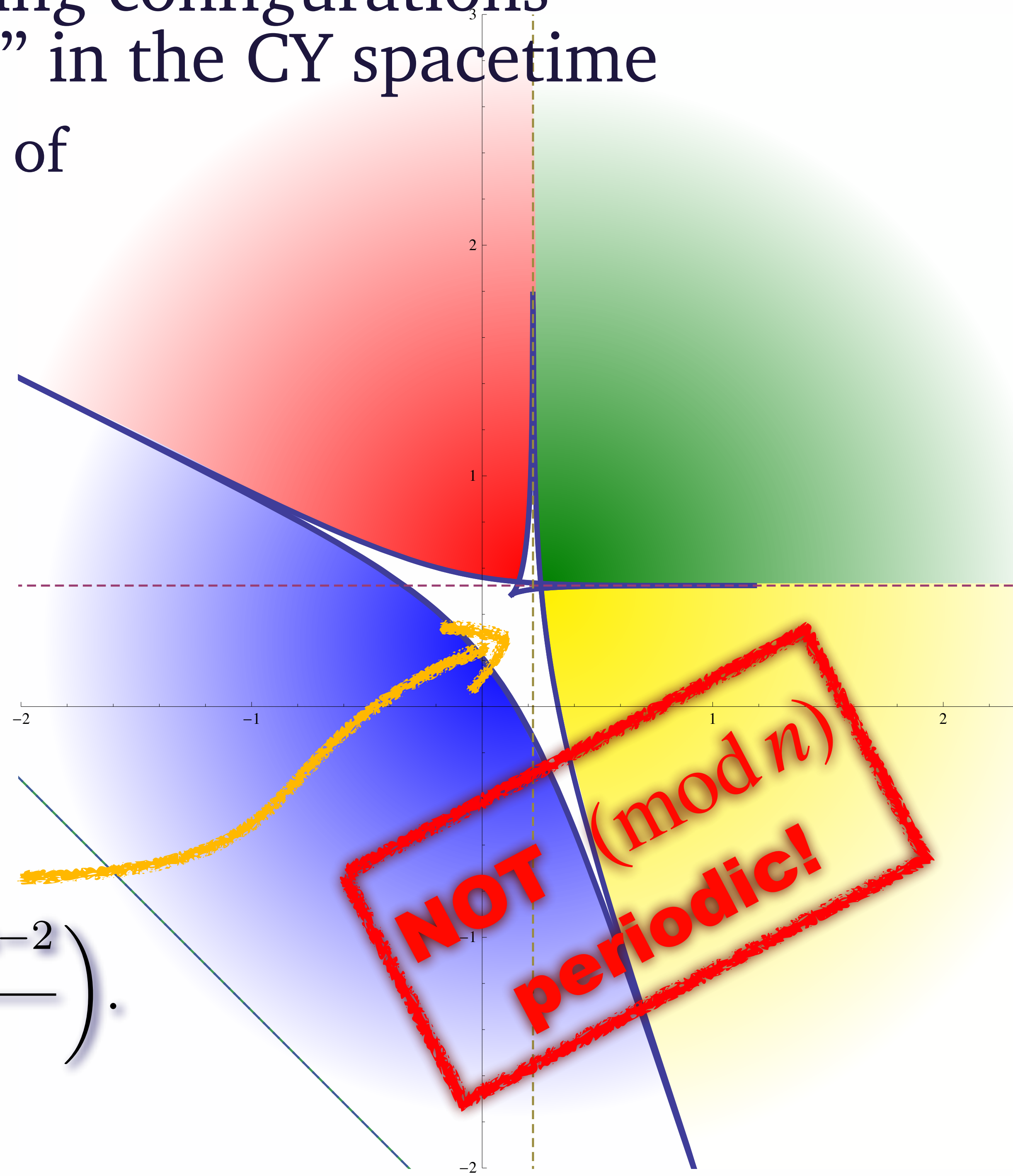
- Now add "instantons": 0-energy string configurations wrapped around "tunnels" & "holes" in the CY spacetime
  - Near $(r_1, r_2) = (0,0)$, classical analysis of Kähler (metric) phase-space fails [M&P: arXiv:hep-th/9412236]
- With

| | $X_0$ | $X_1$ | $X_2$ | $\cdots$ | $X_n$ | $X_{n+1}$ | $X_{n+2}$ |
|---|---|---|---|---|---|---|---|
| $Q^1$ | $-n$ | $1$ | $1$ | $\cdots$ | $1$ | $0$ | $0$ |
| $Q^2$ | $m-2$ | $-m$ | $0$ | $\cdots$ | $0$ | $1$ | $1$ |

- the instanton resummation gives:

$$r_1 + \frac{\hat\theta_1}{2\pi i} = -\frac{1}{2\pi}\log\left(\frac{\sigma_1^{n-1}(\sigma_1 - m\,\sigma_2)}{[(m-2)\sigma_2 - n\sigma_1]^n}\right),$$

$$r_2 + \frac{\hat\theta_2}{2\pi i} = -\frac{1}{2\pi}\log\left(\frac{\sigma_2^2\,[(m-2)\sigma_2 - n\sigma_1]^{m-2}}{(\sigma_1 - m\,\sigma_2)^m}\right).$$



# …and a Mirror Motet

## (Yes, the $B^3H^2K$-mirrors)

# Mirror Motets


arXiv: "real soon, now"

## The Discriminant

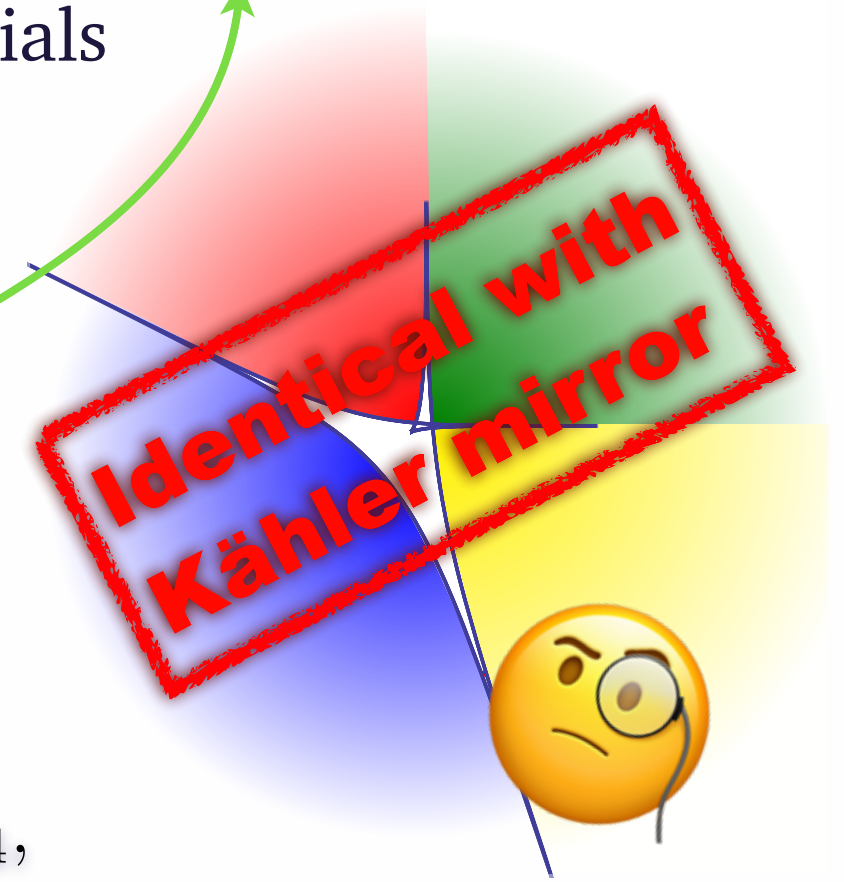
*—Proof-of-Concept—*

- Now compare with the complex structure of the B³H²K-mirror
  - Restricted to the "cornerstone" defining polynomials

$$f(x) = a_0 \prod_{\nu_i\in\Delta^\star} (x_{\nu_i})^{\langle\nu_i,\mu_0\rangle+1} + \sum_{\mu_I\in\Delta} a_{\mu_I} \prod_{\nu_i\in\Delta^\star} (x_{\nu_i})^{\langle\nu_i,\mu_I\rangle+1}$$

Batyrev

$$g(y) = b_0 \prod_{\mu_I\in\Delta} (y_{\mu_I})^{\langle\mu_I,\nu_0\rangle+1} + \sum_{\nu_i\in\Delta^\star} b_{\nu_i} \prod_{\mu_I\in\Delta} (y_{\mu_I})^{\langle\mu_I,\nu_i\rangle+1}$$




- In particular,

$$g(y) = \sum_{i=0}^{n+2} b_i\,\phi_i(y) \;= b_0\,\phi_0 + b_1\,\phi_1 + b_2\,\phi_2 + b_3\,\phi_3 + b_4\,\phi_4,$$

$$\phi_0 := y_1\cdots y_4,\quad \phi_1 := y_1^2\,y_2^2,\quad \phi_2 := y_3^2\,y_4^2,\quad \phi_3 := \frac{y_1^{m+2}}{y_3^{m-2}},\quad \phi_4 := \frac{y_2^{m+2}}{y_4^{m-2}},$$

$$z_1 = -\frac{\beta\,[(m{-}2)\beta + m]}{m{+}2},\qquad z_2 = \frac{(2\beta{+}1)^2}{(m+2)^2\,\beta^m},\qquad \beta := \left[\frac{b_1\,\phi_1}{b_0\,\phi_0}\Big/ {}^{A}\mathscr{J}(g)\right],$$

$\phi_0^2 = \phi_1\phi_2$ etc.

# Mirror Motets

## The Discriminant

*—Proof-of-Concept—*


arXiv: "real soon, now"

- So: $\mathscr{M}({}^{\nabla}\!F_m^{(n)}[c_1]) \overset{\text{mm}}{\approx} \mathscr{W}(F_m^{(n)}[c_1])$ — easy: 2-dimensional
- In fact, also: $\mathscr{W}({}^{\nabla}\!F_m^{(n)}[c_1]) \overset{\text{mm}}{\approx} \mathscr{M}(F_m^{(n)}[c_1])$

✅…restricted to no (MPCP) blow-ups & "cornerstone" polynomial

- Then, $\dim \mathscr{W}({}^{\nabla}\!F_m^{(n)}[c_1]) = n = \dim \mathscr{M}(F_m^{(n)}[c_1])$
- Same method:

$$e^{2\pi i\, \tilde{\tau}_\alpha} = \prod_{I=0}^{2n} \Big( \sum_{\beta=1}^{2} \tilde{Q}_I^\beta\, \tilde{\sigma}_\beta \Big)^{\tilde{Q}_I^\alpha}$$

$$\tilde{z}_a = \prod_{I=0}^{2n} (a_I\, \varphi_I(x))^{\tilde{Q}_I^\alpha} \Big/ {}^{A}\mathscr{J}$$

$n{=}4$

| $I$ | $(\sum_\beta \tilde{Q}_I^\beta\, \tilde{\sigma}_\beta)$ | $(a_I\, \varphi_I)/{}^{A}\mathscr{J}_{(210)}(f)$ |
|---|---|---|
| 0 | $-2(m{+}2)(\tilde{\sigma}_1 + \tilde{\sigma}_2)$ | $-2\big((a_3\, \varphi_3) + (a_4\, \varphi_4)\big)$ |
| 1 | $m\, \tilde{\sigma}_1 + 2\, \tilde{\sigma}_2$ | $\frac{m\,(a_3\, \varphi_3) + 2\,(a_4\, \varphi_4)}{m+2}$ |
| 2 | $2\, \tilde{\sigma}_1 + m\, \tilde{\sigma}_2$ | $\frac{2\,(a_3\, \varphi_3) + m\,(a_4\, \varphi_4)}{m+2}$ |
| 3 | $(m{+}2)\, \tilde{\sigma}_1$ | $(a_3\, \varphi_3)$ |
| 4 | $(m{+}2)\, \tilde{\sigma}_2$ | $(a_4\, \varphi_4)$ |

# Laurent GLSM Coda


arXiv: "real soon, now"

## Summary

—*Proof-of-Concept*—

- CY($n$−1)-folds in Hirzebruch $n$-folds
  - Euler characteristic ✅
  - Chern class, term-by-term ✅
  - Hodge numbers ✅ (*subtlety!*)
  - Cornerstone polynomials & mirror ✅
  - Phase-space regions & mirror ✅
  - Phase-space discriminant & mirror ✅
  - The "other way around" ✅ (*limited!*)
  - Yukawa couplings ✅
  - World-sheet instantons ✅
  - Gromov-Witten invariants 🔜?✅
    - *Will there be anything else? …being ML-datamined*

$d(\theta^{(k)}) := k!\,\mathrm{Vol}(\theta^{(k)})$ [BH: *signed* by orientation!]

- Oriented polytopes
- Trans-polar$^\nabla$ constr.
- Newton $\Delta_X := (\Delta^\star_X)^\nabla$
- VEX polytopes s.t.: $((\Delta)^\nabla)^\nabla = \Delta$
- Star-triangulable w/flip-folded faces
- Polytope *extension* ⇔ Laurent monomials

B$^3$H$^2$K mirrors

Toric *& GLSM* textbooks to be 😇😈 *…extended*

# Thank You!

https://tristan.nfshost.com/


*Departments of Physics & Astronomy and Mathematics, Howard University, Washington DC*
*Department of Physics, Faculty of Natural Sciences, Novi Sad University, Serbia*
*Department of Mathematics, University of Maryland, College Part, MD*