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Mathematical Methods I
2nd Midterm Exam

Dec. '10.
Solutions (T. Hübsch)

The solutions are presented here with much more detail than was expected of the students' answers in the exam. Hopefully, this will provide additional information and help better understand the material.

1. Equip the collection of real numbers in the interval $(-1, 1)$ with the binary operation

$$v * v' = \frac{v + v'}{1 + v\,v'}, \qquad v, v' \in (-1, 1) \ . \tag{1}$$

a. Prove that $G = \{v \in (-1, 1); *\}$ forms a group. [=10pt]

a.*i*: Closure: we note that the the change of variables

$$v =: \tanh(\nu) \ , \qquad v' =: \tanh(\nu') \tag{2}$$

transforms the defining formula for the binary operation into:

$$v * v' = \frac{v + v'}{1 + vv'} = \frac{\tanh(\nu) + \tanh(\nu')}{1 + \tanh(\nu)\tanh(\nu')} = \tanh(\nu + \nu') \ , \tag{3}$$

where the last equality follows from the addition formulae for the area-functions. Since $|\tanh(\cdots)| < 1$, it follows that $|v * v'| < 1$ also.

Otherwise, you can also argue from the defining formula (1), seeing that $|v + v'| < 2$ and $|1 + vv'| < 2$, after which you need to find an argument that $|1 + vv'| > |v + v'|$.

a.*ii*: Unit element: It is straightforward that

$$0 * v = v = v * 0 \ , \qquad \textbf{\textit{for every}}\ v \in G, \tag{4}$$

so that 0 is the unit element.

a.*iii*: Inverse element: We write v^{-1} for the inverse and require:

$$0 = v^{-1} * v = \frac{v^{-1} + v}{1 + v^{-1}\,v} \ , \quad \Rightarrow \quad v^{-1}* = -v \ , \qquad \text{left-inverse}, \tag{5}$$

$$0 = v * v^{-1} = \frac{v + v^{-1}}{1 + v\,v^{-1}} \ , \quad \Rightarrow \quad *v^{-1} = -v \ , \qquad \text{right-inverse}, \tag{6}$$

so that the 'left-inverse' and the 'right-inverse' of v agree, and equal $-v$.

a.*iv*: Associativity is straightforward, since all the operations (plain multiplication and plain addition) used to define the $*$-operation are.

a.*v*: Although not required, we notice that $v * v' = v' * v$: "$*$" is commutative since "$+$" and "$\cdot$" are.

b. Now we need to explore the limiting cases $(v.v') = (1, 1)$, $(1, -1)$ and $(-1, -1)$, which by the $v \leftrightarrow v'$ symmetry will also include the $(-1, 1)$ case.

b.*i*: Two of the marginal, $v, v' = \pm 1$ cases are straightforward:

$$(+1) * (+1) = \frac{1 + 1}{1 + 1{\cdot}1} = +1, \qquad \text{and} \qquad (-1) * (-1) = \frac{-1 - 1}{1 + (-1){\cdot}(-1)} = -1. \tag{7}$$

b.ii: The two mixed cases are harder:

$$(+1) * (-1) = \frac{1-1}{1+(+1)(-1)} = \frac{0}{0}, \quad \text{undefined.} \tag{8}$$

To explore this, we use the hyperbolic tangent mapping, and note that $v = \tanh(\xi) \to \pm 1$ implies that $\xi \to \pm\infty$. Thus, we may define:

$$(+1) * (-1) := \lim_{\substack{\lambda\to+\infty \\ \mu\to\infty}} \tanh(\lambda - \mu) = \lim_{\lambda\to+\infty} \tanh(\lambda\,(1+\nu)), \qquad \nu := \frac{\mu}{\lambda}. \tag{9}$$

From this, we obtain

$$(+1) * (-1) := \lim_{\substack{\lambda\to+\infty \\ \mu\to-\infty}} \tanh(\lambda + \mu) = \begin{cases} +1 & \text{if} \quad (1+\nu) > 0, \quad \textit{i.e.}, \ -\mu < \lambda, \\ 0 & \text{if} \quad (1+\nu) = 0, \quad \textit{i.e.}, \ -\mu = \lambda, \\ -1 & \text{if} \quad (1+\nu) < 0, \quad \textit{i.e.}, \ -\mu > \lambda. \end{cases} \tag{10}$$

This narrows the $\frac{0}{0}$ expression to three possible results, but remains undetermined amongst them and cannot be resolved unambiguously from within the $*$-structure given here. Knowing, however, that this refers to the relativistic addition of velocities, we can further restrict the possibilities by noting that whatever physical agent travels at the speed of light, $|v| = 1$, cannot have a nonzero rest mass, and must always travel at the speed of light. This rules out the "middle" possibility: that is, $(+1) * (-1) \neq 0$.

If we further assume that a physical object, traveling at the speed of light ($v = +1$), emits light in the reverse direction ($v' = -1$), we *can* that beam of emitted light will be traveling at the speed of light in the reverse direction: $v * v' = -1$. So, additional physical information *can* resolve the ambiguous cases of $(+1) * (-1)$ and $(-1) * (+1)$, but nothing in the definition of "$*$" can.

2. Consider the equilateral triangle in the (x, y)-plane with vertices at $(-\frac{1}{2}, 0), (\frac{1}{2}, 0), (0, \frac{\sqrt{3}}{2})$.

a. List *all* the symmetries of this triangle. [=10pt]

b. Construct the multiplication table of rotational symmetries and prove that they form a group. [=10pt]

c. Construct the complete multiplication table of symmetries and prove that they form a group. [=10pt]

This problem is shamelessly minor modification of Arfken's discussion p. 296–298: the only difference is the positioning of the triangle. The one in the exam is the one in Arfken's Fig. 4.13, shifted up by $\frac{\sqrt{3}}{2}$.

3. Determine the convergence (absolute?, conditional?, uniform? — all that are appropriate) of:

a. Test $S \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n\, n^3}{(1-n^3)}$. [=10pt]

This series is ill defined as given:

$$S \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{(-1)^n\, n^3}{(1-n^3)} \quad \text{has} \quad u_0 = 0, \quad \text{and} \quad u_1 = \text{“}\tfrac{-1}{0}\text{”}, \tag{11}$$

and is really well defined only starting from $n = 2$. The second observation is that

$$u_n = \frac{(-1)^n n^3}{1-n^3} < 0 \qquad \forall n \geq 2, \tag{12}$$

and this is contrary to what was assumed of alternating series in the text. Therefore, we extract an overall -1 and have a redefined

$$S' \stackrel{\text{def}}{=} -\sum_{n=2}^{\infty} \frac{(-1)^n n^3}{n^3-1}. \tag{13}$$

a.*i*: Absolute convergence:
Ignoring the alternating sign (now completely captured by the explicit $(-1)^n$), we can use the integral test,

$$u_n = \frac{n^3}{n^3-1} \to \frac{x^3}{x^3-1}, \quad \text{and check} \quad \int_3^\infty \mathrm{d}x \frac{x^3}{x^3-1}. \tag{14}$$

It is easy to see that this integral diverges, even if you can't quite compute

$$\int \mathrm{d}\frac{x^3}{x^3-1} = x - \tfrac{1}{\sqrt{3}} \tan^{-1}\left(\tfrac{1+2x}{\sqrt{3}}\right) + \tfrac{1}{3}\ln(x-1) - \tfrac{1}{6}\ln(1+x+x^2), \tag{15}$$

which clearly diverges at the upper limit. Thus, S' diverges absolutely.

a.*ii*: Regarding possible conditional convergence, we may apply Leibnitz' criterion, whereby conditional convergence is guaranteed if *both* criteria are satisfied:

1. $|u_{n+1}| < |u_n|$, for all $n > N$,
2. $\lim_{n\to\infty} u_n = 0$.

It is very easy to verify that the second criterion does not hold:

$$\lim_{n\to\infty} \frac{n^3}{n^3-1} = \lim_{n\to\infty} \frac{n^3}{n^3} = 1 \neq 0. \tag{16}$$

Thus, the series S' converges not even conditionally—and, we need not try the first criterion at all.

b. Test $S(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{x^n\,(x+n)^{2x}}{(2n)!}$ for $x \geq 0$, and specify the range/interval/radius of convergence. [=10pt]

b.*i*: This is easiest evaluated using the ratio test:

$$\begin{aligned} 1 \stackrel{?}{>} \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}\,(x+(n+1))^{2x}}{(2(n+1))!}}{\frac{x^n\,(x+n)^{2x}}{(2n)!}} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}\,(x+(n+1))^{2x}(2n)!}{x^n\,(x+n)^{2x}(2n+2)!} \right|, \\ = \lim_{n\to\infty} |x| \left| \frac{(x+n+1)^{2x}}{(x+n)^{2x}} \right| \left| \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \right| &= \lim_{n\to\infty} |x||1|^{2x} \left| \frac{1}{(2n+2)(2n+1)} \right|, \\ = |x| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = |x|{\cdot}0, \quad \text{which is indeed less than 1} \quad &\Rightarrow \quad \text{covnerges.} \end{aligned} \tag{17}$$

Thereby, the series $S(x)$ converges point-by-point for all finite x.

b.*ii*: The previous computation can easily be modified, by majorizing:

$$\text{for all } |x| < s < \infty, \qquad \sum_{n=0}^{\infty} \frac{s^n\,(x+n)^{2s}}{(2n)!} < \infty, \tag{18}$$

so that, by Weierstrass' test, the original series $S(x)$ converges absolutely *and* uniformly for $|x| < s < \infty$.

c. Is the sum $\sum_{k=0}^{\infty}(-2)^k$ summable (in Hardy's "Pickwickian" sense)? If so, what is its value? [=10pt]

Applying the standard "maneuvers", we have:

$$\begin{aligned} S \overset{\text{def}}{=} \sum_{k=0}^{\infty} &= 1 + \sum_{k=1}^{\infty}(-2)^k = 1 + \sum_{n=0}^{\infty}(-2)^{n+1} = 1 + (-2)\sum_{n=0}^{\infty}(-2)^n = 1 - 2S, \\ &= 1 - 2S, \qquad \Rightarrow \qquad 3S = 1, \\ \Rightarrow \quad S &= \tfrac{1}{3}. \qquad \text{Yes, it is evidently summable.} \end{aligned} \tag{19}$$

4. Consider the power series $S(x) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \dfrac{x^k}{(k^2{+}1)(k{+}3)}$,

a. Determine the range and rate of convergence, [=10pt]

a.*i*: Let's test for convergence:

$$\begin{aligned} 1 \overset{?}{>} \lim_{k\to\infty} \left| \frac{\frac{x^{k+1}}{((k+1)^2+1)((k+1)+3)}}{\frac{x^k}{(k^2+1)(k+3)}} \right| &= \lim_{k\to\infty} \left| \frac{x^{k+1}\ (k^2+1)(k+3)}{x^k\ (k^2+2k+2)(k+4)} \right| = |x| \lim_{k\to\infty} \left| \frac{k^3}{k^3} \right|, \\ &= |x|, \quad \text{which is less than one when, well,} \quad |x| < 1. \end{aligned} \tag{20}$$

So, the range of convergence is $|x| < 1$.

a.*ii*: The rate of convergence is $\lim_{x\sim\infty} \frac{x^k}{(k^2+1)(k+3)} \sim \frac{x^k}{k^3}$, *i.e.*, $\sim 1/k^3$.

b. Improve the rate of convergence by at least one order. [=10pt]

To this end, consider the related expression,

$$\begin{aligned} (1+\alpha x)S(x) = (1+\alpha x)\sum_{k=0}^{\infty}\frac{x^k}{(k^2{+}1)(k{+}3)} &= \underbrace{\sum_{k=0}^{\infty}\frac{x^k}{(k^2{+}1)(k{+}3)}}_{k\to n=k} + \alpha \underbrace{\sum_{k=0}^{\infty}\frac{x^{k+1}}{(k^2{+}1)(k{+}3)}}_{k\to n=k+1}, \\ &= \sum_{n=0}^{\infty}\frac{x^n}{(n^2{+}1)(n{+}3)} + \alpha\sum_{n=1}^{\infty}\frac{x^n}{((n{-}1)^2{+}1)((n{-}1){+}3)}, \\ &= \frac{x^0}{(0^2{+}1)(0{+}3)} + \sum_{n=1}^{\infty}\frac{x^n}{(n^2{+}1)(n{+}3)} + \alpha\sum_{n=1}^{\infty}\frac{x^n}{(n^2{-}2n{+}2)(n{+}2)}, \\ &= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{(n^2{-}2n{+}2)(n{+}2) + \alpha(n^2{+}1)(n{+}3)}{(n^2{+}1)(n{+}3)(n^2{-}2n{+}2)(n{+}2)}, \\ &= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{(n^2{-}2n{+}2)(n{+}2) + \alpha(n^2{+}1)(n{+}3)}{(n^2{+}1)(n{+}3)(n^2{-}2n{+}2)(n{+}2)}, \\ &= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{(1+\alpha)n^3 + 3\alpha n^2 + (-2+\alpha)n + 4 + 3\alpha}{(n^2{+}1)(n{+}3)(n^2{-}2n{+}2)(n{+}2)}. \end{aligned} \tag{21}$$

This is now the whole point of the exercise: for $\alpha \neq -1$, this (much more complicated-looking) series still converges as $\sim 1/n^3$: the numerator is a polynomial of order 3 in n, the denominator of order 6. ***BUT***, if we set $\alpha = -1$, the leading, n^3-term in the numerator is gone, and we obtain:

$$= \frac{1}{3} - \sum_{n=1}^{\infty} \frac{3n^2+3n-4+3}{(n^2{+}1)(n{+}3)(n^2{-}2n{+}2)(n{+}2)}, \tag{22}$$

where the last series converges at the rate

$$\lim_{x\sim\infty} \frac{3n^2+3n-4+3}{(n^2{+}1)(n{+}3)(n^2{-}2n{+}2)(n{+}2)} \sim \lim_{x\sim\infty} \frac{3n^2+\dots}{(n^2{+}\dots)(n{+}\dots)(n^2{-}\dots)(n{+}\dots)} \sim \lim_{x\sim\infty} \frac{3n^2}{n^6} \tag{23}$$

that is, the new series converges at the rate $\sim 1/n^4$, which is one order faster than the $\sim 1/k^3$ original.

5. Bonus Problem: Consider the matrix $\mathbb{M} = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 3 \end{smallmatrix}\right]$.

a. Determine characteristic polynomial and eigenvalues of $\mathbb{M}$. [=10pt]

a.*i*: The charateristic polynomial is:

$$\chi_{\mathbb{M}}(\lambda) = \det\big(\left[\begin{smallmatrix} 2 & 3 \\ 4 & 3 \end{smallmatrix}\right] - \lambda\mathbb{1}\big) = \det\left[\begin{smallmatrix} 2-\lambda & 3 \\ 4 & 3-\lambda \end{smallmatrix}\right] = (2-\lambda)(3-\lambda) - 3{\cdot}4 = \lambda^2 - 5\lambda - 6. \tag{24}$$

a.*ii*: The eigenvalues are given by:

$$0 \overset{!}{=} \chi_{\mathbb{M}}(\lambda) = \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6), \qquad \Rightarrow \qquad \lambda_1 = -1, \quad \lambda_2 = 6. \tag{25}$$

b. Determine the corresponding eigenvectors of $\mathbb{M}$. [=10pt]

b.*i*: To this end, solve

$$\left[\begin{smallmatrix} 2 & 3 \\ 4 & 3 \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right] \overset{!}{=} (-1)\left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right], \quad \textit{i.e.}, \quad \left[\begin{smallmatrix} 3 & 3 \\ 4 & 4 \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right] \overset{!}{=} 0, \quad \text{so} \quad \left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right] = x_1\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]. \tag{26}$$

You were *not* asked to normalize the eigenvectors, but if you do so, you should get $|{-1}\rangle = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.

b.*ii*: To this end, solve

$$\left[\begin{smallmatrix} 2 & 3 \\ 4 & 3 \end{smallmatrix}\right]\left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right] \overset{!}{=} (+6)\left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right], \quad \textit{i.e.}, \quad \left[\begin{smallmatrix} -4 & 3 \\ 4 & -3 \end{smallmatrix}\right]\left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right] \overset{!}{=} 0, \quad \text{so} \quad \left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right] = \tfrac{x_2}{3}\left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]. \tag{27}$$

If you do normalize this eigenvector, you should get $|{+6}\rangle = \frac{1}{5}\left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$.

c. Calculate $\sqrt{\mathbb{M}}$, *i.e.*, a matrix the square of which equals $\mathbb{M}$. [=10pt]

Since

$$\det\left[\mathbb{M} - \lambda\mathbb{1}\right] = \lambda^2 - 5\lambda - 6, \tag{28}$$

Cayley-Hamilton's theorem implies that

$$\mathbb{M}^2 - 5\,\mathbb{M} - 6\mathbb{1} = 0\ , \qquad \text{that is,} \qquad \mathbb{M}^2 = 6\,\mathbb{1} + 5\,\mathbb{M}\ , \tag{29}$$

whereby all powers higher than the first, and so all power-series (and, by implicit definition, any analytic function of $\mathbb{M}$!), may be expressed as a linear combination of the zeroth and the first power of $\mathbb{M}$. Therefore, we seek $\sqrt{\mathbb{M}}$ in that same form:

$$\sqrt{\mathbb{M}} = \alpha\,\mathbb{1} + \beta\,\mathbb{M}\ , \qquad \text{with } \alpha, \beta \in \mathbb{C}. \tag{30}$$

Squaring this produces:

$$\mathbb{M} \stackrel{!}{=} \alpha^2\,\mathbb{1} + 2\alpha\beta\,\mathbb{M} + \beta^2\,\mathbb{M}^2 \;=\; \alpha^2\,\mathbb{1} + 2\alpha\beta\,\mathbb{M} + \beta^2\big(6\,\mathbb{1} + 5\,\mathbb{M}\big) \\ = (\alpha^2 + 6\beta^2)\mathbb{1} + (2\alpha + 5\beta)\beta\mathbb{M}\ , \tag{31}$$

so that

$$0 \stackrel{!}{=} (\alpha^2 + 6\beta^2)\,\mathbb{1} + (2\alpha\beta - 1 + 5\beta^2)\,\mathbb{M}\ . \tag{32}$$

Since $\mathbb{1}$ is diagonal and $\mathbb{M}$ is not, writing (32) as a matrix-equation and focusing on either of the two off-diagonal terms yields:

$$2\alpha\beta - 1 + 5\beta^2 = 0\ , \qquad \Rightarrow \qquad \alpha = \frac{1 - 5\beta^2}{2\beta}\ . \tag{33}$$

In case you have a hard time picturing this, write it out:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (\alpha^2 + 6\beta^2) + 2(2\alpha\beta - 1 - 3\beta^2) & 3(2\alpha\beta - 1 + 5\beta^2) \\ 4(2\alpha\beta - 1 + 5\beta^2) & (\alpha^2 + 6\beta^2) + 3(2\alpha\beta - 1 + 5\beta^2) \end{bmatrix}, \tag{34}$$

where you see that the vanishing of the off-diagonal elements enforces Eq. (33).

Substituting this back into (34), yields:

$$\frac{1 - 10\beta^2 + 25\beta^4}{4\beta^2} - 6\beta^2 = \frac{1 - 10\beta^2 + 25\beta^4 - 24\beta^4}{4\beta^2} = 0, \qquad \text{or} \qquad \beta^4 - 10\beta^2 + 1 = 0. \tag{35}$$

The solutions of this bi-quadratic equation are:

$$\beta^2 = 5 + 2\sigma_2\sqrt{6}, \qquad \text{so} \qquad \beta = \sigma_1\sqrt{5 + 2\sigma_2\sqrt{6}}, \tag{36}$$

where σ_1, σ_2 are two *independent* sign choices—that is, there are *four* solutions for β! Thereupon,

$$\alpha = \frac{1 - 5(5 + 2\sigma_2\sqrt{6})}{2\sigma_1\sqrt{5 + 2\sigma_2\sqrt{6}}} = \frac{1 - 25 - 10\sigma_2\sqrt{6}}{2\sigma_1\sqrt{5 + 2\sigma_2\sqrt{6}}} = -\sigma_1\frac{12 + 5\sigma_2\sqrt{6}}{\sqrt{5 + 2\sigma_2\sqrt{6}}}\ , \tag{37}$$

where the two *independent* sign-choices, $\sigma_1, \sigma_2 = \pm 1$ provide for a total of *four* solutions:

$$\sqrt{\mathbb{M}} = \begin{bmatrix} \alpha + 2\beta & 3\beta \\ 4\beta & \alpha + 3\beta \end{bmatrix}, \qquad \alpha = -\sigma_1\frac{12 + 5\sigma_2\sqrt{6}}{\sqrt{5 + 2\sigma_2\sqrt{6}}}, \quad \beta = \sigma_1\sqrt{5 + 2\sigma_2\sqrt{6}}, \qquad \sigma_1, \sigma_2 = \pm 1. \tag{38}$$

— ⋆ —

For an arbitrary 2×2 matrix[1], the characteristic polynomial is always $\lambda^2 - \mathrm{Tr}[\mathbb{M}]\lambda + \det[\mathbb{M}]$. Denoting $T := \mathrm{Tr}[\mathbb{M}]$ and $\Delta := \det[\mathbb{M}]$, Cayley-Hamilton's theorem implies that

$$\mathbb{M}^2 = T\,\mathbb{M} - \Delta\mathbb{1}. \tag{39}$$

Then, writing $\sqrt{\mathbb{M}} = \alpha\mathbb{1} + \beta\mathbb{M}$ as before, we get the condition generalizing (32) to be

$$0 \stackrel{!}{=} (\alpha^2 - \Delta\beta^2)\mathbb{1} + (2\alpha\beta - 1 + T\beta^2)\mathbb{M}. \tag{40}$$

[1] This addition is motivated by one Student's solution; you may wish to look up a much more complete treatment in Sam Northshield's *Square Roots of* 2×2 *Matrices*, Contemporary Mathematics 517 (2010) 289-304, also available on=line, at `http://faculty.plattsburgh.edu/sam.northshield/sqrtmat3.pdf`.

As before, we make the second parenthetical coefficient vanish by setting $\alpha = \frac{1-T\beta^2}{2\beta}$, so that the vanishing of the first parenthetical coefficient yields

$$0 \stackrel{!}{=} \frac{(1-T\beta^2)^2}{4\beta^2} - \Delta\beta^2 = \frac{1-2T\beta^2+(T^2-4\Delta)\beta^4}{4\beta^2}, \qquad \Rightarrow \qquad \beta = \frac{\sigma_1}{\sqrt{T+\sigma_2 2\sqrt{\Delta}}}. \tag{41}$$

Then,

$$\alpha = \frac{1-\frac{T}{T+\sigma_2 2\sqrt{\Delta}}}{\frac{2}{\sqrt{T+\sigma_2\sqrt{\Delta}}}} = \frac{\frac{T+\sigma_2 2\sqrt{\Delta}-T}{T+\sigma_2 2\sqrt{\Delta}}}{\frac{2}{\sqrt{T+\sigma_2\sqrt{\Delta}}}} = \frac{\sigma_2\sqrt{\Delta}\,\sqrt{T+\sigma_2\sqrt{\Delta}}}{T+\sigma_2 2\sqrt{\Delta}} = \frac{\sigma_2\sqrt{\Delta}}{\sqrt{T+\sigma_2 2\sqrt{\Delta}}}. \tag{42}$$

To summarize,

$$\sqrt{\mathbb{M}} = \alpha\mathbb{1} + \beta\mathbb{M}, \qquad T := \mathrm{Tr}[\mathbb{M}], \qquad \Delta := \det[\mathbb{M}], \tag{43}$$

$$\alpha = \frac{\sigma_2\sqrt{\Delta}}{\sqrt{T+\sigma_2 2\sqrt{\Delta}}}, \qquad \beta = \frac{\sigma_1}{\sqrt{T+\sigma_2 2\sqrt{\Delta}}}, \qquad \sigma_1,\sigma_2 = \pm 1. \tag{44}$$

Appearances to the contrary, this provides exactly the same four separate solutions (38).

—VERIFY—