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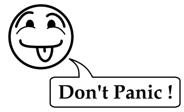

Mathematical Methods I

Midterm 1: 2010, Oct. 4. Solutions

The solutions are presented here with much more detail than was expected of the students' answers in the exam. Hopefully, this will provide additional information and help understanding the material more fully.

1. Given a vector $\vec{A} = \sin(\varphi)\hat{\mathsf{e}}_z$, so specified in cylindrical coordinates ($h_\rho = 1 = h_z$ and $h_\varphi = \rho$),

a. Calculate $\vec{\nabla}\cdot\vec{A}$. [=5*pt*]

Since $\vec{A} = \sin(\varphi)\hat{\mathsf{e}}_z$, we have that $A_\rho = 0 = A_\varphi$ and that $A_z = \sin(\varphi)$. Then, in cylindrical coordinates,

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\Big[\frac{\partial(0\cdot\rho\cdot 1)}{\partial\rho} + \frac{\partial(1\cdot 0\cdot 1)}{\partial\rho} + \frac{\partial\big(1\cdot\rho\cdot\sin(\varphi)\big)}{\partial z}\Big] = 0\ . \tag{1}$$

b. Calculate $\vec{\nabla}\times\vec{A}$. [=5*pt*]

Similarly,

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathsf{e}}_\rho & \rho\,\hat{\mathsf{e}}_\varphi & r\hat{\mathsf{e}}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 1\cdot 0 & \rho\cdot 0 & 1\cdot\sin(\varphi) \end{vmatrix} = \frac{1}{\rho}\big[\hat{\mathsf{e}}_\rho\frac{\partial}{\partial\varphi}\sin(\varphi) - \rho\hat{\mathsf{e}}_\varphi\frac{\partial}{\partial\rho}\sin(\varphi) + \hat{\mathsf{e}}_z\,0\big]\ , \tag{2}$$

$$= \frac{1}{\rho}\cos(\varphi)\,\hat{\mathsf{e}}_\rho. \tag{3}$$

c. Calculate the three components of $\vec{\nabla}^2\vec{A}$. [=20*pt*]

We'll present the calculation both using the identity $\vec{\nabla}^2\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$ and directly, using the formulae in the text. Using the former identity simplifies using the above results:

$$\begin{aligned}\vec{\nabla}^2\vec{A} &= \vec{\nabla}(0) - \vec{\nabla}\times\Big(\frac{1}{\rho}\cos(\varphi)\,\hat{\mathsf{e}}_\rho\Big), \\ &= -\frac{1}{\rho}\begin{vmatrix} \hat{\mathsf{e}}_\rho & \rho\,\hat{\mathsf{e}}_\varphi & \hat{\mathsf{e}}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ \big(\frac{1}{\rho}\cos(\varphi)\big) & 0 & 0 \end{vmatrix} = -\frac{1}{\rho}\Big[-\rho\hat{\mathsf{e}}_\varphi\Big(-\frac{\partial}{\partial z}\frac{1}{\rho}\cos(\varphi)\Big) + \hat{\mathsf{e}}_z\Big(-\frac{\partial}{\partial\varphi}\frac{1}{\rho}\cos(\varphi)\Big)\Big], \\ &= -\frac{1}{\rho}\Big[\hat{\mathsf{e}}_z\big[-\frac{1}{\rho}\big(-\sin(\varphi)\big)\big]\Big] \quad = \quad -\frac{1}{\rho}\sin(\varphi)\,\hat{\mathsf{e}}_z. \end{aligned} \tag{4}$$

In turn, using Arfken's Eqs. (2.37) and that $A_\rho = 0 = A_\varphi$ and $A_z = \sin(\varphi)$,

$$\vec{\nabla}^2\vec{A}|_\rho = 0\ , \tag{5}$$

$$\vec{\nabla}^2\vec{A}|_\varphi = 0\ , \tag{6}$$

$$\vec{\nabla}^2\vec{A}|_z = \big(\vec{\nabla}^2\,\sin(\varphi)\big) = \Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\Big]\sin(\varphi) = -\frac{1}{\rho^2}\sin(\varphi)\ , \tag{7}$$

which is in agreement with (4).

2. Calculate $I := \oint_S \mathrm{d}\vec{\sigma} \times \big(\hat{z}(x^2+y^2)^n\big)$ for $n \in \mathbb{Z}$, where S is a pill-box of radius R and height H, body-centered at the origin:

a. Performing the surface integral directly: [=10*pt*]

There are three surfaces to the pill-box, the flat disc-like surfaces of radius $\rho \in [0, R]$ and positioned at heights $z = \pm\frac{1}{2}H$ and with normals $\pm\hat{z}$, respectively, and the cylinder with radius $\rho = R$, extending $z \in [-\frac{1}{2}H, +\frac{1}{2}H]$, and with normal $\hat{\rho}$, we have:

$$
\begin{aligned}
I &= \Big(\int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi\; \underbrace{\hat{z}\big)\times\big(\hat{z}}_{=0}\; \rho^{2n}\big) + \Big(\int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi\; \underbrace{(-\hat{z})\big)\times\big(\hat{z}}_{=0}\; \rho^{2n}\big) \\
&\quad + \bigg[\Big(\int_{-H/2}^{H/2} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi\; \underbrace{\hat{\rho}\big)\times\big(\hat{z}}_{=-\hat{\varphi}}\; \rho^{2n}\big)\bigg]_{\rho=R,\ \text{on the side}}, \\
&= H\,R^{2n} \int_0^{2\pi} \mathrm{d}\phi\; \hat{\varphi} \;=\; 0, \qquad \text{because } \hat{\varphi} \text{ rotates uniformly around the circle and contributions from the opposite sides cancel each other out.}
\end{aligned} \tag{8}
$$

b. Upon applying an appropriate integration/derivative identity: [=10*pt*]

Using Gauss's theorem, we have that

$$
\begin{aligned}
I &:= \oint_S \mathrm{d}\vec{\sigma} \times \big(\hat{z}(x^2+y^2)^n\big) = \int_V \mathrm{d}^3\vec{r}\; \vec{\nabla} \times \big(\hat{z}\,\rho^{2n}\big) = \int_V \mathrm{d}^3\vec{r}\; \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\,\hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & \rho^{2n} \end{vmatrix}, \\
&= \int_{-H/2}^{H/2} \mathrm{d}z \int_0^R \rho\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi\; \Big(-\hat{\varphi}\,2n\,\rho^{2n-1}\Big) = 2n\,H \int_0^R \rho^{2n}\mathrm{d}\rho \underbrace{\int_0^{2\pi} \mathrm{d}\varphi\;\hat{\varphi}}_{=0},
\end{aligned} \tag{9}
$$

where the last integral vanishes again because $\hat{\varphi}$ rotates uniformly around the circle and contributions from opposite points of the circle precisely cancel.

The unbelieving Student may replace $\hat{\varphi} = \cos(\varphi)\hat{\mathsf{e}}_y - \sin(\varphi)\hat{\mathsf{e}}_x$ and use that $\hat{\mathsf{e}}_x$ and $\hat{\mathsf{e}}_y$ are constant:

$$
\int_0^{2\pi} \mathrm{d}\varphi\; \Big[\cos(\varphi)\hat{\mathsf{e}}_y - \sin(\varphi)\hat{\mathsf{e}}_x\Big] = \hat{\mathsf{e}}_y \int_0^{2\pi} \mathrm{d}\varphi\; \cos(\varphi) - \hat{\mathsf{e}}_x \int_0^{2\pi} \mathrm{d}\varphi\; \sin(\varphi) = 0. \tag{10}
$$

c. Is any value of n exceptional? Explain. [=5*pt*]

The integrand is ill-defined when $2n - 1 < 0$, *i.e.*, when $n < \frac{1}{2}$.

3. Consider a (generalized) coordinate system (ξ, η, ζ) which is related to the Cartesian system (x, y, z) through the relations

$$
x = \xi\,\eta, \quad y = \tfrac{1}{2}(\eta^2 - \xi^2), \quad z = \zeta\;.
$$

a. Calculate the (inverse) transformation matrix $\mathbb{J} = \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}$. [=10*pt*]

The transformation matrix is

$$\mathbb{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \eta & \xi & 0 \\ -\xi & \eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

b. Calculate the metric, $g_{ij}(\xi,\eta,\zeta)$, for the (ξ,η,ζ) coordinate system. [=10*pt*]

The metric is defined as

$$\begin{aligned} g_{jk} &:= \sum_{i=1}^{3} \frac{\partial x^i}{\partial q^j}\frac{\partial x^i}{\partial q^k} = \frac{\partial x}{\partial q^j}\frac{\partial x}{\partial q^k} + \frac{\partial y}{\partial q^j}\frac{\partial y}{\partial q^k} + \frac{\partial z}{\partial q^j}\frac{\partial z}{\partial q^k}, \\ g_{\xi\xi} &= \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \xi} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \xi} = (\eta)(\eta) + (-\xi)(-\xi) + 0{\cdot}0 = \xi^2 + \eta^2, \\ g_{\xi\eta} &= \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} = (\eta)(\xi) + (-\xi)(\eta) + 0{\cdot}0 = 0, \\ g_{\xi\zeta} &= \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \zeta} = (\eta)(0) + (-\xi)(0) + 0{\cdot}1 = 0, \end{aligned}$$

and so on. The end result is:

$$\big[\, g_{ij} \,\big] = \begin{bmatrix} (\xi^2{+}\eta^2) & 0 & 0 \\ 0 & (\xi^2+\eta^2) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{J}^T\,\mathbb{J}. \tag{12}$$

c. Determine if the (ξ,η,ζ) system is orthogonal or not. Explain. [=10*pt*]

The coordinate system (ξ,η,ζ) is orthogonal since $g_{jk}(\xi,\eta,\zeta)$ is diagonal.

d. State the relationship between $\mathbb{J}$ and the matrix $[g_{ij}(\xi,\eta,\zeta)]$. [=5*pt*]

As used above, $[\,g_{jk}(\xi,\eta,\zeta)\,] = \mathbb{J}^T\,\mathbb{J}$. To prove this, just multiply the defining expression of $\mathbb{J}$, given in Eq. (11), with its transpose—and realize that the result equals (12).

4. For $i,j = 1,2,3$, A_i, B^j are components of a covariant and a contravariant vector, and C^{kl} are the components of a type-$(2,0)$ tensor.
These statements amount to the following transformation rules with respect to a change of variables $x^i \to \xi^i(x)$:

$$\begin{gathered} A_i(x) \mapsto A'_i(\xi) = \frac{\partial x^j}{\partial \xi^i} A_j(x), \qquad B^i(x) \mapsto B'^i(\xi) = \frac{\partial \xi^i}{\partial x^j} B^j(x), \\ C^{ij}(x) \mapsto C'^{ij}(\xi) = \frac{\partial \xi^i}{\partial x^k}\frac{\partial \xi^j}{\partial x^\ell} C^{k\ell}(x). \end{gathered} \tag{13}$$

a. Determine the (tensorial) transformation properties of $(A_i C^{ij} B^k)$. [=5*pt*]

Using Eqs. (13)—and ensuring that each dummy index is written as a different letter, we obtain:

$$(A'_i C'^{ij} B'^k) = \Big(\frac{\partial x^\ell}{\partial \xi^i} A_\ell\Big)\Big(\frac{\partial \xi^i}{\partial x^m}\frac{\partial \xi^j}{\partial x^n} C^{mn}\Big)\Big(\frac{\partial \xi^k}{\partial x^p} B^p\Big),$$

$$
\begin{aligned}
&= \Big(\frac{\partial x^\ell}{\partial \xi^i}\frac{\partial \xi^i}{\partial x^m}\Big)\frac{\partial \xi^j}{\partial x^n}\frac{\partial \xi^k}{\partial x^p}(A_\ell C^{mn}B^p),\\
&= \delta^\ell_m\,\frac{\partial \xi^j}{\partial x^n}\frac{\partial \xi^k}{\partial x^p}(A_\ell C^{mn}B^p),\\
&= \frac{\partial \xi^j}{\partial x^n}\frac{\partial \xi^k}{\partial x^p}(A_\ell C^{\ell n}B^p), \qquad \text{rank-2, type-}(2,0)\text{ tensor.}
\end{aligned} \tag{14}
$$

b. Determine the (tensorial) transformation properties of $\varepsilon^{ijk}A_i\big(\frac{\partial B^m}{\partial x^j}\big)$ with respect to *general* coordinate changes. [=5*pt*]

Besides using Eqs. (13), we will also need to figure out the transformation properties of ε^{ijk}. To this end, we note that the determinant of the transformation and its inverse is:

$$
J := \det[\mathbb{J}] \quad = \det\Big[\frac{\partial x}{\partial \xi}\Big] = \tfrac{1}{3!}\,\widetilde{\varepsilon}^{ijk}\varepsilon_{\ell mn}\,\frac{\partial x^\ell}{\partial \xi^i}\frac{\partial x^m}{\partial \xi^j}\frac{\partial x^n}{\partial \xi^k}, \tag{15}
$$

$$
\frac{1}{J} := \det[\mathbb{J}^{-1}] = \det\Big[\frac{\partial \xi}{\partial x}\Big] = \tfrac{1}{3!}\,\varepsilon^{ijk}\widetilde{\varepsilon}_{\ell mn}\,\frac{\partial \xi^\ell}{\partial x^i}\frac{\partial \xi^m}{\partial x^j}\frac{\partial \xi^n}{\partial x^k}, \tag{16}
$$

from which it follows that

$$
\widetilde{\varepsilon}^{\ell mn} = J\,\varepsilon^{ijk}\,\frac{\partial \xi^\ell}{\partial x^i}\frac{\partial \xi^m}{\partial x^j}\frac{\partial \xi^n}{\partial x^k}, \tag{17}
$$

which shows that ε^{ijk} transforms *almost* like a rank-3, type-$(3,0)$ tensor: the only difference is the pre-factor $J = \det[\mathbb{J}]$; ε^{ijk} is not a tensor, but a rank-3, type-$(3,0)$ *tensor density* of weight 1—since J occurs to the power 1 in the transformation rule.

Using these, we obtain:

$$
\begin{aligned}
\widetilde{\varepsilon}^{ijk}\widetilde{A}_i\Big(\frac{\partial \widetilde{B}^m}{\partial \xi^j}\Big) &= \Big(J\,\varepsilon^{\ell np}\,\frac{\partial \xi^i}{\partial x^\ell}\frac{\partial \xi^j}{\partial x^n}\frac{\partial \xi^k}{\partial x^p}\Big)\Big(\frac{\partial x^r}{\partial \xi^i}A_r\Big)\bigg(\frac{\partial x^s}{\partial \xi^j}\frac{\partial}{\partial x^s}\Big(\frac{\partial \xi^m}{\partial x^t}B^t\Big)\bigg),\\
&= J\,\varepsilon^{\ell np}\,\frac{\partial \xi^k}{\partial x^p}\Big(\frac{\partial \xi^i}{\partial x^\ell}\frac{\partial x^r}{\partial \xi^i}\Big)\Big(\frac{\partial \xi^j}{\partial x^n}\frac{\partial x^s}{\partial \xi^j}\Big)A_r\Big(\frac{\partial \xi^m}{\partial x^t}\frac{\partial B^t}{\partial x^s} + \frac{\partial^2 \xi^m}{\partial x^s \partial x^t}B^t\Big),\\
&= J\,\varepsilon^{\ell np}\,\frac{\partial \xi^k}{\partial x^p}\,\delta^r_\ell\,\delta^s_n\,A_r\,\frac{\partial \xi^m}{\partial x^t}\frac{\partial B^t}{\partial x^s} + J\,\varepsilon^{\ell np}\,\frac{\partial \xi^k}{\partial x^p}\,\delta^r_\ell\,\delta^s_n\,A_r\,\frac{\partial^2 \xi^m}{\partial x^s \partial x^t}B^t,\\
&= J\,\frac{\partial \xi^k}{\partial x^p}\frac{\partial \xi^m}{\partial x^t}\Big(\varepsilon^{\ell np}\,A_\ell\,\frac{\partial B^t}{\partial x^n}\Big) + J\,\frac{\partial \xi^k}{\partial x^p}\Big(\varepsilon^{\ell np}\,A_\ell\,\frac{\partial^2 \xi^m}{\partial x^n \partial x^t}B^t\Big).
\end{aligned} \tag{18}
$$

Owing to this second term, the quantity $\varepsilon^{ijk}A_i\big(\frac{\partial B^m}{\partial x^j}\big)$ is not a tensor, and not even a tensor density with respect to *general* coordinate transformations! However, if the transformation $x^i \mapsto \xi^i(x)$ is *linear*, then $\frac{\partial \xi}{\partial x}$ is a matrix of constant elements, and $\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} = 0$, so that this second term vanishes.

Thus, with respect to *general* coordinate changes,

$$
\boldsymbol{X}^{km} := \varepsilon^{ijk}A_i\Big(\frac{\partial B^m}{\partial x^j}\Big) \tag{19}
$$

is neither a tensor nor even a tensor density. However, with respect to *linear* coordinate changes, it is a *scalar density*: it transforms as a rank-0, type-$(0,0)$ tensor density of weight-1 (that's weight-one, not "weight *minus* one").

Note that the tensorial nature of a quantity inextricably depends on the class of transformations with respect to which the tensorial nature is being specified.

c. Write down two *algebraically* independent scalars constructed from A_i, B^j and C^{kl}. [=10*pt*]

By *scalar*—and without qualification as to the class of coordinate transformations, we *must* mean 'invariant with respect to general coordinate transformations'. Then, it is not hard to prove, following the above methods, that

$$Z := A{\cdot}B := (A_i\,B^i), \qquad \text{and} \qquad W := A{\cdot}C{\cdot}A := (A_i\,C^{ij}\,A_j) \tag{20}$$

are two such quantities. In fact, any quantity of the form $Z^\alpha\,W^\beta$—for arbitrary α, β—would be just as invariant. However, all quantities $Z^\alpha\,W^\beta$ are constructed *algebraically* from Z and W, and so are *not algebraically independent* from Z and W. It then follows that Z and W are the only two algebraically independent scalars constructed from A_i, B^j and $C_{k\ell}$.

d. How many independent components does the set of quantities $\varepsilon_{ijk}B^iC^{j\ell}A_\ell$ represent? [=10*pt*]

It is not hard to verify that

$$\widetilde{\boldsymbol{V}}_k := (\tilde{\varepsilon}_{\,ijk}\widetilde{B}^{\,i}\widetilde{C}^{\,j\ell}\widetilde{A}_\ell) = J\frac{\partial x^m}{\partial \xi^k}(\varepsilon_{ijm}B^iC^{j\ell}A_\ell) = J\frac{\partial x^m}{\partial \xi^k}\boldsymbol{V}_m. \tag{21}$$

That is, the quantity $\boldsymbol{V}_k := (\varepsilon_{ijk}B^iC^{j\ell}A_\ell)$ is a contravariant vector density: a rank-1, type-$(1,0)$ tensor density of weight-1. It has a single free index, which may take any value from among $\ell = 1, 2, 3$[1]. Therefore, there are three independent components: $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$.

STURGEON GENERAL'S WARNING: Tests like this one may look deceptively like one of the previous year's tests. The differences are subtle, but make it easy to identify the Student who decided to indiscriminately copy the solution of a past year's test and so make a fool of themself.

[1]The fact that the indices can take precisely three distinct values is implied by the use of the rank-3 Levi-Civita symbol, ε^{ijk}. In general, the number of indices on such a symbol must equal the number of distinct values which those indices can assume, for a formula such as Eq. (15) to even make sense.