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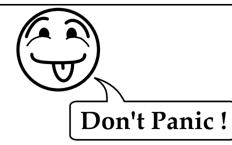

Judicious Changes of Variables

1. Some Known Examples

Several differential equations can be solved rather easily upon a judicious change of variables. For example, consider the d-dimensional Bessel equation:

$$\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{d-1}\frac{\mathrm{d}R}{\mathrm{d}r}\Big) + \big(k^2 - \frac{Q}{r^2}\big)R \;=\; 0\;, \tag{1.1}$$

where $R = R(r)$ is the sought-for function or r, and k, Q certain suitable constants. This equation occurs, for example, upon separating the d-dimensional Helmholz equation, *i.e.*, the d-dimensional analogue of $\big[\vec{\nabla}^2 - k^2\big]\psi = 0$ (for the analysis of the latter one, see chapter 2 of Arfken [1]).

Straightforward differentiation of the above equation yields

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \Big(k^2 - \frac{Q}{r^2}\Big)R \;=\; 0\;. \tag{1.2}$$

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Consider now a change of the dependent function: $R(r) = r^\alpha P(r)$, whereby

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}r} &= r^\alpha\Big(\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{\alpha}{r}P\Big)\;,\\ \frac{\mathrm{d}^2R}{\mathrm{d}r^2} &= r^\alpha\Big(\frac{\mathrm{d}^2P}{\mathrm{d}r^2} + \frac{2\alpha}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{\alpha(\alpha-1)}{r^2}P\Big)\;,\end{aligned}$$

Substituting this into Eq. (1.2), and upon some straightforward simplifications, we obtain

$$r^\alpha\bigg[\frac{\mathrm{d}^2P}{\mathrm{d}r^2} + \frac{2\alpha+d-1}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \Big(k^2 - \frac{Q-\alpha(d+a-2)}{r^2}\Big)P\bigg] \;=\; 0\;. \tag{1.4}$$

Now, r^α will vanish only for $r=0$ if $\alpha>0$, and for $r=\infty$ if $\alpha<0$, so we may drop it and examine the remaining differential equation for $P(r)$:

$$\frac{\mathrm{d}^2P}{\mathrm{d}r^2} + \frac{2\alpha+d-1}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \Big(k^2 - \frac{Q-\alpha(d+a-2)}{r^2}\Big)P \;=\; 0\;. \tag{1.5}$$

1.1. The very special cases

We notice that the equation (1.5) simplifies when $\alpha = -\frac{d-1}{2}$, in which case the first order derivative drops out:

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \Big(k^2 - \frac{4Q + (d-1)(d-3)}{4r^2}\Big)P \;=\; 0\;. \tag{1.6}$$

Finally, we find the extremely simple case where also $Q = -(d-1)(d-3)/4$, and when

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + k^2 P \;=\; 0\;. \tag{1.7}$$

This is a simple differential equation with constant coefficients, and is solved by $P(r) = A\sin(kr+\delta)$, where A, δ are two undetermined constants. Going back to Eq. (1.1), we find that its special case

$$\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{d-1}\frac{\mathrm{d}R}{\mathrm{d}r}\Big) + \big(k^2 + \frac{(d-1)(d-3)}{4r^2}\big)R \;=\; 0\;, \tag{1.8}$$

are solved by $R(r) = Ar^{-\frac{(d-1)}{2}}\sin(kr+\delta)$, which does look like a considerable feat.

On the other hand, when $\alpha = 1 - \frac{d}{2}$, the equation (1.5) simplifies into:

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \Big(k^2 - \frac{4Q + (2-d)^2}{4r^2}\Big)P \;=\; 0\;, \tag{1.9}$$

which is solved by $P(r) = AJ_\mu(kr) + BN_\mu(kr)$, where $\mu = \pm\sqrt{Q + (\frac{d}{2}-1)^2}$, and $J_\mu(kr)$ and $N_\mu(kr)$ are the *cylindrical* Bessel functions of the first and second kind.

Finally, when $\alpha = -\frac{d-3}{2}$, the equation (1.5) simplifies into:

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \Big(k^2 - \frac{4Q + (d-1)(d-3)}{4r^2}\Big)P \;=\; 0\;, \tag{1.10}$$

which is solved by $P(r) = Aj_\mu(kr) + Bn_\mu(kr)$, where $\mu = \pm\sqrt{Q + (d-1)(d-3)/4}$, and $j_\mu(kr)$ and $n_\mu(kr)$ are the *spherical* Bessel functions of the first and second kind.

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There is another relatively simple case, which can be seen straight from the original equation (1.1). When $k = 0$, the equation becomes homogeneous. That is, should we replace $r \to \lambda r$ for some non-zero constant λ and leave everything else the same, the equation stays the same. More precisely, both terms in Eq (1.1) pick up a multiplicative constant λ^{-2}. This however can be factored out, and being non-zero, can be cancelled. For a homogeneous equation, it is consistent to look for $R(r) = r^\alpha$; indeed, each term in the equation now scales as $\lambda^{\alpha-2}$, which again is an overall constant and can easily be cancelled [1)]. Straightforward calculation yields:

$$\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{d-1}\frac{\mathrm{d}r^\alpha}{\mathrm{d}r}\Big) - \frac{Q}{r^2}r^\alpha \;=\; 0\;,$$

[1)] Were it not for this homogeneity, looking for $R(r) = r^\alpha$ could not possibly work.

$$
\begin{aligned}
\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\Big(\alpha r^{\alpha+d-2}\Big) - Q r^{\alpha-2} &= 0 \ , \\
\frac{\alpha(\alpha+d-2)}{r^{d-1}} r^{\alpha+d-3} - Q r^{\alpha-2} &= 0 \ , \\
\Big(\alpha(\alpha+d-2) - Q\Big) r^{\alpha-2} &= 0 \ .
\end{aligned}
$$

Thus, the exponent α is determined as the solution of

$$
\alpha^2 + (d-2)\alpha - Q \ = \ 0 \ , \tag{1.12}
$$

that is,

$$
R(r) = r^{\alpha}, \qquad \alpha_{\pm} \ = \ \frac{2-d}{2} \pm \sqrt{\frac{(d-2)^2}{4} + Q} \ , \tag{1.13}
$$

are the two solutions of Eq. (1.1), when $k = 0$.

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