On Rockets

1. The General Equation

Begin with the straightforward application of the third Newton's law

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} \ . \tag{1.1}$$

If we assume that the momentum (and its change) are both in the direction of the (total) force acting on the rocket, the vector notation can be omitted.

Expanding the derivative on the left, where $p = mv$, and assuming that the two forces acting on the rocket are the thrust force (proportional to the change of the mass owing to the depletion of the fuel) and the gravitiational force given in the familiar Newton-Coulomb form, we obtain

$$\Big(\frac{\mathrm{d}m}{\mathrm{d}t}\Big)\Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big) + m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -f\frac{\mathrm{d}m}{\mathrm{d}t} - \frac{gR^2m}{(R+x)^2} \ . \tag{1.2}$$

The sign of the thrust force is set negative, so that its value would be positive as the fuel is *depleted*, *i.e.*, when $\frac{\mathrm{d}m}{\mathrm{d}t}$ is negative. We have also used that the Newton-Coulomb formula for gravity is

$$F_{\text{grav.}}(x) = G_N\frac{Mm}{(R+x)^2} \ , \tag{1.3}$$

where G_N is Newton's constant, M the mass and R the radius of the planet off which the rocket is launched. Since at the surface ($x = 0$), the gravitational force is mg, (with g the planet's gravitational acceleration), we have

$$F_{\text{grav.}}(0) = G_N\frac{Mm}{R^2} = mg \ , \tag{1.4}$$

whereupon $G_N M = gR^2$, which we have used in (1.2). The coefficient f stands for the factor of efficiency and has the dimensions of speed.

Notice that Eq. (1.2) is a second order non-linear differential equation for $x(t)$, depending on the function $m(t)$; alternatively, it may be viewed as a linear, first order differential equation for $m(t)$, depending on the function $x(t)$.

2. Constant Acceleration

For reasons or relative comfort of the passengers, we may want to ensure that the rocket moves at constant acceleration, say ag, where a is some constant. As the acceleration is set to be this constant, we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = ag \ , \tag{2.1a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = agt + v_0 \qquad = agt \ , \tag{2.1b}$$

$$x(t) = \tfrac{1}{2}agt^2 + v_0t + x_0 = \tfrac{1}{2}agt^2 \ , \tag{2.1c}$$

where we have set $v_0 = 0 = x_0$ so as to describe lift-off from the surface and with no initial speed. Put in (1.2), this produces

$$\Big(\frac{\mathrm{d}m}{\mathrm{d}t}\Big)agt + mag = -f\frac{\mathrm{d}m}{\mathrm{d}t} - \frac{gR^2m}{(R+\frac{1}{2}agt^2)^2}\ . \tag{2.2}$$

This differential equation is separable and we soon obtain

$$\frac{\mathrm{d}m}{m} \;=\; -g\mathrm{d}t\frac{(a+(1+\frac{ag}{2R}t^2)^{-2})}{agt+f}\ . \tag{2.3}$$

This can be solved by strainghtorward integration on both sides (albeit the integral on the right hand side is really pesky), and we will quote the *Mathematica* solution below.

Before that, however, the differential equation (and the right-hand-side integral above) simplifies in two regimes, for very early times ($t \approx 0^+$) and for very long times ($t \to +\infty$).

In the first case, we only keep the leading terms, and so obtain

$$\frac{\mathrm{d}m}{m} \;=\; -g\mathrm{d}t\frac{(a+(1+\frac{ag}{2R}t^2)^{-2})}{agt+f} \;\approx\; -g\mathrm{d}t\frac{(a+1)}{f}\ , \tag{2.4}$$

whereby we find that $m(t)$ should start out as $m(t) \approx m_0 e^{-(a+1)gt/f}$, where $m_0 = m(0)$ is the initial total mass of the rocket.

On the other hand, for very tong times, we have

$$\frac{\mathrm{d}m}{m} \;=\; -g\mathrm{d}t\frac{(a+(1+\frac{ag}{2R}t^2)^{-2})}{agt+f} \;\approx\; -g\mathrm{d}t\frac{(a+0)}{agt} = -\frac{\mathrm{d}t}{t}\ , \tag{2.5}$$

so that the mass of the rocket approaches asymptotically $m(t) \approx \mu_\infty/t$, where μ_∞ is a suitable constant.

Finally, it is clear form (2.3), that $m(t)$ is a monotonically decreasing function of time, as expected on physical grounds.

Mathematica solves (2.3) and (after a little simplification) produces

$$\begin{aligned}\exp\Bigg\{\frac{1}{2}\Bigg(&\frac{-4gR^2}{(f^2+2agR)(2R+agt^2)} - \frac{2fgRt}{(f^2+2agR)(2R+agt^2)}\\ &- \frac{4fgR\sqrt{2agR}\arctan(\sqrt{\frac{ag}{2R}}\,t)}{(f^2+2agR)^2} - \frac{f\sqrt{2gR}\arctan(\sqrt{\frac{ag}{2R}}\,t)}{\sqrt{a}(f^2+2agR)}\\ &+ 2\mathrm{C}(1) - 2\log(f+agt)\\ &+ \frac{4f^2gR\log(f+agt)}{(f^2+2agR)^2} - \frac{2f^2gR\log(2R+agt^2)}{(f^2+2agR)^2}\\ &- \frac{4gR\log(f+agt)}{f^2+2agR} + \frac{2gR\log(2R+agt^2)}{f^2+2agR}\Bigg)\Bigg\}\ .\end{aligned} \tag{2.6}$$

This can be manipulated into a more reasonable and compact form:

$$
\begin{aligned}
m(t) &= \frac{C}{A(t)} \left[\frac{B(t)}{[A(t)]^2}\right]^{\frac{4gR^3}{D^2T^2}} e^{-\frac{gR(2R+ft)}{D\,B(t)}}\; e^{-\frac{gfT(D+8R^2/T^2)}{2\,D^2}\arctan(t/T)} \\
A(t) &\stackrel{\rm def}{=} f + 2Rt/T^2\;, \qquad B(t) \stackrel{\rm def}{=} 2R(1+t^2/T^2)\;, \\
D &\stackrel{\rm def}{=} f^2 + 4R^2/T^2\;, \qquad T \stackrel{\rm def}{=} \sqrt{\frac{2R}{ag}}\;, \qquad C \stackrel{\rm def}{=} m_0\, f\, e^{\frac{gR}{D}} \left[\frac{f^2}{2R}\right]^{\frac{4gR^3}{D^2T^2}}\;,
\end{aligned}
\tag{2.7}
$$

where the constant—C(1) in *Mathematica*'s raw result, equal to $\ln C$ of Eq. (2.7)—is fixed so that indeed $m(0) = m_0$. Note the appearance of the planet's characteristic time-scale, $T \stackrel{\rm def}{=} \sqrt{2R/ag}$—it depends only on the constants characteristic of the planet. Also, the constant $\sqrt{D} \stackrel{\rm def}{=} \sqrt{f^2+4R^2/T^2}$ may be thought of as a shifted factor of efficiency; the shift roughly corresponding to the increase in the fuel depletion required to countermand the gravitational force (at initial time).

The $t \to \infty$ limiting behavior of (2.7) is easy to read off, as B/A^2 tend to a constant, $ft/B \to 0$, and so A^{-1} remains, recovering the qualitative behavior of the solution of (2.5). The exponential behavior at $t \to 0$ is less straightforward, as it appears at a sub-leading orders when expanding (2.7); so there we learn more (and more easily) from the limiting form of the differential equation, (2.4), than from the solution (2.7).

References

[1] G. Arfken: *Mathematical Methods for Physicists*, (Academic Press, New York, 1985).