☞ **$2 \times 1 = 1 \times 2$**

Or, how a system of two first-order differential equations equals a single second-order differential equation.

Consider the pet example system of two first-order differential equations

$$\frac{\mathrm{d}w}{\mathrm{d}t} = a\,w - c\,wr\,, \tag{1a}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -b\,r + k\,wr\,. \tag{1b}$$

A solution of this system consists of a pair of functions of time, $\big(w(t), r(t)\big)$. In fact, since we have two first-order differential equations, we need to perform two integrations and so expect two constants. Thus, the general solution will be a pair of two functions $\big(w(t; C_1, C_2), r(t; C_1, C_2)\big)$ which both depend on time, t, but also the two integration constants, C_1, C_2.

Suppose that we can somehow solve for one of the two functions, say $w(t)$. Then, Eq. ($1a$) determines the other function, $r(t)$, *completely.* That is, given a solution for $w(t)$, we can solve (algebraically, i.e, without integration) Eq. ($1a$) for $r(t)$ and obtain that

$$c\,wr = a\,w - \frac{\mathrm{d}w}{\mathrm{d}t}\,,$$

whence

$$c\,r(t) = a - \frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t}\,. \tag{2}$$

So, if we only knew $w(t)$, and of course its derivative, $r(t)$ would be completely determined.

To obtain a (differential) equation for $w(t)$ alone, we will try to elliminate $r(t)$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$ in terms of $w(t)$ and its derivatives. To elliminate two quantities, we should need two relations and that's precisely what we have in ($1a, b$); so, if it were possible to elliminate both r and $\frac{\mathrm{d}r}{\mathrm{d}t}$ from these two equations alone, we would be left with no equation for w.

To introduce a new equation, take the derivative of Eq. ($1a$):

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} = a\,\frac{\mathrm{d}w}{\mathrm{d}t} - c\Big[\frac{\mathrm{d}w}{\mathrm{d}t}r + w\frac{\mathrm{d}r}{\mathrm{d}t}\Big]\,. \tag{1c}$$

Next note that Eq. ($1b$) determins $\frac{\mathrm{d}r}{\mathrm{d}t}$ in terms of $w(t)$, its derivative and $r(t)$. On the other hand, Eq. ($1a$) provides $r(t)$ in terms of only $w(t)$ and its derivative[1)], as solved in (2). Therefore, we first elliminate $\frac{\mathrm{d}r}{\mathrm{d}t}$ from Eq. ($1c$), using Eq. ($1b$):

$$\begin{aligned}\frac{\mathrm{d}^2w}{\mathrm{d}t^2} &= a\,\frac{\mathrm{d}w}{\mathrm{d}t} - c\Big[\frac{\mathrm{d}w}{\mathrm{d}t}r + w(-b\,r + k\,wr)\Big]\,,\\ &= a\,\frac{\mathrm{d}w}{\mathrm{d}t} - (c\,r)\Big[\frac{\mathrm{d}w}{\mathrm{d}t} - b\,w + k\,w^2\Big]\,.\end{aligned} \tag{3}$$

[1)] **Caution:** Eq. (2) can be used to elliminate $r(t)$ only if $c \neq 0$! If $c = 0$, Eq. ($1a$) is already independent of $r(t)$ and may be solved readily for $w(t)$.

Finally, we elliminate $c\,r(t)$ from here, using Eq. (2):

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} \;=\; a\,\frac{\mathrm{d}w}{\mathrm{d}t} \;-\; \Big[a - \frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t}\Big]\Big[\frac{\mathrm{d}w}{\mathrm{d}t} \;-\; b\,w \;+\; k\,w^2)\Big]\;, \tag{4}$$

that is,

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} \;=\; ab\,w \;-\; ak\,w^2 \;-\; b\frac{\mathrm{d}w}{\mathrm{d}t} \;+\; k\,w\frac{\mathrm{d}w}{\mathrm{d}t} \;+\; \frac{1}{w}\Big(\frac{\mathrm{d}w}{\mathrm{d}t}\Big)^2\;. \tag{5}$$

So, indeed, the system of two first-order differential equations ($1a,b$)—with $c \neq 0$—is shown to be equivalent to a single second-order differential equation (5). Solving this, we would determine one of the functions, $w(t)$ and introduce two integration constants; the other function, $r(t)$ is then determined completely from Eq. (2).

Now, the second-order differential equation (5) indeed looks formidable and it is dubious whether this offers any practical simplification in comparison to the innocent-looking system ($1a,b$). The important point is however, the equivalence between a system of two first order differential equations and a single second-order one.

You may think of running this procedure "backwards" also. Suppose you were given the unpleasantly looking second order differential equation (5), for a single function $w(t)$. Then Eq. (2) *defines* a new (auxiliary) function, $r(t)$, with the use of which we can produce the system ($1a,b$). Of course, viewed this way, the choice of the new function (2) may seem somewhat unsuspected and artificial; note, however that Eq. (5) may be re-written as (3), whence the introduction of the new variable $r(t)$ as in (2) does seem …well,… natural.

Sometimes, the system of first-order equations is relatively easily found and more easily analyzed then the original second-order equation. Sometimes, the second order equation is easier to solve or analyze. Finally, some questions pertain to the derivatives of $w(t)$ and $r(t)$, rather then the functions themselves. For example, the equilibrium states are most easily found by setting, in Eq. ($1a,b$), $\frac{\mathrm{d}w}{\mathrm{d}t} = 0$, $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$. Issues pertaining to the curvature (concave *vs.* convex, etc.) are most easily dealt with using the second-order differential equation (5).

In general, this method of converting a system of two first-order differential equations into a single second-order equation can be repeated for much more general forms of the r.h.s. of Eqs. ($1a,b$), as long as *either* Eq. ($1a$) can be solved for r in terms of w and $\frac{\mathrm{d}w}{\mathrm{d}t}$ *or* Eq. ($1b$) can be solved for w in terms of r and $\frac{\mathrm{d}r}{\mathrm{d}t}$.