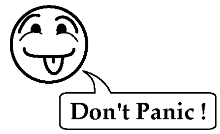
HOWARD UNIVERSITY
WASHINGTON, D.C. 20059

DEPARTMENT OF PHYSICS AND ASTRONOMY
(202)-806-6245 (Main Office)
(202)-806-5830 (FAX)

2355 Sixth Str., NW, TKH Rm.215
thubsch@howard.edu
(202)-806-6257



Mathematical Methods II 6th March '98.
Quizz Solution by T. Hübsch

1. Given the generating function

$$g(x,t;a,b) \;=\; \frac{e^{ax^2t}}{1+bt} \;\stackrel{\rm def}{=}\; \sum_{n=0}^{\infty} A_n(x;a,b)t^n \;,$$

where a, b are some constants (see below), determine at least 20 points' worth of the following:

a. a contour-integral formula for the A_n; [$=5pt$]

b. a series representation for the A_n (expanding e^{ax^2t} and $(1+bt)^{-1}$ separately, then combining the two sums); [$=5pt$]

c. a recurrence relation by operating with $\frac{\partial}{\partial t}$ on the generating function; [$=5pt$]

d. a recurrence relation by operating with $\frac{\partial}{\partial x}$ on the generating function; [$=5pt$]

e. the differential equation that the A_n satisfy; [$=5pt$]

f. the integral $\int_{-\infty}^{\infty} {\rm d}x A_m A_n$, by integrating $g(x,t;a,b)g(x,s;a,b)$—assuming $\Re e\, a < 0$ (why?), and re-expanding the answer in s, t. (No orthogonality of the A_n's is established!) [$=15pt$]

---

a. From the definition of the A_n's, we see that they are proportional to the coefficients in the Taylor (MacLaurin) series:

$$A_n(x;a,b) = \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n} g(x,t;a,b)\right]_{t=0} \;,$$

which is easy to rewrite, using Cauchy's integral formula, as

$$A_n(x;a,b) = \frac{1}{2\pi i}\oint_C \frac{{\rm d}z\; g(x,z;a,b)}{z^{n+1}} = \frac{1}{2\pi i}\oint_C \frac{{\rm d}z\; e^{ax^2z}}{(1+bz)z^{n+1}} \;.$$

The contour C is chosen in any convenient way, as long as it circumscribes $z=0$ once, in a counter-clockwise manner.

b. The individual expansions of e^{ax^2t} and of $(1+bt)^{-1}$ are straightforward:

$$g(x,t;a,b) = \sum_{k,l=0}^{\infty} \frac{(ax^2t)^k}{k!}\;(-bt)^l \;.$$

Seeing that the powers of t combine into t^{k+l}, we introduce $n \stackrel{\rm def}{=} (k{+}l)$, and substitute $l = n{-}k$. Now, since $l \geq 0$, it follows that also $n{-}k \geq 0$, so that $n \geq k$, and k acquires a finite upper limit:

$$g(x,t;a,b) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{a^k x^{2k}(-b)^{n-k}}{k!}\right]\; t^n \;.$$

Comparing with the defining equation for the A_n's, the quantity in the square brackets is seen to equal the A_n. Introducing a new summation variable $m \stackrel{\text{def}}{=} (n{-}l)$:

$$A_n(x;a,b) = \sum_{k=0}^{n} \frac{a^k x^{2k}(-b)^{n-k}}{k!} \ ,$$

or

$$A_n(x;a,b) = (-b)^n \sum_{k=0}^{n} (-1)^k \frac{(ax^2/b)^k}{k!} \ .$$

This makes it obvious that $A_n(x;a,b)$ is a degree-$2n$ polynomial in x.

c. Operating with $\frac{\partial}{\partial t}$ on the left-hand side of the defining equation:

$$\begin{aligned} \frac{\partial}{\partial t}\frac{e^{ax^2t}}{1+bt} &= ax^2\frac{e^{ax^2t}}{1+bt} - b\frac{e^{ax^2t}}{(1+bt)^2} = \frac{ax^2+abx^2t-b}{1+bt}\cdot\frac{e^{ax^2t}}{1+bt} \ , \\ &= \frac{ax^2+abx^2t-b}{1+bt}\sum_{n=0}^{\infty} A_n t^n \ . \end{aligned} \tag{1}$$

On the other hand,

$$\frac{\partial}{\partial t}\sum_{n=0}^{\infty} A_n t^n = \sum_{n=0}^{\infty} nA_n t^{n-1} \ .$$

Equating these two and multiplying through by $1{+}bt$, we obtain:

$$(ax^2-b)\sum_{n=0}^{\infty} A_n t^n + abx^2\sum_{n=0}^{\infty} A_n t^{n+1} = \sum_{n=0}^{\infty} nA_n t^{n-1} + b\sum_{n=0}^{\infty} nA_n t^n \ ,$$

or (shifting the dummy summation indices to identify like powers of t):

$$\sum_{n=-1}^{\infty}(n{+}1)A_{n+1}t^n + \sum_{n=0}^{\infty}(bn+b-ax^2)A_n t^n - abx^2\sum_{n=1}^{\infty} A_{n-1}t^n \ = \ 0 \ .$$

Setting the (combined) coefficients of the various powers of t to zero, we obtain

$$\begin{aligned} @t^{-1} &: 0{\cdot}A_0 \ = \ 0 \ , \\ @t^{0} &: 1{\cdot}A_1 + [b{\cdot}0 + b - ax^2]A_0 \ = \ 0 \ , \\ @t^{k} &: (k{+}1)A_{k+1} + [b(k{+}1) - ax^2]A_k - abx^2 A_{k-1} \ = \ 0 \ , \qquad k \geq 1 \ . \end{aligned} \tag{2}$$

This yields the recurrence relation:

$$(k{+}1)A_{k+1} = [ax^2 - b(k{+}1)]A_k + abx^2 A_{k-1} \ , \tag{3}$$

which includes all of the above if we set $A_k \equiv 0$ for $k < 0$.

d. Similarly:

$$\frac{\partial}{\partial x}\frac{e^{ax^2t}}{1+bt} = 2axt\,\frac{e^{ax^2t}}{1+bt} = 2ax\sum_{n=0}^{\infty} A_n t^{n+1} \ . \tag{4}$$

On the other hand,

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} A_n t^n = \sum_{n=0}^{\infty} A'_n t^n \ .$$

Equating these two, we obtain:

$$2ax\sum_{n=0}^{\infty} A_n t^{n+1} = \sum_{n=0}^{\infty} A'_n t^n \ ,$$

or (shifting the dummy summation indices to identify like powers of t):

$$2ax\sum_{n=1}^{\infty} A_{n-1} t^{n} = \sum_{n=0}^{\infty} A'_n t^n \ ,$$

Setting the (combined) coefficients of the various powers of t to zero, we obtain

$$@t^0 \ : 0 \ = \ A'_0 \ ,$$
$$@t^k \ : 2axA_{k-1} \ = \ A'_k \ , \qquad k > 0 \ , \tag{5}$$

the latter of which is the second required recurrence relation.

e. From (5),

$$A_{k-1} \ = \frac{1}{2ax} A'_k \ , \tag{5a}$$
$$A'_{k+1} \ = 2axA_k \ . \tag{5b}$$

Substituting (5a) into (3), we obtain

$$(k{+}1)A_{k+1} = [ax^2 - b(k{+}1)]A_k + \frac{bx}{2}A'_k \ , \tag{3a}$$

the derivative of which is

$$(k{+}1)A'_{k+1} = 2axA_k + [ax^2 - b(k{+}1)]A'_k + \frac{b}{2}A'_k + \frac{bx}{2}A''_k \ . \tag{3b}$$

In this, we eliminate A'_{k+1}, using (5b) and obtain the desired (2nd order) differential equation for the $A_k(x;a,b)$:

$$\frac{bx}{2}A''_k + [ax^2 - b(k{+}\tfrac{1}{2})]A'_k - 2akxA_k \ = \ 0 \ . \tag{3c}$$

Note that the 'no derivative' term depends on parameters a and k, both of which appear in the derivative terms. Hence, this can be identified with Sturm-Liouville type equation

$$\mathscr{L}u + \lambda w(x)u \ = \ 0 \ ,$$

only in the too restricted sense, when $\lambda = 0$ and (before making $\mathscr{L}$ self-adjoint)

$$\mathscr{L} = \frac{bx}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + [ax^2 - b(k{+}\tfrac{1}{2})]\frac{\mathrm{d}}{\mathrm{d}x} - 2akx \ .$$

Since this includes only the $\lambda = 0$ case, the major utility of the Sturm-Liouville theory (the use of the complete set of solutions, labeled by λ) is lost. In particular, we cannot infer any orthogonality relation between the A_k's.

f. We multiply the desired integral, $\int_{-\infty}^{\infty} \mathrm{d}x\, A_m(x)A_n(x)$ by $t^m s^n$ and sum over n, m:

$$\sum_{m,n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\ A_m(x)t^m\, A_n(x)s^n = \int_{-\infty}^{\infty} \mathrm{d}x\, \Big[\sum_{m=0}^{\infty} A_m(x)t^m\Big]\Big[\sum_{n=0}^{\infty} A_n(x)s^n\Big]\ , \tag{6a}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x\ \frac{e^{ax^2 t}}{1+bt}\frac{e^{ax^2 s}}{1+bs} = \frac{1}{1+bt}\frac{1}{1+bs}\int_{-\infty}^{\infty} \mathrm{d}x\ e^{ax^2(t+s)}\ , \tag{6b}$$

$$= \frac{1}{1+bt}\frac{1}{1+bs}\sqrt{\frac{\pi}{-a(t+s)}}\ . \tag{6c}$$

From (6a) to (6b), we used that the A_n's were defined as the coefficient functions in the expansion of the generating functions. From (6b) to (6c), we evaluated the integral by substituting $\xi = -ax^2(t{+}s)$ which brings the integral into the Γ-function form:

$$\int_{-\infty}^{\infty} \mathrm{d}x\ e^{ax^2(t+s)} = 2\int_{0}^{\infty} \mathrm{d}x\ e^{ax^2(t+s)} = 2\int_{0}^{\infty} \frac{\xi^{\frac{1}{2}-1}\mathrm{d}\xi}{2\sqrt{-a(t+s)}}\ e^{-\xi} = \frac{\Gamma(\frac{1}{2})}{\sqrt{-a(t+s)}}$$

In (6b), it is clear that $\Re e\, a < 0$ ensures convergence of these integrals. It now remains to re-expand the right-hand side of Eq. (6c) as a power series in s, t and identify the coefficients of the various s, t-monomials as the values of the corresponding integrals on the left-hand side of (6a). This we leave as an easy exercise for the diligent student.

Notice however, that *the right-hand side of* (6c) *is not analytic*: it blows up when $t{+}s = 0$! Yet, the left-hand side started out as a manifestly analytic power series (no negative powers). Therefore, it must be that the integrals diverge. Indeed, this must be the case, since the A_n's are polynomials, and so are the products $A_m A_n$; the integrals over infinite limits necessarily diverge. It is easy to remedy this by considering instead integrals of the type

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-cx^2} A_m(x)A_n(x)\ , \qquad \Re e\, c > 0\ .$$

Then, the calculation as above produces

$$\sum_{m,n=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-cx^2} A_m(x)t^m\, A_n(x)s^n = \frac{1}{1+bt}\frac{1}{1+bs}\sqrt{\frac{\pi}{c-a(t+s)}}\ , \tag{7}$$

which now is finite at $t, s = 0$ and has a well-defined (double) Taylor series in s, t. The above integrals are now calculated by re-expanding the left-hand side of (7) and identifying the coefficients in the double power series with the corresponding integrals on the left.

The rate at which the original integrals in (6a) diverge can now be determined by taking the limit $c \to 0$.

2. Attempt solving the differential equation

$$\alpha x^3 y'' + \beta x y' - \gamma(1+x)y \ = \ 0$$

in the form $y = \sum_{k=0}^{\infty} c_k x^{k+s}$ although $x{=}0$ is an essential singularity.

a. Determine the values(s) of s. [=2*pt*]

b. Determine the recursion relation. [=4*pt*]

c. Find a choice of α, β, γ such that the above series is a valid solution. [=4*pt*]

We substitute the series form of the solution and obtain

$$\alpha\sum_{k=0}^{\infty} c_k(k{+}s)(k{+}s{-}1)x^{k+s+1} + \beta\sum_{k=0}^{\infty} c_k(k{+}s)x^{k+s} - \gamma\sum_{k=0}^{\infty} c_k x^{k+s} - \gamma\sum_{k=0}^{\infty} c_k x^{k+s+1} = 0\ .$$

In the first and the last sum, we shift $k \to k{-}1$ and combine:

$$\sum_{k=1}^{\infty} c_{k-1}\big[\alpha(k{+}s{-}1)(k{+}s{-}2) - \gamma\big]x^{k+s} + \sum_{k=0}^{\infty} c_k\big[\beta(k{+}s) - \gamma\big]x^{k+s} = 0\ .$$

Compared to the first sum, the second one has an extra term when $k = 0$, which we can write separately and combine the rest:

$$c_0\big[\beta s{-}\gamma\big]x^s + \sum_{k=1}^{\infty}\Big[c_{k-1}\big[\alpha(k{+}s{-}1)(k{+}s{-}2){-}\gamma\big] - \big[\gamma{-}\beta(k{+}s)\big]\Big]x^{k+s} = 0\ .$$

Different powers of x being linearly independent, the numerical coefficient in front of each power has to vanish separately.

a. Since $c_0{\neq}0$ (the series must begin somewhere), the vanishing of the coefficient of x^s imposes

$$\beta s - y = 0\ , \qquad \textit{i.e.}, \qquad s = \frac{\gamma}{\beta}\ .$$

b. The vanishing of the coefficients in front of x^{k+s} in the infinite sum produces the recursion relation

$$c_k = c_{k-1}\frac{\alpha(k{+}s{-}1)(k{+}s{-}2){-}\gamma}{\gamma{-}\beta(k{+}s)}\ .$$

Using that $s = \gamma/\beta$ in the denominator and shifting $k \to k{+}1$, this simplifies a little:

$$c_{k+1} = -c_k\frac{\alpha(k{+}s)(k{+}s{-}1){-}\gamma}{\beta(k+1)}\ . \tag{8}$$

c. It is easy to see that for large k, the ratio of successive terms in the series becomes

$$\lim_{k\to\infty}\left|\frac{c_{k+1}x^{k+s+1}}{c_k x^{k+s}}\right| = \lim_{k\to\infty}\left|\frac{\alpha(k{+}s)(k{+}s{-}1){-}\gamma}{\beta(k+1)}\right| = \frac{\alpha}{\beta}\lim_{k\to\infty} k = \infty\ ,$$

whence the series diverges, as expected since we expanded about an essential singularity. However, if for some $k = K$, the numerator in the recursion relation (8) should happen to vanish, so would c_{K+1}, and thereupon all higher coefficients. The series would terminate and become a finite polynomial (up to an overall negative power of x), a perfectly well-defined expression. So,

$$\alpha\big(K{+}\frac{\gamma}{\beta}\big)\big(K{+}\frac{\gamma}{\beta}{-}1\big){-}\gamma \;=\; 0\ ,$$

or

$$K_{\pm} = \frac{1}{2} - \frac{\gamma}{\beta} \pm \sqrt{\frac{1}{4} + \frac{\gamma}{\alpha}} \ .$$

Since k ranges over integers, at least one of the two solutions $K_{\pm}$ must also be an integer, which happens only for select values of α, β, γ.

For example, this happens when $\alpha = 3$, $\beta = -2$ and $\gamma = 6$. Then $K_{-} = 2$ and $K_{+} = 5$, $s = -3$ and the recursion relation is

$$c_{k+1} = c_k \frac{3}{2} \frac{(k-2)(k-5)}{(k+1)} \ ,$$

so the coefficients turn out to be: $c_1 = 15c_0$, $c_2 = 3c_1 = 45c_0$, $c_3 = 0$ and so $c_k = 0$ for $k \geq 3$. That is, the differential equation

$$3x^3 y'' - 2xy' - 6(1+x)y \ = \ 0$$

is solved by $y = c_0(\frac{1}{x^3} + \frac{15}{x^2} + \frac{45}{x})$. This then is and example of the limitation (termination) of the series solution.

Note that although we obtained two solutions, $K_{\pm}$, for possible limiting values of k, only one of them (K_{-}) corresponds to a solution to the differential equation, and so we only obtain one solution. The other limiting value cannot be reached as the series already becomes limited by the lower one. On the other hand, if K_{-} turned out to be negative, only K_{+} would correspond to a solution to the differential equation. The other solution must be obtained either by expanding about a different point or by using the general (integral) formula for the second solution.

3. For the differential equation

$$x^2(x^2-1)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \ + \ (x+1)\frac{\mathrm{d}y}{\mathrm{d}x} \ - \ 4!\,y \ = \ 0 \ , \tag{9}$$

answer the following questions (circle the correct option in 1–3, write ‘yes’ or ‘no’ for 4, 5):

1: Is the point $x = 0$ a *smooth*, or a *regular-singular*, or an *irregular singular* point ?

2: Is the point $x = 1$ a *smooth*, or a *regular-singular*, or an *irregular singular* point ?

3: Is the point $x = 4$ a *smooth*, or a *regular-singular*, or an *irregular singular* point ?

4: Can Eq. (9) be solved by the method of series in the form $y = \sum_{k=0}^{\infty} a_k x^{k+s}$?

5: Can Eq. (9) be solved by the method of series in the form $y = \sum_{k=0}^{\infty} a_k (x-1)^{k+s}$?

4. For the differential equation $R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = 0$,

a. find $I = I(t)$; [=5*pt*]

b. determine the constant of integration in part a. so that $C = 10^{-2}F$, $R = 10^6\Omega$ and $I(0) = 1A$. [=5*pt*]

---

a. The differential equation is separable, for it can be rewritten as

$$R\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{1}{C}I \;, \qquad i.e., \qquad \frac{\mathrm{d}I}{I} = -\frac{1}{RC}\mathrm{d}t \;,$$

which is easy to integrate directly

$$\int\frac{\mathrm{d}I}{I} = -\frac{1}{RC}\int\mathrm{d}t \;, \qquad \Rightarrow \qquad \ln(I) = K - \frac{t}{RC} \;,$$

where K is the integration constant. Exponentiating both sides, we obtain the general solution,

$$I(t) = \mathrm{e}^{K-t/RC} \;. \tag{10}$$

b. To determine the integration constant, we substitute the $t{=}0$ values as given (remember that "$\stackrel{!}{=}$" denotes an equality that we are imposing):

$$1A \stackrel{!}{=} I(0) = \mathrm{e}^{K-\frac{0}{RC}} = \mathrm{e}^{K} \;,$$

so $K = 0$ seems to be the numerical value of the integration constant. However, there is a problem with this! The expression e^K must be dimensionless, as must K be. To see this, note that $\mathrm{e}^K = 1 + K + \frac{1}{2}K^2 + \ldots$, so that K must have the units of the number '1'—that is, no units at all. It is than plain impossible to equate a dimensionless quantity e^K to the constant $1A$.

Instead, a little maneuver in writing the general solution will save our face. Write $K = \ln(\kappa)$, whereupon the general solution becomes $I(t) = \kappa\mathrm{e}^{-t/RC}$. Now this makes perfect (physics/engineering/...) sense, since κ is again the integration constant in a simple disguise, but can easily be assigned a value with the required dimensions. Indeed:

$$1A \stackrel{!}{=} I(0) = \kappa\mathrm{e}^{-\frac{0}{RC}} = \kappa \;,$$

Thus, in this case, $\kappa = 1A$, and has the physical meaning of the value of the current at the time $t = 0$. More generally, we can write $\kappa = I(0)$, so that the general solution is more appropriately written as

$$I(t) = I(0)\mathrm{e}^{-t/RC} \;. \tag{11}$$

5. Find at least one singular point of the differential equation

$$\sin(\frac{\theta}{2})\frac{\mathrm{d}^2f(\theta)}{\mathrm{d}\theta^2} + \cot(\theta)f(\theta) \;=\; 0 \;, \tag{12}$$

and determine whether it is a regular or an essential singularity. $(0 \leq \theta \leq \pi)$

A. First bring the second order differential equation into the "standard" form:

$$\frac{\mathrm{d}^2f}{\mathrm{d}\theta^2} + \Big[\frac{\cot(\theta)}{\sin(\frac{\theta}{2})}\Big]f \;=\; 0 \;, \tag{13}$$

from where we identify the "standard" coefficients

$$P(\theta) \equiv 0 \;, \qquad Q(\theta) = \frac{\cot(\theta)}{\sin(\frac{\theta}{2})} = \frac{\cos(\theta)}{\sin(\theta)\sin(\frac{\theta}{2})} \;. \tag{14}$$

$P(\theta)$ is clearly finite, while $Q(\theta)$ diverges whenever $\sin(\theta)\sin(\frac{\theta}{2})$ vanishes—which happens at $\theta = 0$ (both factors vanish) and at $\theta = \pi$ (only the first factor vanishes). Therefore, the points $\theta = 0, \pi$ are singular.

To determine the type of the singularity, check if $(\theta - \theta_0)^2 Q(\theta)$ is finite as $\theta \to \theta_0$. At $\theta_0 = 0$,

$$(\theta - \theta_0)^2 Q(\theta) \;=\; \theta^2 \frac{\cos(\theta)}{\sin(\theta)\sin(\frac{\theta}{2})} \;\xrightarrow{\theta\to 0}\; \theta^2 \frac{(1+\ldots)}{(\theta+\ldots)(\frac{\theta}{2}+\ldots)} \;\to\; 2 \;, \tag{15}$$

which is finite and wherefore $\theta = 0$ is a *regular singular point.*

At $\theta_0 = \pi$, introduce $\vartheta \stackrel{\text{def}}{=} (\theta - \pi)$, so

$$\begin{aligned}(\theta - \pi)^2 Q(\theta) \;&=\; \vartheta^2 \frac{\cos(\vartheta+\pi)}{\sin(\vartheta+\pi)\sin(\frac{\vartheta+\pi}{2})} \;=\; \vartheta^2 \frac{[-\cos(\vartheta)]}{[-\sin(\vartheta)][\cos(\frac{\vartheta}{2})]} \;, \\ &\xrightarrow{\theta\to 0}\; \vartheta^2 \frac{(1+\ldots)}{(\vartheta+\ldots)(1+\ldots)} \;\to\; 0 \;,\end{aligned} \tag{16}$$

which is finite, and $\theta = \pi$ also is a *regular singular point.*

Instead of the small-angle expansions $\sin\theta \approx \theta$ and $\cos\theta \approx 1 + \ldots$, one could have applied L'Hospital's rule (taking the second derivative of the numerator and of the denominator at $\theta = 0$ and taking the first derivatives at $\theta = \pi$). The result is the same.

6. Find the singular points of the differential equation

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + k^2 y \;=\; 0 \;, \tag{17}$$

and find a solution in form of a power series.

A. First bring the second order differential equation into the "standard" form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}k^2 y \;=\; 0 \;, \tag{18}$$

from where we identify the "standard" coefficients

$$P(x) \equiv 0 \;, \qquad Q(x) = \frac{1}{x} \;. \tag{19}$$

$P(x)$ is clearly finite, while $Q(x)$ diverges only at $x = 0$. Therefore, among finite values of x, only the point $x = 0$ is singular. Furthermore, the coefficients are $P(x) \equiv 0$, while $Q(x)$ blows up, linearly. Therefore, $x^2 Q(x)$ is finite,

For $x = \infty$, we substitute $z = 1/x$, and use that

$$\begin{aligned}\widetilde{P}(z) &= \Big[\frac{2}{z} - \frac{P(\frac{1}{z})}{z^2}\Big] = \Big[\frac{2}{z} - \frac{0}{z^2}\Big] = \frac{2}{z} \;, \\ \widetilde{Q}(z) &= \frac{Q(\frac{1}{z})}{z^4} = \frac{z}{z^4} = z^{-3} \;.\end{aligned} \tag{20}$$

At $x=\infty$, $z=0$, and there $\widetilde{P}(0)\equiv 0$ is finite, but $\widetilde{Q}(z)$ blows up cubically. Therefore $z^2\widetilde{Q}(z)$ also blows up and $x=\infty$ ($z=0$) is an essential singularity.

To determine the type of the singularity, check if $(\theta-\theta_0)^2Q(\theta)$ is finite as $\theta\to\theta_0$. At $\theta_0=0$,

$$(\theta-\theta_0)^2Q(\theta) \;=\; \theta^2\frac{\cos(\theta)}{\sin(\theta)\sin(\frac{\theta}{2})} \;\xrightarrow{\theta\to 0}\; \theta^2\frac{(1+\ldots)}{(\theta+\ldots)(\frac{\theta}{2}+\ldots)} \;\to\; 2\;, \tag{21}$$

which is finite and wherefore $\theta=0$ is a *regular singular point.*

At $\theta_0=\pi$, introduce $\vartheta \stackrel{\text{def}}{=} (\theta-\pi)$, so

$$\begin{aligned}(\theta-\pi)^2Q(\theta) \;&=\; \vartheta^2\frac{\cos(\vartheta+\pi)}{\sin(\vartheta+\pi)\sin(\frac{\vartheta+\pi}{2})} \;=\; \vartheta^2\frac{[-\cos(\vartheta)]}{[-\sin(\vartheta)][\cos(\frac{\vartheta}{2})]}\;,\\ &\xrightarrow{\theta\to 0}\; \vartheta^2\frac{(1+\ldots)}{(\vartheta+\ldots)(1+\ldots)} \;\to\; 0\;,\end{aligned} \tag{22}$$

which is finite, and $\theta=\pi$ also is a *regular singular point.*

Instead of the small-angle expansions $\sin\theta\approx\theta$ and $\cos\theta\approx 1+\ldots$, one could have applied L'Hospital's rule (taking the second derivative of the numerator and of the denominator at $\theta=0$ and taking the first derivatives at $\theta=\pi$). The result is the same.

7. For the differential equation

$$x^2(x^2-1)\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \;+\; (x+1)\,\frac{\mathrm{d}y}{\mathrm{d}x} \;-\; 4!\,y \;=\; 0\;, \tag{23}$$

answer the following questions (circle the correct option in 1–3, write 'yes' or 'no' for 4, 5):

1: Is the point $x=0$ a *smooth*, or a *regular-singular*, or an *irregular singular* point ?

2: Is the point $x=1$ a *smooth*, or a *regular-singular*, or an *irregular singular* point ?

3: Is the point $x=4$ a *smooth*, or a *regular-singular*, or an *irregular singular* point ?

4: Can Eq. (23) be solved by the method of series in the form $y=\sum_{k=0}^{\infty}a_k x^{k+s}$?

5: Can Eq. (23) be solved by the method of series in the form $y=\sum_{k=0}^{\infty}a_k (x-1)^{k+s}$?

Use the space below for calculations.

8. Solve the partial differential equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} - 2\frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{\partial^2 f(x,y)}{\partial y^2} + 9\, f(x,y) = 0 \,, \qquad (*)$$

using Fourier (integral) transforms:

a: Find the x-Fourier transform of Eq. (∗). [=3pt]

b: Find the y-Fourier transform of Eq. (∗). [=3pt]

c: Find the (double) x, y-Fourier transform of Eq. (∗). [=3pt]

d: State the relation between k_x and k_y (the inverse variables for x and y, respectively) as implied by the double Fourier transform of Eq. (∗). [=3pt]

e: Write down the general solution to Eq. (∗), as obtained by the double-inverse transform, and implementing the condition from part d. [=3pt]

Use the space below and on the back for calculations.