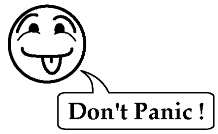
HOWARD UNIVERSITY
WASHINGTON, D.C. 20059

DEPARTMENT OF PHYSICS AND ASTRONOMY
(202)-806-6245 (Main Office)
(202)-806-5830 (FAX)

2355 Sixth Str., NW, TKH Rm.215
thubsch@howard.edu
(202)-806-6257



Mathematical Methods II April '98.
2nd Midterm Exam Solutions (T. Hübsch)

— DISCLAIMER —
The completeness and detail presented herein were by no means expected in the Student's solutions for full credit. The additional information given here is solely for the Student's convenience and education.

1. The solution to this problem follows closely the examples done in class.

a. We begin by noting that $g(x,t) = \sum_{n=-\infty}^{\infty} A_n(x) t^n$ *defines* the $A_n(x)$ as the coefficients of the Laurent [1] expansion of $g(x,t)$—the power series in t about $t=0$. Therefore,

$$A_n(x) \;=\; \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n} e^{x(t+x/2t)}\right]_{t=0}, \qquad n \geq 0\;, \tag{1}$$

should have been a give-away.

Recall Cauchy's integral formula

$$\frac{\mathrm{d}^n f}{\mathrm{d}z^n} = \frac{n!}{2\pi i}\oint_C \frac{\mathrm{d}\zeta\; f(\zeta)}{(\zeta - z)^{n+1}}\;, \tag{2}$$

where C is any contour that encircles $\zeta = z$ precisely once, and in the counterclockwise fashion. Then, the above derivative formula is easy to rewrite as:

$$A_n(x) \;=\; \frac{n!}{2\pi i}\oint_C \frac{\mathrm{d}\zeta\; e^{x(\zeta + x/2\zeta)}}{(\zeta - z)^{n+1}}\;, \qquad \text{for all } n\;. \tag{3}$$

b. The series representation is obtained by rewriting $g(x,t) = e^{xt}e^{x^2/2t}$, expanding the exponentials and then re-summing:

$$g(x,t) \;=\; \sum_{k,l=0}^{\infty} \frac{(xt)^k}{k!}\frac{(x^2/2t)^l}{l!} \;=\; \sum_{k,l=0}^{\infty} \frac{x^{k+2l}\; t^{k-l}}{2^l\, k!\, l!}\;, \tag{4a}$$

$$= \sum_{n=-\infty}^{\infty}\left[\sum_{l\geq 0,-n} \frac{x^{3l+n}}{2^l (n+l)!\, l!}\right] t^n\;, \qquad \begin{cases} k - l = n\;,\; k = l+n\;, \\ k \geq 0 \;\Rightarrow\; l \geq -n\;. \end{cases} \tag{4b}$$

Comparing with the definition of the $A_n(x)$ as the expansion coefficients of $g(x,t)$:

$$A_n(x) \;=\; \sum_{l\geq 0,-n} \frac{x^{3l+n}}{2^l (n+l)!\, l!}\;. \tag{5}$$

(At this point the observant student should have noticed that the expansion in powers of t must extend over positive and negative powers!)

---

[1] Note the typo in the problem: $e^{x(t+x/2t)}$ has an essential singularity at $t = 0$, whence the summation must extend over positive and negative powers of t.

c. Acting with $\frac{\partial}{\partial t}$ both on the left-hand-side and on the right-hand-side of

$$e^{x(t+x/2t)} \;=\; \sum_{n=0}^{\infty} A_n(x) t^n \;, \tag{6}$$

we obtain:

$$\left(x - \frac{x^2}{2t^2}\right) e^{x(t+2)} \;=\; \sum_{n=-\infty}^{\infty} A_n(x)\, n\, t^{n-1} \;, \tag{7a}$$

$$\underbrace{x \sum_{n=-\infty}^{\infty} A_n(x) t^n}_{n\,\to\, m-1} - \frac{x^2}{2} \underbrace{\sum_{n=-\infty}^{\infty} A_n(x) t^{n-1}}_{n\,\to\, m+1} = \underbrace{\sum_{n=0}^{\infty} n\, A_n(x)\, t^{n-1}}_{n\,\to\, m} \;, \tag{7b}$$

$$\sum_{m=-\infty}^{\infty} \left[x A_{m-1}(x) - \frac{x^2}{2} A_{m+1}(x) - m A_m(x) \right] t^{m-1} \;=\; 0 \;. \tag{7c}$$

As the different powers of t are linearly independent, the quantity in the square brackets must vanish:

$$2m A_m \;=\; 2x A_{m-1} - x^2 A_{m+1} \;. \tag{8}$$

Next, acting with $\frac{\partial}{\partial x}$ on both sides of (6), we have

$$\left(t + \frac{x}{t}\right) e^{-x(t+2)} \;=\; \sum_{n=-\infty}^{\infty} A_n'(x) t^n \;, \tag{9a}$$

$$\underbrace{\sum_{n=-\infty}^{\infty} A_n(x) t^{n+1}}_{n\to m-1} + x \underbrace{\sum_{n=-\infty}^{\infty} A_n(x) t^{n-1}}_{n\to m+1} = \underbrace{\sum_{n=-\infty}^{\infty} A_n'(x) t^n}_{n\to m} \;, \tag{9b}$$

$$\sum_{m=-\infty}^{\infty} \left[A_{m-1}(x) + x A_{m+1}(x) - A_m'(x) \right] t^m \;=\; 0 \;. \tag{9c}$$

Again, because of the linear independence of different powers of t, we conclude that

$$A_m' \;=\; A_{m-1} + x A_{m+1} \;. \tag{10}$$

Combining (8) and (10), we obtain:

$$3x\, A_{m-1}(x) \;=\; 2m\, A_m(x) + x\, A_m'(x) \;, \tag{11a}$$

$$3x^2 A_{m+1}(x) \;=\; 2x\, A_m'(x) - 2m\, A_m(x) \;, \tag{11b}$$

an alternate set of two (independent) recursion relations satisfied by the $A_m(x)$.

d. To obtain a differential equation for $A_n(x)$ at any given order n, without involving other orders, is fairly easy. The general strategy is to shift the index in, say, (11b):

$$3x^2 A_m \;=\; 2x\, A_{m-1}' - 2(m-1)\, A_{m-1} \;, \tag{12}$$

and then elliminate A'_{m-1} using the derivative by x of the $(3x)^{-1}$-multiple of $(11a)$:

$$A'_{m-1} \;=\; \tfrac{1}{3}A''_m + \tfrac{2m}{3x}A'_m - \tfrac{2m}{3x^2}A_m \;, \tag{13}$$

while A_{m-1} is elliminated using $(11a)$ itself. We obtain

$$2x^2A''_m + 2(m{+}1)xA'_m - (9x^2{+}4m^2)A_m \;=\; 0 \;. \tag{14}$$

This can be identified with a a Sturm-Liouville differential equation only in the limited sense so that

$$\mathscr{L}A_m - 9x^2A_m = 0 \;, \qquad \mathscr{L} \stackrel{\rm def}{=} 2x^2\frac{{\rm d}^2}{{\rm d}x^2} + 2(m{+}1)x\frac{\rm d}{{\rm d}x} - 4m^2 \;. \tag{15}$$

With the differential operator $\mathscr{L}$ itself depending on m (in the coefficient of the 1st derivative), this permits only the identification of x^2 as the weight function and we only have a single eigenvalue: -9. Clearly, a full cancellation of all m-dependent terms in the 1st and 2nd derivative terms (whence the m-dependent term can be identified as the eigenvalue term) occurs only for special generating functions, and this is not one of them.

2. The Laplace equation $\vec{\nabla}^2V = 0$ separates in cylindrical coordinates into three equations [Arfken, p.473–474]

$$\frac{{\rm d}^2Z}{{\rm d}z^2} \;=\; \ell^2 Z \;, \tag{16z}$$

$$\frac{{\rm d}^2\Phi}{{\rm d}\phi^2} \;=\; -m^2\Phi \;, \tag{16ϕ}$$

$$r\frac{\rm d}{{\rm d}r}\Big(r\frac{{\rm d}R}{{\rm d}r}\Big) \;=\; (m^2-\ell^2r^2)R \;, \tag{16r}$$

where as usual, $Z(z) \propto e^{\pm\ell z}$ and $\Phi(\phi) \propto e^{\pm im\phi}$; allowing m, ℓ to range over positive and negative values will include both solutions and the sign in the exponents may be chosen positive. $R(r)$ is a linear combination of $J_m(\ell r)$ and $N_m(\ell r)$. Thus, we have the general solution for the potential

$$V(r,\phi,z) \;=\; \begin{cases} \sum_{\ell,m} V^{\rm in}_{\ell,m}\, J_m(\ell r)\, e^{im\phi}e^{\ell z} \;, & \text{for } r \leq a, \\ \sum_{\ell,m} \big[V^{\rm out,J}_{\ell,m}\, J_m(\ell r) + V^{\rm out,N}_{\ell,m}\, N_m(\ell r)\big]\, e^{im\phi}e^{\ell z} \;, & \text{for } r \geq a. \end{cases} \tag{17}$$

Owing to regularity around $r = 0$, we have omitted $N_m(\ell r)$ from the solution to be used inside, as $N_m(\ell r)$ blows up at $r{=}0$.

Next, the boundary conditions. The most general solution is expected to be of the form $V(r,\phi,z) = \mathring{V}(r,\phi,z) + V_p(r,\phi,z)$, such that e.g. the radial boundary condition becomes

$$\mathring{V}(a,\phi,z) = 0 \;, \qquad \text{and} \qquad V_p(a,\phi,z) = V_0\sin(2\theta) \;. \tag{18}$$

Periodicity in $\phi \simeq \phi + 2\pi$ implies that the m's must be integers, for both $\mathring{V}$ and V_p.

Note that the only boundary condition we have, $V(a,\phi,z) = V_0\sin(2\phi)$ is independent of z. Therefore, $V_p(r,\phi,z)$ will also have to be independent of z, and for this ("particular") part we must set ℓ=0. Looking back at Eq. (16r), we see that the Bessel equation degenerates into the homogeneous equation (all terms scale equally with respect to a rescaling $r \to \lambda r$), and this equation is solved by the lowest-order terms in the Maclaurin expansion of $J_m(x)$: the pure powers $r^{\pm m}$. Alternatively, this could have been obtained by taking the. Since $r^{-|m|}$ blows up at r=0, and $r^{+|m|}$ blows up at $r \to \infty$, we must choose:

$$V_p(r,\phi,z) \;=\; \begin{cases} \sum_{m=0}^{\infty} \left(\frac{r}{a}\right)^m \big[a_m \cos(m\phi) + b_m \sin(m\phi)\big]\;, & \text{for } r \leq a, \\ \sum_{m=0}^{\infty} \left(\frac{a}{r}\right)^m \big[c_m \cos(m\phi) + d_m \sin(m\phi)\big]\;, & \text{for } r \geq a. \end{cases} \tag{19}$$

By virtue of the balancing powers of a, all constants $a_m, \cdots, d_m$ to have the same dimensions (units) as $V_p(r,\phi,z)$. We have also switched from $e^{\pm im\phi}$ to $\sin(m\phi)$ and $\cos(m\phi)$ to facilitate the subsequent (final) step. The boundary condition (18) is now used to determine a_m–d_m, by setting r=a and multiplying the second equation in (18) in turn by $\cos(n\phi)$ and integrating from 0 to 2π, and then by $\sin(n\phi)$ and integrating. It is easy to obtain

$$V_0{\cdot}0 \;=\; \sum_{m=0}^{\infty} \big[\,a_m\,\pi\,\delta_{m,n} + b_m\,0\,\big] = \pi a_n\;, \tag{20a}$$

$$V_0{\cdot}\pi\,\delta_{2,n} \;=\; \sum_{m=0}^{\infty} \big[\,a_m\,0 + b_m\,\pi\,\delta_{m,n}\,\big] = \pi b_n\;, \tag{20b}$$

$$V_0{\cdot}0 \;=\; \sum_{m=0}^{\infty} \big[\,c_m\,\pi\,\delta_{m,n} + d_m\,0\,\big] = \pi c_n\;, \tag{20c}$$

$$V_0{\cdot}\pi\,\delta_{2,n} \;=\; \sum_{m=0}^{\infty} \big[\,c_m\,0 + d_m\,\pi\,\delta_{m,n}\,\big] = \pi d_n\;. \tag{20b}$$

That is, $a_m = 0 = c_m$, and $b_m = V_0\delta_{m,2} = d_m$, so that

$$V_p(r,\phi,z) \;=\; \begin{cases} V_0\left(\frac{r}{a}\right)^2 \sin(2\phi)\;, & \text{for } r \leq a, \\ V_0\left(\frac{a}{r}\right)^2 \sin(2\phi)\;, & \text{for } r \geq a. \end{cases} \tag{21}$$

Note that, for $r \to 0$, $V \sim r^2 < \infty$, and that for $r \to \infty$, $(rV\frac{\mathrm{d}}{\mathrm{d}r}V) \sim r^{-4} < \infty$.

The first part, $\mathring{V}(r,\phi,z)$, vanishes on the cylindrical surface. For the 'inside' solution (where the N_m are ruled out since they diverge for $r \to 0$), we therefore must have

$$J_m(\ell a) = 0\;, \qquad \textit{i.e.}, \qquad \ell = \frac{\alpha_{m,L}}{a}\;, \quad L = 1,2,3,\ldots \tag{22}$$

where $\alpha_{m,L}$ is the L^{th} zero of $J_m(x)$. For the outside solution, we have that

$$J_m(\ell a) + \frac{V^{\text{out,N}}_{\ell,m}}{V^{\text{out,J}}_{\ell,m}} N_m(\ell a) = 0\;, \qquad \textit{i.e.}, \qquad \ell = \frac{\gamma_{m,L}}{a}\;, \quad L = 1,2,3,\ldots \tag{23}$$

which defines another collection of points, $\gamma_{m,L}$, where this combination of functions vanishes; note that the list of $\gamma_{m,L}$'s depends on the ratio $(V^{\text{out,N}}_{\ell,m}/V^{\text{out,J}}_{\ell,m})$, and so can only be determined pending on this detail of application. Thus,

$$\mathring{V}(r,\phi,z) \;=\; \begin{cases} \sum\limits_{m=0,L=1}^{\infty} V^{\rm in}_{L,m}\, J_m\big(\alpha_{m,L}\tfrac{r}{a}\big)\, e^{im\phi} e^{\alpha_{m,L}\frac{z}{a}}\;, & \text{for } r\leq a,\\[2ex] \sum\limits_{m=0,L=1}^{\infty} M_m\big(\gamma_{m,L}\tfrac{r}{a}\big)\, e^{im\phi} e^{\gamma_{m,L}\frac{z}{a}}\;, & \text{for } r\geq a. \end{cases} \tag{24}$$

where

$$M_m\big(\gamma_{m,L}\tfrac{r}{a}\big) \;\stackrel{\rm def}{=}\; V^{\rm out,J}_{L,m}\, J_m\big(\gamma_{m,L}\tfrac{r}{a}\big) + V^{\rm out,N}_{L,m}\, N_m\big(\gamma_{m,L}\tfrac{r}{a}\big) \tag{25}$$

is a custom-mixed Bessel function.

This determines the part of the solution that vanishes on the boundary, $\mathring{V}(r,\phi,z)$, in which the constants $V^{\rm in}_{L,m}$ and $V^{\rm out,J}_{L,m}$ and $V^{\rm out,N}_{L,m}$ remain unspecified by the boundary conditions. Comparing (21) with (24)–(25), we see that the particular part, $V_p(r,\phi,z)$ is much more precisely determined. This is also what is usually required in problems. The present solution however is meant to show that there always exists a much less well determined 'null' part, $\mathring{V}(r,\phi,z)$, which may be added freely to the 'particular' solution without affecting the boundary conditions.

3.a. Since the boundary conditions refer to a sphere, we use shperical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big[r^2\frac{\partial V}{\partial r}\Big] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big[\sin\theta\frac{\partial V}{\partial\theta}\Big] + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} \;=\; 0\;. \tag{26}$$

Following the hint, we write $V(\vec r,t) = H(\vec r)e^{i\omega t}$ and obtain (writing $k=\omega/v$)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big[r^2\frac{\partial H}{\partial r}\Big] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big[\sin\theta\frac{\partial H}{\partial\theta}\Big] + \frac{1}{r^2}\frac{\partial^2 H}{\partial\phi^2} + k^2 H \;=\; 0\;. \tag{27}$$

This now is the well-studeid Helmholtz equation, and direct comparison with Arfken [§ 11 and 12] yields:

$$V(r,\theta,\phi,t) = \sum_{k,q,s} c^+_{kqs} j_q(kr) Y^s_q(\theta,\phi) e^{ikvt} + \sum_{k,q,s} c^-_{kqs} j_q(kr) Y^s_q(\theta,\phi) e^{-ikvt}\;. \tag{28}$$

The mathematical properties of the spherical harmonics $Y^s_q(\theta,\phi)$ require that either $q=Q$ or $q=Q{+}\frac{1}{2}$, with Q an integer, and $s=-q,(1{-}q),\cdots,(q{-}1),q$. The von Neumann functions, $n_q(kr)$, could not be used since they diverge at $r\to 0$, whereas the displacement of the jelly cannot.

b. Next we impose all boundary conditions that there are. The periodicity requirement in ϕ, that is, $V(r,\theta,\phi{+}2\pi,t) = V(r,\theta,\phi,t)$ and for arbitrary r,θ,t, implies[2] that $e^{is(\phi+2\pi)} = e^{is\phi}$ and so $e^{is2\pi}=1$ or that $s=0,\pm1,\pm2,\pm3\ldots$. Therefore, q also must be an integer. Next, we impose the condition that there is no displacement at the spherical boundary, so that $V(a,\theta,\phi,t)=0$ for arbitrary θ,ϕ,t. Thus, it must be that $j_q(ka)=0$, whereupon ak must equal one of the zeros (say, the n^{th}) of the q^{th} spherical Bessel function: $ak=\alpha_{q,n}$, so that $j_q(ka)=j_q(\alpha_{q,n})=0$. Finally, since $\omega=kv$, and there is a k for each q,n,

[2] Remember that $Y^s_q(\theta,\phi)\propto P^s_q(\cos\theta)e^{is\phi}$.

the frequencies of the drum are: $\omega_{q,n} = \alpha_{q,n}\frac{v}{a}$, and $q = 0, 1, 2, 3\ldots$ while $n = 1, 2, 3, \ldots$ Therefore, the general solution becomes

$$V(r,\theta,\phi,t) \;=\; \sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\sum_{s=-q}^{q} c^{\pm}_{n,q,s}\; j_q\big(\alpha_{q,n}\frac{r}{a}\big)\; P^s_q(\cos\theta)e^{is\phi}\; e^{\pm i(\alpha_{q,n}v/a)t}\;. \tag{29}$$

c. The list of frequencies has already been obtained:

$$\omega_{q,n} \;=\; \alpha_{q,n}v/a\;, \qquad q = 0, 1, 2, \ldots\;, \quad n = 1, 2, 3, \ldots \tag{30}$$

Now, the lowest frequency manifestly occurs for $q = 0$, and we use the fact that $j_0(kr) = \frac{\sin(kr)}{kr}$, the zeroes of which are $\alpha_{0,n} = n\pi$. The lowest frequency then is

$$\omega_{0,1} \;=\; \alpha_{0,1}\frac{v}{a} \;=\; \frac{\pi v}{a}\;. \tag{31}$$

d. Most of the time it is not possible to hear the type of vibration. This is because the frequency is determined by q and n, but not by s. So whenever $q \neq 0$, there are several distinct modes of vibration, labeled by $s = -q, \cdots, q$, which all have the same frequency. For the special cases when $q = 0$, also $s = 0$, so that the frequency uniquely determins the mode of vibration.

e. Upon inserting the non-slip partitions, the boundary conditions are changed, but only in the ϕ-direction. Since the partitions divide the jelly into four non-interacting parts, periodicity is no longer required of $V(\vec{r},t)$. However, since the jelly cannot slip at the partitions, it must be that $V(r,\theta,\phi,t) = 0$ when $\phi = 0, \frac{2\pi}{N}$. We may rewrite (29) in terms of $\sin(s\phi)$ and $\cos(s\phi)$, whereupon the $\cos(s\phi)$ terms are immediately ruled out. Furthermore,

$$\sin(s\tfrac{2\pi}{N}) = 0 \quad \text{implies} \quad s = \pm S\tfrac{N}{2}\;, \quad S = 0, 1, 2, 3, \ldots \tag{32}$$

However, recall that $\max(s) = q$, so we have

$$\omega_{q,n} \;=\; \alpha_{q,n}v/a\;, \qquad q = \tfrac{N}{2}Q, \quad Q = 0, 1, 2, \ldots\;, \quad n = 1, 2, 3, \ldots \tag{33}$$

So, in particular, if say $N = 7$, we have $q = 0, \frac{7}{2}, 7, \frac{21}{2}, \ldots$, which then determins the list of Bessel functions the zeroes of which determine the list of frequencies $\omega_{q,n} = \alpha_{q,n}v/a$.

4. The given function $f(x) = 1-(\frac{x}{\pi})^2$ is obviously symmetric, $f(-x) = f(x)$, whereupon in the general expression for the Fourier transform:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\big[a_k\cos(kx) + b_k\sin(kx)\big]\;, \tag{34}$$

all $b_k = 0$. The coefficients a_k, $k = 1, 2\ldots$ are determined by the integral:

$$a_k \;= \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}x\; f(x)\cos(kx)\;, \tag{35a}$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \mathrm{d}x\; f(x)\cos(kx) + \frac{1}{\pi}\int_{0}^{\pi} \mathrm{d}x\; f(x)\cos(kx)\;, \tag{35b}$$

$$= \frac{2}{\pi} \int_0^\pi \mathrm{d}x \ (1 - (\frac{x}{\pi})^2) \cos(kx) \ , \tag{35c}$$

$$= \frac{2}{\pi} \int_0^\pi \mathrm{d}x \ \cos(kx) - \frac{2}{\pi^3} \int_0^\pi \mathrm{d}x \ x^2 \cos(kx) \ , \tag{35d}$$

$$= \frac{2}{\pi} \Big[\frac{1}{k} \sin(kx) \Big]_0^\pi - \frac{2}{\pi^3} \Big\{ \Big[\frac{x^2}{k} \sin(kx) \Big]_0^\pi - \frac{2}{k} \int_0^\pi \mathrm{d}x \ x \sin(kx) \Big\} \ , \tag{35e}$$

$$= \frac{2}{\pi} 0 - \frac{2}{\pi^3} \Big\{ 0 - \frac{2}{k} \Big\{ \Big[-\frac{x}{k} \cos kx \Big]_0^\pi - \frac{1}{k} \int_0^\pi \mathrm{d}x \ \big(-\cos(kx) \big) \Big\} \Big\} \ , \tag{35f}$$

$$= -\frac{4}{\pi^3 k} \Big\{ -\frac{\pi}{k} (-1)^k + \frac{1}{k^2} \big[\sin(kx) \big]_0^\pi \Big\} \ = \frac{4}{\pi^2 k^2} (-1)^k \ , \tag{35g}$$

Finally, for the case $k = 0$, the starting integral above was appropriate, but the integrations by part illegitimate; instead, we have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{d}x \ f(x) \ = \ \frac{2}{\pi} \int_0^\pi \mathrm{d}x \ (1 - (\frac{x}{\pi})^2) \ = \ \frac{2}{\pi} \Big\{ \Big[x - \frac{x^3}{3\pi^2} \Big]_0^\pi \Big\} \ = \ \frac{4}{3} \ . \tag{36}$$

Thus,

$$f(x) = \Big(x^2 - \frac{\pi^2}{3} \Big) \ = \ \frac{4}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx) \ , \tag{37}$$

which does converge, quadratically.