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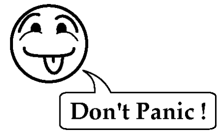

Mathematical Methods II 13th Feb. '98.

1st Midterm Exam Solutions (T. Hübsch)

— DISCLAIMER —
The completeness and detail presented herein were by no means expected in the Student's solutions for full credit. The additional information given here is solely for the Student's convenience and education.

1.a. The equation $\frac{\mathrm{d}V}{\mathrm{d}P} = -\big(\frac{V}{P}\big)^\alpha$ separates for all values of α, as it can easily be rewritten as

$$\frac{\mathrm{d}V}{V^\alpha} = -\frac{\mathrm{d}P}{P^\alpha} \ . \tag{1}$$

This can be integrated straightforwardly, to produce

$$\frac{1}{1-\alpha} V^{1-\alpha} = -\frac{1}{1-\alpha} P^{1-\alpha} + C \ , \tag{2}$$

where C is the constant of integration. Straightforwardly then,

$$\frac{V^{1-\alpha} + P^{1-\alpha}}{1-\alpha} \ = \ C \ , \tag{3}$$

is the algebraic combination of V and P that is constant.

1.b. In the limit $\alpha \to 1$, the left hand side of Eq. (3) becomes of type $\frac{1+1}{0}$, which simply diverges. Equivalently, upon multiplying Eq. (3) by $(1-\alpha)$ and then taking the limit, one obtains that $1+1=0$, which is clearly nonsense! Of course, we have assumed that C is a constant, independent not only of P, V but also of α. Clearly, this latter (implicit and well hidden [1]!) assumption must have been wrong. Indeed, as a constant of integration, C must definitely be independent of the variables in the differential equation, P and V. However, there is no reason why one should be obtaining the same constant for all the different differential equations, parametrized by the choice of ϵ, *i.e.*, α.

Rewriting Eq. (3) by using $\epsilon \stackrel{\text{def}}{=} (1-\alpha)$:

$$\frac{V^\epsilon + P^\epsilon}{\epsilon} \ = \ C \ , \tag{4}$$

we conclude that C ought to be a function of ϵ, and have a pole of order 1 at $\epsilon{=}0$. Thus, we write

$$\frac{V^\epsilon + P^\epsilon}{\epsilon} \ = \ \frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + \dots \ , \tag{5}$$

or

$$V^\epsilon + P^\epsilon \ = \ c_{-1} + c_0\epsilon + c_1\epsilon^2 + \dots \ . \tag{6}$$

[1] …and not expected to be uncovered by the Student under the pressures of the exam…

The constants c_k, $k=-1,0,2\ldots$ are now determined by expanding the left hand side into a power series in ϵ and comparing like terms. Thus:

$$c_{-1} \stackrel{\text{def}}{=} \Big[V^\epsilon + P^\epsilon\Big]_{\epsilon\to0} = 2\ , \tag{7a}$$

$$c_0 \stackrel{\text{def}}{=} \Big[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big(V^\epsilon + P^\epsilon\Big)\Big]_{\epsilon\to0} = \Big[V^\epsilon \ln V + P^\epsilon \ln P\Big]_{\epsilon\to0} = \ln PV\ , \tag{7b}$$

and so on. Exponentiating Eq. ($7b$), we obtain

$$PV = \mathrm{e}^{c_0} = const.,$$

as found from the original equation (1) when α=1. This indeed is the required standard Boyle's gas law. In our study of the limit $\alpha\to1$, this final result occurs as the subleading term, after the leading term has been ensured to give $1+1=2$ rather than the nonsensical $1+1=0$ of the naïve limit of our result (3).

2. Before we launch into attempting to solve $x^3y''-2\alpha xy'-\beta(1+x)y=0$ in a series form, we check for singular points of the equation, and in particular, for essentially singular points. Here, $P(x)=-\frac{2\alpha}{x^2}$ diverges at $x=0$, where moreover $(x-0)P(x)$ also diverges. This, then, is an essential singularity (no need to check $Q(x)$ too; whatever its behavior, it won't cure the divergence of $P(x)$.). This is the reason why an attempted solution in the form of $\sum_{k=0}^\infty c_k x^{k+s}$ cannot succeed producing a convergent series. Nevertheless, we proceed as instructed:

$$\begin{aligned}
0 &= x^3y''-2\alpha xy'-\beta(1+x)y\ ,\\
&= \sum_{k=0}^\infty c_k(k+s)(k+s-1)x^{k+s+1} - 2\alpha\sum_{k=0}^\infty c_k(k+s)x^{k+s}\\
&\qquad -\beta\sum_{k=0}^\infty c_k x^{k+s} - \beta\sum_{k=0}^\infty c_k x^{k+s+1}\ ,\\
&= \sum_{m=1}^\infty c_{m-1}(m+s-1)(m+s-2)x^{m+s} - 2\alpha\sum_{m=0}^\infty c_m(m+s)x^{m+s}\\
&\qquad -\beta\sum_{m=0}^\infty c_m x^{m+s} - \beta\sum_{m=1}^\infty c_{m-1}x^{m+s}\ ,\\
&= -[2\alpha s+\beta]c_0(s)x^s\\
&\qquad +\sum_{m=1}^\infty\Big[c_{m-1}\big[(m+s-1)(m+s-2)-\beta\big]-\big[2\alpha(m+s)-\beta\big]c_m\Big]x^{m+s}\ .
\end{aligned} \tag{8}$$

Now we are ready to answer the questions.

2.a. The vanishing of the coefficient of x^s guarantees that $2\alpha s+\beta=0$, so that $s=-\beta/2\alpha$.

2.b. The vanishing of the remaining therms then produces the recursion relation (having used that $s=-\beta/2\alpha$)

$$c_m = \frac{(m+s-1)(m+s-2)-\beta}{2\alpha m}\, c_{m-1}\ , \qquad m\geq 1\ , \tag{9}$$

or, shifting back to $k=m+1$:

$$c_{k+1} = \frac{(k+s)(k+s-1)-\beta}{2\alpha(k+1)}\, c_k\ , \qquad k\geq 0\ . \tag{10}$$

2.c. It should be clear that this series diverges:

$$1 > \lim_{k\to\infty}\left|\frac{c_{k+1}x^{k+1+s}}{c_k x^{k+s}}\right| = |x|\lim_{k\to\infty}\left|\frac{(k+s)(k+s-1)-\beta}{2\alpha(k+1)}\right| = |x|\lim_{k\to\infty}\frac{k^2}{2k} \tag{11}$$

is true only for $x=0$. Thus, our would-be solution simply makes no sense for $x\neq 0$, and for general values of α,β.

2.d. However, if $\beta=(n+s)(n+s+1)$ for some integer n, the infinite series truncates to a polynomial of order n since $c_{n+1}=0$, and then $c_k=0$ for all $k>n$. Substituting our earlier result for s, the condition $\beta=(n+s)(n+s+1)$ reads:

$$n^2-\Big(\frac{\beta}{\alpha}-1\Big)n-\Big[\beta-\frac{\beta}{2\alpha}\Big(\frac{\beta}{2\alpha}-1\Big)\Big]=0\ .$$

For n to be a limiting value for the summation variable k, it must be an integer, and so α,β must be such that at least one solution of this quadratic equation,

$$n_\pm = \Big(\frac{\beta}{2\alpha}-\frac{1}{2}\Big)\pm\sqrt{\beta+\frac{1}{4}}\ ,$$

must be an integer.

3. To use the general formula

$$y_2 \;=\; y_1\int \mathrm{d}x\frac{e^{\int \mathrm{d}x P(x)}}{(y_1)^2}\ , \tag{12}$$

we divide the differential equation $x^2y''-2y=0$ by x^2 and identify $P(x)\equiv 0$. Now, straightforwardly:

$$y_2 \;=\; x^2\int \mathrm{d}x\frac{e^0}{(x^2)^2} \;=\; x^2\int\frac{\mathrm{d}x}{x^4} \;=\; -\tfrac{1}{3}x^{-1}\ . \tag{13}$$

That this (dropping the overall constant $-\frac{1}{3}$) indeed is a solution, we calculate

$$y_2' = -x^{-2}\ , \qquad y_2''=+2x^{-3}\ , \tag{14}$$

whereby $y_2=x^{-1}$ also solves $x^2y''-2y=0$, and so is the second solution, as sought. Furthermore, it should be obvious that x^2 and x^{-1} are linearly independent. However, to prove this, we evaluate the Wronskian:

$$W[y_1,y_2] \;\stackrel{\text{def}}{=}\; y_1\,y_2'-y_1'\,y_2 \;=\; (x^2)(-x^{-2})-(2x^1)(x^{-1}) \;=\; -3\;\neq\;0\ ; \tag{15}$$

the non-vanishing of the Wronsian proves the linear independence of y_1, y_2.

4.a. Since $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}}$ differs by the factor $\frac{1}{2}$ from the coefficient of the first derivative, the equation

$$\sqrt{1+x}\frac{\mathrm{d}^2 f_\alpha}{\mathrm{d}x^2} + \frac{1}{\sqrt{1+x}}\frac{\mathrm{d}f_\alpha}{\mathrm{d}x} + \alpha f_\alpha = 0 \tag{16}$$

is not self-adjoint as it is. It can be made self-adjoint by (pre-)multiplying all terms with

$$\begin{aligned} \frac{1}{p_0}\exp\int \mathrm{d}x\,\frac{p_1}{p_0} = \frac{1}{\sqrt{1+x}}\exp\Big\{\int\frac{\mathrm{d}x}{(1+x)}\Big\} &= \frac{1}{\sqrt{1+x}}\exp\Big\{\ln(1+x)\Big\}\ , \\ = \frac{(1+x)}{\sqrt{1+x}} &= \sqrt{1+x}\ , \end{aligned} \tag{17}$$

upon which the differential equation becomes

$$(1+x)\frac{\mathrm{d}^2 f_\alpha}{\mathrm{d}x^2} + \frac{\mathrm{d}f_\alpha}{\mathrm{d}x} + \alpha\sqrt{1+x}f_\alpha = 0\ , \tag{18}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(1+x)\frac{\mathrm{d}f_\alpha}{\mathrm{d}x}\Big] + \alpha\sqrt{1+x}f_\alpha = 0\ . \tag{19}$$

The differential equation is now indeed in the self-adjoint form.

4.b. In the preceeding equation, $\frac{\mathrm{d}}{\mathrm{d}x}[(1+x)\frac{\mathrm{d}}{\mathrm{d}x}]$ may be identified as the Sturm-Liouville operator. This leaves α to be the eigenvalue and $\sqrt{1+x}$ the weight function. As for the limits, we must verify that the "integrated term" $\big[p_0 W[v^*, u]\big]_a^b$ vanishes.

The Wronskian was calculated, as in Chapter 8, without knowing the solutions u, v. It is $W = W_0\exp\{-\int \mathrm{d}x P(x)\}$, where $P(x) = p_1(x)/p_0(x)$. For the self-adjoint operator, $p_1(x) = p_0'(x)$, so $\int \mathrm{d}x\, p_0'/p_0 = \ln[p_0]$ and $W = W_0\exp\{-\ln[p_0(x)]\} = W_0/p_0(x)$. It is now easy to see that $\big[p_0 W[v^*, u]\big]_a^b = [1]_a^b \equiv 0$ for any two limits a, b — provided the inverse of $p_0(x)$ is well defined for all $x \in [a, b]$ so that the calculation of W would make sense. This indeed is true for $x \in (-1, 1]$, and special care may need to be taken at $x{=}-1$.

4.c. Adopting the limits ± 1 (recalling that the evaluation at -1 may need to be done carefully), we can write

$$\langle f_\alpha | f_\beta\rangle = \int_{-1}^{1}\mathrm{d}x\,\sqrt{1+x}\ f_\alpha(x)\, f_\beta(x) = \delta_{\alpha,\beta}\ , \tag{20}$$

where we have assumed a suitable normalization for $f_\alpha(x)$.

5. With the weight function being e^{-x} and for $0 \le x \le +\infty$, the orthonormality relation must be

$$\langle p_n | p_m\rangle \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}x\, e^{-x}\, p_n(x)\, p_m(x) \stackrel{!}{=} \delta_{n,m}\ . \tag{21}$$

The Student should have recognized that the Gamma function type integrals,

$$\int_0^\infty \mathrm{d}x\ x^k\, e^{-x} = k! \tag{22}$$

will be handy.

The equation (21) implies all the conditions that we will need to check; since the normalizations are not required, we only need the $n \neq m$ cases. We begin with $p_1 = \alpha x + \beta$, and request that $\langle p_1 | p_0 \rangle = 0$:

$$\langle p_1 | p_0 \rangle = \int_0^\infty \mathrm{d}x \, e^{-x} \, (\alpha x + \beta)(1) = (\alpha \, 1! + \beta \, 0!) = (\alpha + \beta) \; . \tag{23}$$

Vanishing of this fixes $\beta = -\alpha$ and so $p_1 = \alpha(x - 1)$.

Then we'll turn to $p_2 = \gamma x^2 + \delta x + \epsilon$, and request that

$$\langle p_2 | p_0 \rangle \;=\; 0 \; , \qquad \langle p_2 | p_1 \rangle \;=\; 0 \; . \tag{24}$$

So, we have

$$\begin{aligned} \langle p_2 | p_0 \rangle &= \int_0^\infty \mathrm{d}x \, e^{-x} \, (\gamma x^2 + \delta x + \epsilon)(1) = \Big(\gamma \, 2! + \delta \, 1! + \epsilon \, 0!\Big) \\ &= (2\gamma + \delta + \epsilon) \;\stackrel{!}{=}\; 0 \; , \end{aligned} \tag{25}$$

and

$$\begin{aligned} \langle p_2 | p_1 \rangle &= \alpha \int_0^\infty \mathrm{d}x \, e^{-x} \, (\gamma x^2 + \delta x + \epsilon)(x - 1) \\ &= \alpha \int_0^\infty \mathrm{d}x \, e^{-x} \, (\gamma x^3 + (\delta - \gamma) x^2 + (\epsilon - \delta) x - \epsilon) \\ &= \alpha \Big(\gamma \, 3! + (\delta - \gamma) 2! + (\epsilon - \delta) 1! - \epsilon \, 0!\Big) \\ &= \alpha (4\gamma + \delta) \;\stackrel{!}{=}\; 0 \; , \end{aligned} \tag{26}$$

from which $\delta = -4\gamma$ and then (from the previous relation) $\epsilon = 2\gamma$, so we obtain $p_2 = \gamma(x^2 - 4x + 2)$.