1. A Nifty Integral

The integral

$$\int_0^\infty \mathrm{d}x\; e^{-x} x^{k+2} \big[L_n^k(x)\big]^2$$

can be evaluated using the recursion relation

$$x\,L_n^k = (2n{+}k{+}1)\,L_n^k - (n{+}k)\,L_{n-1}^k - (n{+}1)\,L_{n+1}^k\ . \tag{1.1}$$

One way to do so [1)] is to use the recursion relation iteratively, but knwoing the orthogonality relation

$$\int_0^\infty \mathrm{d}x\; e^{-x}\, x^k\, L_n^k(x)\, L_m^k(x) \;=\; \frac{(n{+}k)!}{n!}\delta_{n,m}\ , \tag{1.2}$$

keep only the terms that remain non-zero. To this end, note that (by shifting n) the equation (1.1) also implies

$$x\,L_{n-1}^k = (2n{+}k{-}1)\,L_{n-1}^k - (n{+}k{-}1)\,L_{n-2}^k - n\,L_n^k\ , \tag{1.3}$$

and

$$x\,L_{n+1}^k = (2n{+}k{+}3)\,L_{n+1}^k - (n{+}k{+}1)\,L_n^k - (n{+}2)\,L_{n+2}^k\ . \tag{1.4}$$

Then

$$\begin{aligned} x^2\,L_n^k &= (2n{+}k{+}1)\,x\,L_n^k - (n{+}k)\,x\,L_{n-1}^k - (n{+}1)\,x\,L_{n+1}^k\ ,\\ &= (2n{+}k{+}1)\big[\,(2n{+}k{+}1)\,L_n^k + \ldots\,\big]\\ &\qquad - (n{+}k)\big[\,\ldots - n\,L_n^k\,\big]\\ &\qquad\quad - (n{+}1)\big[\,\ldots - (n{+}k{+}1)\,L_n^k - \ldots\,\big]\ ,\\ &= \big[(2n{+}k{+}1)^2 + (n{+}k)n + (n{+}1)(n{+}k{+}1)\big]\,L_n^k + \ldots \end{aligned} \tag{1.5}$$

where the ellipses denote terms that do not contribute to the integral because of the orthogonality condition. The original integral now may be rewritten as

$$\begin{aligned} &\big[(2n{+}k{+}1)^2 + (n{+}k)n + (n{+}1)(n{+}k{+}1)\big]\int_0^\infty \mathrm{d}x\; e^{-x} x^k \big[L_n^k(x)\big]^2\\ &= \big[(2n{+}k{+}1)^2 + (n{+}k)n + (n{+}1)(n{+}k{+}1)\big]\frac{(n{+}k)!}{n!}\ . \end{aligned}$$

1) For another method, ask Ms. Diagne or Mr. Bendidi.