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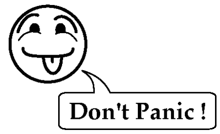

Mathematical Methods II 18th April '97.

Quizz, Solutions

Consider solving the integral equation

$$f(x) = \int_a^b \frac{\mathrm{d}t\ \varphi(t)}{(x-t)^\alpha} \tag{$*$}$$

by identifying the kernel, $K(x,t) = (x-t)^{-\alpha}$, with the product $w(t)g(t,x)$ where $g(t,x)$ is the generating function for the functions $F_k(t)$:

$$g(t,x) = \sum_k F_k(t)\, x^k \ ,$$

which satisfy an orthogonality relation:

$$\int_a^b \mathrm{d}t\ w(t)\, F_k^*(t)\, F_\ell(t) \ = \ \delta_{k,\ell} \ . \tag{\#}$$

If this identification is possible, we'll have:

$$\begin{aligned} f(x) &= \int_a^b \mathrm{d}t\ \varphi(t)\ \Big[w(t)\ \sum_k F_k(t)\, x^k\Big] \ , \\ &= \int_a^b \mathrm{d}t\ \Big[\ \sum_\ell c_\ell F_\ell^*(t)\Big]\ \Big[\ w(t)\ \sum_k F_k(t)\, x^k\ \Big] \ , \\ &= \sum_{\ell,k}\ c_\ell\, x^k \int_a^b \mathrm{d}t\ w(t)\ F_\ell^*(t)\, F_k(t) \ , \\ &= \sum_{\ell,k}\ c_\ell\, x^k \delta_{\ell,k} \ = \ \sum_k c_k x^k \ . \end{aligned}$$

Here we have used that the collection of functions $F_k(t)$—and therefore also of their conjugates, $F_k^*(t)$, is a complete set, so that any function, including $\varphi(t)$, can be expanded as was done on the second line. This result give the known function $f(x)$ in terms of a power series. The latter being unique, we can determine the coefficients from the Taylor series formula:

$$c_k \ = \ \frac{1}{k!} f^{(k)}(0) \ \stackrel{\text{def}}{=} \ \frac{1}{k!}\Big[\frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x)\Big]_{x=0} \ ,$$

whereupon

$$\varphi(t) = \sum_k \frac{1}{k!} f^{(k)}(0) F_k^*(t) \ .$$

It now remains to find out if such a collection of functions $F_k(t)$ exists.

To begin with, assume that $w(t)=1$, and note that

$$K(x,t) = (x{-}t)^{-\alpha} = (-t)^{-\alpha}(1-x/t)^{-\alpha} \ ,$$
$$= (-t)^{-\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{x^k}{t^k} \ = \sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{e^{-i\alpha\pi}}{t^{k+\alpha}}\ x^k \ ,$$

where $\binom{-\alpha}{k} \stackrel{\text{def}}{=} \frac{(-\alpha)!}{(-\alpha-k)!\,k!} = \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-k+1)}{k!}$, and $(-1)^{-\alpha}=(e^{i\pi})^{-\alpha}=e^{-i\alpha\pi}$. So, if $w(t)=1$, then $F_k(t)=\binom{-\alpha}{k}\frac{e^{-i\alpha\pi}}{t^{k+\alpha}}$. Notice that these functions are homogeneous (simple powers of the argument). Therefore, straightforward calculation yields

$$t\frac{\mathrm{d}}{\mathrm{d}t}\,t\frac{\mathrm{d}}{\mathrm{d}t}F_k(t) \ = \ (k{+}\alpha)^2F_k(t) \ .$$

That is, the $F_k(t)$ satisfy the Sturm-Liouville type equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[t\frac{\mathrm{d}F_k}{\mathrm{d}t}\Big] - \frac{(k{+}\alpha)^2}{t}F_k(t) \ = \ 0 \ ,$$

where the eigenvalue is $-(k{+}\alpha)^2$ and the weight function is $w(t)=\frac{1}{t}$.

We therefore try $K(x,t)=w(t)g(t,x)$, *i.e.*, $g(t,x)=tK(x,t)$, so that

$$g(t,x) = t(x{-}t)^{-\alpha} = t(-t)^{-\alpha}(1-x/t)^{-\alpha} \ ,$$
$$= t(-t)^{-\alpha}\sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{x^k}{t^k} \ = \sum_{k=0}^{\infty}\binom{-\alpha}{k}\frac{e^{-i\alpha\pi}}{t^{k+\alpha-1}}\ x^k \ .$$

Modifying a little our earlier tentative result, this implies that

$$F_k(t) = \binom{-\alpha}{k}\frac{e^{-i\alpha\pi}}{t^{k+\alpha-1}} \ .$$

The precise determination of compatible limits (if such exist!) a, b in the original integral equation (*) is done so that the orthogonality relation (#) would hold; this is left as an exercise for the diligent Student.