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Mathematical Methods II — 16th April '97.

Quizz, — Solutions

For the differential equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} \;-\; 2\frac{\partial^2 f(x,y)}{\partial x \partial y} \;+\; \frac{\partial^2 f(x,y)}{\partial y^2} \;+\; 9\, f(x,y) \;=\; 0\;, \tag{$*$}$$

Q.a. find the x-Fourier transform.

A.a. Applying the Fourier transform to the equation means multiplying with the kernel, $e^{ik_x x}$, and integrating over $-\infty<x<\infty$:

$$\int_{-\infty}^{\infty} \mathrm{d}x\; e^{ik_x x}\Big[\frac{\partial^2 f(x,y)}{\partial x^2} \;-\; 2\frac{\partial^2 f(x,y)}{\partial x \partial y} \;+\; \frac{\partial^2 f(x,y)}{\partial y^2} \;+\; 9\, f(x,y)\Big] \;=\; 0\;,$$

which, upon using Arfken's [1)] (15.41):

$$(-ik_x)^2 \tilde{f}(k_x,y) \;-\; 2(-ik_x)\frac{\partial \tilde{f}(k_x,y)}{\partial y} \;+\; \frac{\partial^2 \tilde{f}(k_x,y)}{\partial y^2} \;+\; 9\, \tilde{f}(k_x,y) \;=\; 0\;.$$

Q.b. find the y-Fourier transform.

A.b. Applying the Fourier transform to the equation means multiplying with the kernel, $e^{ik_y y}$, and integrating over $-\infty<y<\infty$:

$$\int_{-\infty}^{\infty} \mathrm{d}y\; e^{ik_y y}\Big[\frac{\partial^2 f(x,y)}{\partial x^2} \;-\; 2\frac{\partial^2 f(x,y)}{\partial x \partial y} \;+\; \frac{\partial^2 f(x,y)}{\partial y^2} \;+\; 9\, f(x,y)\Big] \;=\; 0\;,$$

which, analogously to the previous case yields:

$$\frac{\partial^2 \tilde{f}(x,k_y)}{\partial x^2} \;-\; 2(-ik_y)\frac{\partial \tilde{f}(x,k_y)}{\partial x} \;+\; (-ik_y)^2 \tilde{f}(x,k_y) \;+\; 9\, \tilde{f}(x,k_y) \;=\; 0\;.$$

Q.c. find the (double) x,y-Fourier transform.

A.c. Applying the double Fourier transform to the equation means multiplying with both kernels, $e^{ik_k k}$ and $e^{ik_y y}$, and integrating over both $-\infty<y<\infty$ and $-\infty<y<\infty$:

$$\int_{-\infty}^{\infty} \mathrm{d}x\; e^{ik_x x}\int_{-\infty}^{\infty} \mathrm{d}y\; e^{ik_y y}\Big[\frac{\partial^2 f(x,y)}{\partial x^2} \;-\; 2\frac{\partial^2 f(x,y)}{\partial x \partial y} \;+\; \frac{\partial^2 f(x,y)}{\partial y^2} \;+\; 9\, f(x,y)\Big] \;=\; 0\;,$$

which, combining the previous two cases, yields:

$$\begin{aligned}(-ik_x)^2 \tilde{\tilde{f}}(k_x,k_y) \;-\; 2(-ik_x)(-ik_y)\tilde{\tilde{f}}(k_x,k_y) \;+\; (-ik_y)^2 \tilde{\tilde{f}}(k_x,k_y) \;+\; 9\, \tilde{\tilde{f}}(k_x,k_y)&\\ = -\big[\,k_x^2 - 2k_x k_y + k_y^2 \;-\; 9\,\big]\tilde{\tilde{f}}(k_x,k_y)&\\ = -\big[\,(k_x-k_y)^2 \;-\; 9\,\big]\tilde{\tilde{f}}(k_x,k_y) = 0\;.&\end{aligned}$$

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1) and following the convention of his Eq. (15.37), rather than mine, given in class.

Q.d. State the relation between k_x and k_y (the inverse variables for x and y, respectively) as implied by the double Fourier transform.

A.d. Since $\tilde{\tilde{f}}(k_x, k_y) = 0$ is the trivial solution, we are bound to conclude that a nontrivial solution exists only if $(k_x - k_y)^2 = 9$, *i.e.*, that

$$k_y = k_x \pm 3 \ . \qquad (\diamondsuit)$$

Q.e. Write down the general solution to Eq. $(*)$, as obtained by the double-inverse transform, and implementing the condition from part d.

A.e. Now, the general solution, $f(x, y)$ is of course, the double inverse transform of $\tilde{\tilde{f}}(k_x, k_y)$:

$$f(x,y) \stackrel{\text{def}}{=} \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_x \ e^{-ik_x x} \int_{-\infty}^{\infty} \mathrm{d}k_y \ e^{-ik_y y} \ \tilde{\tilde{f}}(k_x, k_y) \ .$$

Since $\tilde{\tilde{f}}(k_x, k_y)$ itself has never been determined, this is simply a general double Fourier (continuous, integral) expansion, where the values of the undetermined function $\tilde{\tilde{f}}(k_x, k_y)$ play the roles of the Fourier coefficients.

To implement the condition $(\diamondsuit)$, we may stick into the integral a linear combination of two delta functions—one for each of the signs:

$$\begin{aligned} f(x,y) &\stackrel{\text{def}}{=} \tfrac{1}{2\pi} \iint_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y \ e^{-ik_x x} e^{-ik_y y} \ \tilde{\tilde{f}}(k_x, k_y) \big[c_+ \delta(k_y - k_x - 3) + c_- \delta(k_y - k_x + 3) \big] \ , \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_x \ e^{-ik_x x} \big[c_+ e^{-i(k_x+3)y} \ \tilde{\tilde{f}}(k_x, k_x+3) + c_- e^{-i(k_x-3)y} \ \tilde{\tilde{f}}(k_x, k_x-3) \big] \ , \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_x \ e^{-ik_x(x+y)} \big[c_+ e^{-3iy} \ \tilde{\tilde{f}}(k_x, k_x+3) + c_- e^{-3iy} \ \tilde{\tilde{f}}(k_x, k_x-3) \big] \ . \end{aligned}$$

The solution is thus determined in the form of a double (continuous, integral) Fourier expansion. The 'coefficient' function $\tilde{\tilde{f}}(k_x, k_y)$ is left undetermined so far, as no boundary conditions were given.

Q. Solve the system of differential equations with initial conditions:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \alpha A(t) + \beta B(t)\ , \qquad A(0) = 1\ ,$$

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \gamma A(t)\ , \qquad B(0) = 0\ ,$$

using the Laplace transform.

A. Applying the Laplace transform on the system of equations, denoting $a(s) \stackrel{\text{def}}{=} \mathscr{L}\{A(t)\}$, $b(s) \stackrel{\text{def}}{=} \mathscr{L}\{B(t)\}$, we have

$$s\,a(s) - A(0^+) = \mathscr{L}\left\{\frac{\mathrm{d}A(t)}{\mathrm{d}t}\right\} = \alpha a(s) + \beta b(s)\ ,$$

$$s\,b(s) - B(0^+) = \mathscr{L}\left\{\frac{\mathrm{d}B(t)}{\mathrm{d}t}\right\} = \gamma a(s)\ ,$$

where in the far-left equations, we have used Arfken's (15.123). Now we implement the boundary conditions given above, and rewrite the system equating the far-left and far-right parts:

$$(s-\alpha)\,a(s) \ - \ \beta\,b(s) = \ 1\ ,$$

$$-\gamma\,a(s) \ + \ s\,b(s) = \ 0\ .$$

The second equation implies that $b(s) = \gamma\,a(s)/s$, whereupon the first equation yields

$$(s-\alpha)\,a(s) \ - \ \frac{\beta\,\gamma}{s}\,a(s) = \ 1\ ,$$

i.e.,

$$a(s) = \frac{s}{s^2-\alpha s-\beta\gamma} \qquad \text{and} \qquad b(s) = \frac{\gamma}{s^2-\alpha s-\beta\gamma}\ .$$

This can be rewritten as

$$a(s) = \frac{\sigma}{\sigma^2-\kappa^2} + \frac{\alpha}{2\kappa}\,\frac{\kappa}{\sigma^2-\kappa^2} \qquad \text{and} \qquad b(s) = \frac{\gamma}{\kappa}\,\frac{\kappa}{\sigma^2-\kappa^2}\ .$$

where $\sigma \stackrel{\text{def}}{=} s - \frac{1}{2}\alpha$ and $\kappa^2 \stackrel{\text{def}}{=} \frac{1}{4}\alpha^2+\beta\gamma$. Here we have manipulated the solutions for $a(s)$ and $b(s)$ to resemble some of the entries in Arfken's Table 15.12 (p.915). Subject to the limitation of $\sigma > \kappa$, *i.e.*, $s > \frac{1}{4}\alpha^2+\frac{1}{2}\alpha+\beta\gamma$, we then have (using entries 6. and 7. from said table, and operation 4. from Table 15.1, on p.914):

$$A(t) = e^{\frac{1}{2}\alpha t}\cosh\left[(\tfrac{1}{4}\alpha^2+\beta\gamma)t\right] + \frac{2\alpha\,e^{\frac{1}{2}\alpha t}}{\alpha^2+4\beta\gamma}\,\sinh\left[(\tfrac{1}{4}\alpha^2+\beta\gamma)t\right]\ ,$$

and

$$B(t) = \frac{4\gamma\,e^{\frac{1}{2}\alpha t}}{\alpha^2+4\beta\gamma}\,\sinh\left[(\tfrac{1}{4}\alpha^2+\beta\gamma)t\right]\ .$$