1 Using a Generating Function

Throughout the Chapters on the Bessel and Legendre functions, and also the first section of Chapter 13—the Hermite functions—the discussion started and was centered around the generating function. Please do work through the following calculations, line by line, and check all signs, coefficients, *etc.*

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As the technicalities in the derivation might have obscured the general idea behind it, let us consider a frivolous example where the generating function (a function of two variables) is taken to be:

$$g(x,t) \;=\; \frac{e^{xt}}{1+2t}\;, \tag{1}$$

chosen rather at random (with one exception, which I will try to make clear below). By definition of a generating function, it generates some orthogonal functions which appear when we expand $g(x,t)$ as a function of t:

$$g(x,t) \;=\; \sum_{n=0}^{\infty} A_n(x)t^n\;. \tag{2}$$

So far, this merely defines the functions $A_n(x)$ as the coefficients (up to an $n!$) in the Taylor series of $g(x,t)$, having expanded in t and about $t=0$. That is, equating the right-hand-sides of the above two equations and taking the k^{th} partial derivative with respect to t, we have

$$\frac{\partial^k}{\partial t^k}\frac{e^{xt}}{1+2t} = \sum_{n=k}^{\infty} A_n(x)\, n(n{-}1)\cdots(n{-}k{+}1)\, t^{n-k}\;. \tag{3}$$

Note that the lower limit of the summation has been shifted since

$$\frac{\partial^k}{\partial t^k}\,t^n \;=\; \frac{\mathrm{d}^k}{\mathrm{d}t^k}\,t^n \;=\; 0 \quad \text{if} \quad k>n\;. \tag{4}$$

Setting now $t=0$ in (3)—*after* having calculated the derivatives—we notice that the sum on the r.h.s. of Eq. (3) starts with the 0^{th} power (=1) of t and for $n=k$; all other terms are positive powers of t. Setting $t=0$ therefore sets all terms with $n>k$ to zero and leaves only the $n=k$ term on the r.h.s.[1]:

$$\Big[\frac{\partial^k}{\partial t^k}\frac{e^{xt}}{1+2t}\Big]_{t=0} \;=\; A_k(x)\,k!\;, \tag{5}$$

whence

$$A_k(x) \;\stackrel{\text{def}}{=}\; \frac{1}{k!}\Big[\frac{\partial^k}{\partial t^k}\frac{e^{xt}}{1+2t}\Big]_{t=0}\;. \tag{6}$$

With a derivative formula like this, it is straightforward to obtain a contour-integral representation, which than also extends the definition of $A_n(x)$ to complex x:

$$A_k(x) \;=\; \frac{1}{2\pi i}\oint_C \mathrm{d}t\,\frac{g(x,t)}{t^{n+1}} \;=\; \frac{1}{2\pi i}\oint_C \mathrm{d}t\,\frac{e^{xt}}{(1+2t)t^{n+1}}\;, \tag{7}$$

where the contour C encircles $t=0$ (as that is where the $1/t^{n+1}$ factor in the integrand produces the residue). Note however that the generating function itself has a pole of the first order at $t=-\frac{1}{2}$,

[1] r.h.s.='right hand side'; w.r.t.='with respect to'; w.l.o.g.='without loss of generality'…

so that there exist different classes of contours, labeled by the number of times they wind around the $t = -\frac{1}{2}$ pole, which introduces some arbitrariness in the complex extension of $A_n(x)$.

Notice that the generating function can be expanded into a double series, by expanding the exponential in the conventional series while leaving the $1/(1+2t)$ factor as it is, and than expanding $1/(1+2t)$ into a geometric series. Since the latter has a singularity at $t = -\frac{1}{2}$, we may expand either for $|t| < \frac{1}{2}$ or for $|t| > 2$ at a time. The two expansions however can be related by analytic continuation (and using the above contour-integral representation). We therefore concentrate here at the $|t| < \frac{1}{2}$ case. So,

$$\frac{e^{xt}}{1+2t} = \sum_{k=0}^{\infty} \frac{(xt)^k}{k!(1+2t)} \ , \tag{8}$$

$$= \sum_{k=0}^{\infty} \frac{(xt)^k}{k!} \sum_{l=0}^{\infty} (-2t)^l \ = \ \sum_{k,l=0}^{\infty} (-2)^l \frac{x^k}{k!}\, t^{k+l} \ . \tag{9}$$

Comparing with the defining expansion (2), we introduce the summation variable $n = k + l$, for which we wish to sum over $n = 0, \ldots, \infty$. If we leave the other summation variable to be k, than owing to the fact that originally $0 \le l = n - k$, it would follow that $k \le n$, and the summation over n would have to run not from 0, but from k. Thus we must replace k also. There are many ways to do this, one of which is to introduce:

$$n = k + l \ , \qquad\qquad k = \tfrac{1}{2}(n+m) \ , \tag{10}$$

$$m = k - l \ , \qquad\qquad l = \tfrac{1}{2}(n-m) \ . \tag{11}$$

Since $k \ge 0$, $m \ge -n$, and similarly, since $l \ge 0$, $m \le +n$; that is, $-n \le m \le +n$. Furthermore, notce that if k, l are chosen so that n is even, m is also even, and if n is odd, m is also odd. Therefore the summation over m is not independent of n, which we must specify explicitly:

$$\frac{e^{xt}}{1+2t} \ = \ \sum_{n=0}^{\infty} \Bigg[\sum_{\substack{m=-n \\ m\pm n \text{ is even}}}^{n} (-2)^{\frac{1}{2}(n-m)} \frac{x^{\frac{1}{2}(n+m)}}{[\frac{1}{2}(n+m)]!} \Bigg] \, t^n \ , \tag{12}$$

where "$m{\pm}n$ is even" assures that m is even (odd) when n is even (odd). Owing to this constraint on m, the exponents and the argument of the factorial are all integers despite appearances. Consequently, the quantity in the large square brackets is a polynomial of order n in x. Comparing with Eq. (2), it may be identified with $A_n(x)$:

$$A_n(x) \ = \ \sum_{\substack{m=-n \\ m+n \text{ is even}}}^{n} (-2)^{\frac{1}{2}(n-m)} \frac{x^{\frac{1}{2}(n+m)}}{[\frac{1}{2}(n+m)]!} \ . \tag{13}$$

Alternatively instead of Eq. (11), we can use l itself. Thus, we have

$$n = k + l \qquad\qquad \Rightarrow k = n - l \ , \tag{14}$$

and, since $0 \le k \le \infty$, we have that

$$0 \le [k = (n-l)] \le \infty \qquad\qquad \Rightarrow l \le n \le \infty \ . \tag{15}$$

Therefore,

$$\frac{e^{xt}}{1+2t} = \sum_{n=0}^{\infty} \left[\sum_{l=0}^{n} (-2)^l \frac{x^{n-l}}{(n-l)!} \right] t^n \ , \tag{16}$$

$$A_n(x) = \sum_{l=0}^{n} (-2)^l \frac{x^{n-l}}{(n-l)!} = (-2)^n \sum_{m=0}^{n} (-2)^{-m} \frac{x^m}{m!} \ , \tag{17}$$

which is indeed simpler-looking than the formula (12), but the Reader should check that the two formulae provide identical expressions for each $A_n(x)$.

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Now to the calculation of recursion relations. Consider first taking the partial derivative w.r.t. t. On one hand, taking $\frac{\partial}{\partial t}$ of Eq. (1) produces

$$\frac{\partial}{\partial t} \frac{e^{xt}}{1+2t} \ = \ \frac{xe^{xt}}{1+2t} - \frac{e^{xt}}{(1+2t)^2}(2) \ = \ \big(x - \frac{2}{1+2t}\big)\big(\frac{e^{xt}}{1+2t}\big) \ , \tag{18}$$

or

$$\frac{\partial}{\partial t}\, g(x,t) \ = \ \big(x - \frac{2}{1+2t}\big)\, g(x,t) \ . \tag{19}$$

Most importantly, the derivative of $g(x,t)$ is proportional to $g(x,t)$ itself. This is the only requirement that one must impose on the choice of a generating function; without this, the next step would be impossible, and with it also the rest of the 'generating function magic'. *Voila*:

$$\frac{\partial}{\partial t}\, g(x,t) \ = \ \big(x - \frac{2}{1+2t}\big) \sum_{n=0}^{\infty} A_n(x) t^n \ . \tag{20}$$

Indeed, was the derivative not proportional to $g(x,t)$, the expansion (2) would not be possible to insert, and so the derivative of $g(x,t)$ would have no bearing on the $A_n(x)$'s.

On the other hand, taking the derivative of Eq. (2), we have

$$\frac{\partial}{\partial t}\, g(x,t) \ = \ \sum_{n=0}^{\infty} A_n(x) n t^{n-1} \ . \tag{21}$$

The l.h.s's of the last two equations being the same, we may equate their respective r.h.s's and obtain

$$\big(x - \frac{2}{1+2t}\big) \sum_{n=0}^{\infty} A_n(x) t^n \ = \ \sum_{n=0}^{\infty} A_n(x) n t^{n-1} \ , \tag{22}$$

or, multiplying through (left and right) by $1+2t$:

$$(1+2t)x \sum_{n=0}^{\infty} A_n(x) t^n \ - \ 2 \sum_{n=0}^{\infty} A_n(x) t^n \ = \ (1+2t) \sum_{n=0}^{\infty} A_n(x) n t^{n-1} \ . \tag{23}$$

Multiplying through by the powers of t (and writing each term separately):

$$x \sum_{n=0}^{\infty} A_n(x) t^n \ + \ 2x \sum_{n=0}^{\infty} A_n(x) t^{n+1} \ - \ 2 \sum_{n=0}^{\infty} A_n(x) t^n \tag{24}$$

$$= \ \sum_{n=0}^{\infty} A_n(x) n t^{n-1} \ + \ 2 \sum_{n=0}^{\infty} A_n(x) n t^n \ . \tag{25}$$

Combining sums with like powers, we have

$$\sum_{n=0}^{\infty}(x-2-2n)\,A_n(x)\,t^n \;+\; 2x\sum_{n=0}^{\infty}A_n(x)t^{n+1} \;=\; \sum_{n=0}^{\infty}A_n(x)nt^{n-1}\;. \tag{26}$$

Since this is an equation of power series and different powers of t are linearly independent, this will hold only if the equality holds for each different power of t separately. So, equating the coefficients of t^k than produces one of the desired recursion relations:

$$(x-2k-2)\,A_k(x) \;+\; 2x\,A_{k-1}(x) \;=\; (k{+}1)\,A_{k+1}(x)\;. \tag{27}$$

Note that this is a “three-orders/no-derivatives” relation.

Another recursion relation is obtained by taking the (first) derivative w.r.t. x:

$$\text{from Eq. (1)} \quad : \qquad \frac{\partial}{\partial x}\,g(x,t) \;=\; t\,\frac{e^{xt}}{1+2t} \;=\; t\sum_{n=0}^{\infty}A_n(x)t^n\;, \tag{28}$$

which is again—most importantly—proportional to $g(x,t)$ itself! On the other hand,

$$\text{from Eq. (2)} \quad : \qquad \frac{\partial}{\partial x}\,g(x,t) \;=\; \sum_{n=0}^{\infty}A'_n(x)t^n\;. \tag{29}$$

Equating the r.h.s’s of these two representations of the derivative:

$$\sum_{n=0}^{\infty}A_n(x)t^{n+1} \;=\; \sum_{n=0}^{\infty}A'_n(x)t^n\;. \tag{30}$$

Again, this is an equation of power series and different powers of t are linearly independent, whence the equality has to hold for each individual power and we may equate the coefficients of (say) the k^{th} power. This produces the “other” recursion relation:

$$A_{k-1}(x) \;=\; A'_k(x)\;. \tag{31}$$

Note that this is a “two-orders/one-derivatives” relation.

With the help of (31), we may modify (27) into another “two-orders/one-derivative” relation:

$$(x-2k-2)\,A_k(x) \;+\; 2x\,A'_k(x) \;=\; (k{+}1)\,A_{k+1}(x)\;. \tag{32}$$

Shifting $k\to k{-}1$ in the last relation, we obtain

$$(x-2k)\,A_{k-1}(x) \;+\; 2x\,A'_{k-1}(x) \;=\; k\,A_k(x)\;, \tag{33}$$

in which—owing to Eq. (31)—we can replace A_{k-1} with A'_k and A'_{k-1} with A''_k, and so obtain

$$(x-2k)\,A'_k(x) \;+\; 2x\,A''_k(x) \;=\; k\,A_k(x)\;, \tag{34}$$

which is a “one-order/two-derivatives” relation, better known as the second order differential equation for $A_n(x)$:

$$2x\,A''_k(x) \;+\; (x-2k)\,A'_k(x) \;-\; k\,A_k(x) \;=\; 0\;, \tag{35}$$

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It is perhaps hardly a surprise that you've never seen this differential equation and that it does not have a name: I have just made it up (when I chose the generating function) so as to illustrate the method. Remember—the generating function was quite arbitrary, except that its derivatives had to be proportional to the function itself. Of course, this is not really a restriction, since for any $f(x,t)$:

$$\frac{\partial}{\partial t}\,f(x,t) \;\equiv\; \Big[\frac{1}{f(x,t)}\frac{\partial}{\partial t}\,f(x,t)\Big]\,f(x,t) \;\equiv\; \Big[\frac{\partial}{\partial t}\,\ln\big(f(x,t)\big)\Big]\,f(x,t) \tag{36}$$

and so a derivative can *always* be made proportional to the function. The trouble is that, for an arbitrary $f(x,t)$, the 'coefficient' of proportionality, $\frac{\partial}{\partial t}\ln\big(f(x,t)\big)$, may not be a ratio of polynomials. However, if it is, the method may still be carried through, albeit perhaps not by mortal humans.

Next, I cheated a bit when choosing the generating function, in that I chose it so that the differential equation would come out to be of second order. By contrast, consider

$$G(x,t) \;=\; \frac{e^{xt}}{1+t^2} \;=\; \sum_{n=0}^{\infty} Z_n(x)t^n. \tag{37}$$

the derivative by t now yields

$$(1+t^2)x\sum_{n=0}^{\infty} Z_n(x)t^n \;-\; 2t\sum_{n=0}^{\infty} Z_n(x)t^n \;=\; (1+t^2)\sum_{n=0}^{\infty} Z_n(x)nt^{n-1}\ , \tag{38}$$

and so

$$x\sum_{n=0}^{\infty} Z_n(x)t^n \;+\; x\sum_{n=0}^{\infty} Z_n(x)t^{n+2} \;-\; 2\sum_{n=0}^{\infty} Z_n(x)t^{n+1} \tag{39}$$

$$=\; \sum_{n=0}^{\infty} Z_n(x)nt^{n-1} \;+\; \sum_{n=0}^{\infty} Z_n(x)nt^{n+1}\ . \tag{40}$$

This being an equation of power series where different powers of t are linearly independent, we again equate the coefficients of a specified (say, the k^{th}) power of t and obtain a recursion relation:

$$x\,Z_k(x) \;+\; x\,Z_{k-2}(x) \;-\; 2\,Z_{k-1}(x) \;=\; (k{+}1)\,Z_{k+1}(x) \;+\; (k{-}1)\,Z_{k-1}(x)\ , \tag{41}$$

or

$$x\,Z_k(x) \;+\; x\,Z_{k-2}(x) \;=\; (k{+}1)\,Z_{k+1}(x) \;+\; (k{+}1)\,Z_{k-1}(x)\ , \tag{42}$$

which is a "*four*-orders/no-derivatives" relation! With a little forethought, let us shift $k \to k{-}1$:

$$x\,Z_{k-1}(x) \;+\; x\,Z_{k-3}(x) \;=\; k\,Z_k(x) \;+\; k\,Z_{k-2}(x)\ . \tag{43}$$

Now, the x-derivative still produces the rather simple recursion relation

$$Z_{k-1}(x) \;=\; Z'_k(x)\ . \tag{44}$$

Shifting $k \to k-1$ in (44), we have

$$Z_{k-2}(x) \;=\; Z'_{k-1}(x)\ , \tag{45}$$

where the r.h.s. can be related back to (the second derivative) of $Z_k(x)$[2] using (44),

$$Z_{k-2}(x) \;=\; Z''_k(x)\ . \tag{46}$$

Similarly,

$$Z_{k-3}(x) \;=\; Z'''_k(x)\ . \tag{47}$$

Using these relations, we can turn the "four-order/no-derivative" recursion relation into the *third* order differential equation:

$$x\,Z'_k(x) \;+\; x\,Z'''_k(x) \;=\; k\,Z_k(x) \;+\; k\,Z''_k(x)\ . \tag{48}$$

This again is not something you are likely to find in typical physics problems, but I hope does illustrate the method.

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Actually, we have thus far merely obtained recursion relations (including the differential equation). There is rather more to it (as you know).

Once the differential equation is obtained, check if it is in the self-adjoint form. If it is not (as is the case with the above two equations), find a suitable prefactor to multiply the whole equation and turn it into a self-adjoint one. (This can always be done, and was discussed in detail in the first month of the course.)

Once the differential equation is put into the self-adjoint form, you can *read-off* the weight-function $w(x)$ and therefore *know* that there must be an orthogonality relation of the type

$$\int_{x_1}^{x_2} \mathrm{d}x\; w(x)\,A_n(x)\,A_m(x) \;=\; N_n\,\delta_{m,n}\ , \tag{49}$$

where N_n is the normalization constant (to be determined shortly) and the limits of integration, x_1 and x_2, are chosen so that the differential operator in the self-adjoint form of the equation is indeed self-adjoint. That is, if $\mathcal{L}$ is that operator, we must have

$$\int_{x_1}^{x_2} \mathrm{d}x\; w(x)\,A_n(x)\,\mathcal{L}\big[A_m(x)\big] \;=\; \int_{x_1}^{x_2} \mathrm{d}x\; w(x)\,\mathcal{L}\big[A_n(x)\big]\,A_m(x)\ . \tag{50}$$

The condition on the limits comes from 'passing' the differential operator $\mathcal{L}$ from one function onto the other using integration by parts—the 'boundary' terms, usually called "the uv-terms" in

$$\int_{x_1}^{x_2} u\,\mathrm{d}v \;=\; \big[\,u\,v\,\big]_{x_1}^{x_2} \;-\; \int_{x_1}^{x_2} v\,\mathrm{d}u\ , \tag{51}$$

must vanish; $[\,u\,v\,]_{x_1}^{x_2} = 0$, either because "$uv$" vanishes in both limits or because the values cancel.

[2] You should notice that we are persistently "trading orders for derivatives"...

Finally, the normalization constant can often be calculated using again the generating functions, as follows. One generating function such as (1)–(2) expands into a series containing a single orthogonal function $A_n(x)$. So, to obtain a product of two orthogonal functions, $A_n(x)A_m(x)$, we multiply two generating functions with however different 'formal' variables:

$$g(x,t)\,g(x,s) \;=\; \Big[\sum_{n=0}^{\infty} A_n(x)t^n\Big]\Big[\sum_{m=0}^{\infty} A_m(x)s^m\Big] \;=\; \sum_{m,n=0}^{\infty} t^n s^m\, A_n(x)A_m(x)\;. \tag{52}$$

To obtain the expression within the orthogonality integral, multiply left and right by the weight-function, $w(x)$, and integrate:

$$\int_{x_1}^{x_2} \mathrm{d}x\; w(x)\; g(x,t)\,g(x,s) \;=\; \sum_{m,n=0}^{\infty} t^n s^m \int_{x_1}^{x_2} \mathrm{d}x\; w(x)\; A_n(x)A_m(x)\;. \tag{53}$$

Using the orthogonality of the $A_n(x)$'s (and inserting the so far unknown normalization constant N_n):

$$\int_{x_1}^{x_2} \mathrm{d}x\; w(x)\; g(x,t)\,g(x,s) \;=\; \sum_{m,n=0}^{\infty} t^n s^m\, N_n\,\delta_{m,n} \;=\; \sum_{n=0}^{\infty} (ts)^n\, N_n\;. \tag{54}$$

We may now proceed only if it is possible to evaluate the integral on the left[3]. Whatever the result of this integration, expanding it as a power-series in (st) and comparing coefficients of like powers is then guaranteed to produce a result for N_n[4].

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Finally (now really), recall again that generating functions appear not infrequently in physics applications in the guise of 'partition functions' and potentials, and that an expansion into orthogonal functions such as (2) produces a useful tool for calculating. For example, the Legendre polynomials appear in such an expansion of the Coulomb potential—which therefore is the generating function for $P_n(\cos\theta)$. These than are useful in the "multipole expansion"—expanding the potential or electromagnetic field into the Legendre polynomials, where the $P_n(\cos\theta)$-term has the physical interpretation of approximating the complete system of charges producing the field by a *multipole*—a simple arrangement of 2^n equal charges.

[3]. . . which is why we spent so much time last semester on complex variables and Γ-function techniques of integration. . .

[4]The adventurous reader should try their hand in completing these calculations for the above two generating functions.