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The BLT (Bessel-Legendre-Trigonometric) System

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The following notes are meant to complement, rather than supplant the material given in Ref. [1].

1 Introduction

We are interested in solutions to the Helmholz equation

$$\big[\,\vec{\nabla}^2_{(D)} + k^2\,\big]\psi(\vec{r}_{(D)}) \;=\; 0, \tag{1.1}$$

where $\vec{r}$ and $\vec{\nabla}^2_{(D)}$ are the position vector and the Laplacian operator, respectively, in D-dimensional space. In Cartesian coordinate systems,

$$\vec{r}_{(D)} \;:=\; \sum_{i=1}^{D} x^i\,\hat{\mathrm{e}}_i \;=\; x\,\hat{\mathrm{e}}_x + y\,\hat{\mathrm{e}}_y + z\,\hat{\mathrm{e}}_z + \ldots, \tag{1.2}$$

$$\vec{\nabla}^2_{(D)} \;:=\; \sum_{i=1}^{D} \frac{\mathrm{d}^2}{\mathrm{d}x^{i\,2}} \;=\; \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \ldots \tag{1.3}$$

It is frequently convenient to use (hyper-)spherical coordinates[1] $(r, \theta_1, \cdots, \theta_{D-2}, \phi)$:

$$r = \sqrt{\textstyle\sum_{i=1}^{D}(x^i)^2}, \qquad x^D = r\,\cos\theta_1, \tag{1.4}$$

$$\theta_1 = \arctan\left(\frac{\sqrt{\sum_{i=1}^{D-1}(x^i)^2}}{x^d}\right) \qquad x^{D-1} = r\,\sin\theta_1\,\cos\theta_2, \tag{1.5}$$

$$\vdots \qquad\qquad \vdots \tag{1.6}$$

$$\theta_j = \arctan\left(\frac{\sqrt{\sum_{i=1}^{D-j}(x^i)^2}}{x^{D-j+1}}\right) \qquad x^{D-j} = r\,\sin\theta_1\,\sin\theta_2\,\cos\theta_{j+1}, \tag{1.7}$$

$$\vdots \qquad\qquad \vdots \tag{1.8}$$

$$\theta_{D-2} = \arctan\left(\frac{\sqrt{(x^1)^2+(x^2)^2}}{x^3}\right) \qquad x^2 = r\,\sin\theta_1\cdots\sin\theta_{D-2}\,\cos\phi, \tag{1.9}$$

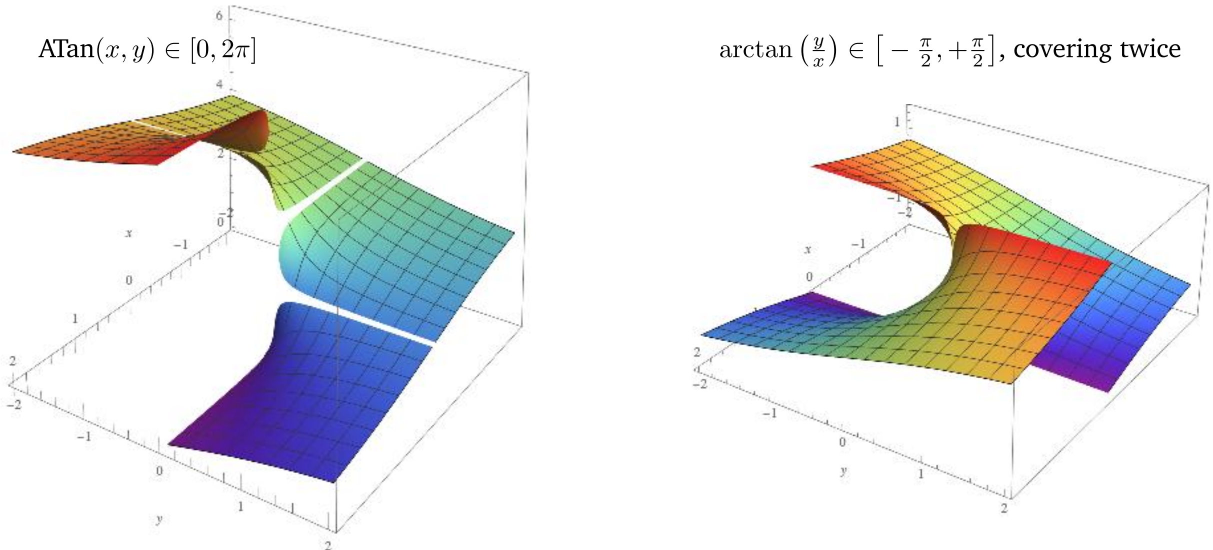
$$\phi = \mathrm{ATan}(x^1, x^2) \qquad x^1 = r\,\sin\theta_1\cdots\sin\theta_{D-2}\,\sin\phi. \tag{1.10}$$

Note that all the numerators in the argument of the arctan functions are positive by definition, so that the arguments take on values $(-\infty, +\infty)$, corresponding to which the computed angles take values $\theta_i \in [0, \pi]$ for $i = 1, \cdots, D{-}2$, just as needed. This succession would imply the assignment $\phi = \theta_{D-1} = \arctan(\frac{x^1}{x^2})$ for

[1] See <http://en.wikipedia.org/wiki/Hypersphere> for more info, in a slightly different notation.

the last angle, which is however doubly-valued since $\frac{x^1}{x^2} = \frac{-x^1}{-x^2}$. To remedy this, we've used the function:

$$\mathrm{ATan}(x,y) := \begin{cases} \arctan(y/x) & \text{for} \quad 0 \le x, y, \\ \pi + \arctan(y/x) & \text{for} \quad x < 0, \\ 2\pi + \arctan(y/x) & \text{for} \quad y < 0 \le x. \end{cases} \tag{1.11a}$$



(1.11b)

to define the azimuthal angle ϕ, which therefore takes on twice as many values as do the θ_i. It is not hard to verify by direct integration of the volume element, obtained by changing variables from $\prod_i \mathrm{d}x^i$, that these ranges of the angles give the correct volume, without double-counting.

In such (hyper-)spherical coordinates, $\vec{r}_{(D)} = r\,\hat{\mathrm{e}}_r$ and

$$\vec{\nabla}^2_{(D)} f(\vec{r}_{(D)}) \;=\; \frac{1}{r^{D-1}}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}f(\vec{r}_{(D)})}{\mathrm{d}r}\Big)\Big] - \frac{1}{r^2}\mathscr{L}^2_{(D)}\, f(\vec{r}_{(D)}), \tag{1.12}$$

where $\mathscr{L}^2_{(D)}$ is a 2^{nd} order partial differential operator but only with respect to the angular variables, $\theta_1, \cdots, \theta_{D-2}, \phi$, and may be interpreted as the square of the angular momentum operator. For the lowest few values of D, we have:

$$\mathscr{L}^2_{(1)} = 0, \tag{1.13a}$$

$$\mathscr{L}^2_{(2)} = -\frac{\partial^2}{\partial\phi^2}, \tag{1.13b}$$

$$\mathscr{L}^2_{(3)} = -\frac{1}{\sin\theta}\Big[\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)\Big] - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{1.13c}$$

$$\vdots \qquad \vdots$$

In fact, $\mathscr{L}^2_{(2)}$ is the square of a 2-dimensional *scalar* operator and $\mathscr{L}^2_{(3)}$ the square of a 3-dimensional *vector* operator: in D dimensions, $\mathscr{L}^2_{(D)}$ is the so-called tensor-square of the angular momentum rank-$(D{-}2)$ tensor operator, most easily represented in Cartesian coordinates[2]:

$$[\,\mathscr{L}_{(D)}\,]_{i_1\cdots i_{(D-2)}} \;:=\; -\,i\,\varepsilon_{i_1\cdots i_{(D-2)}jk}\, g^{kl}\, x^j \frac{\partial}{\partial x^l}. \tag{1.14}$$

[2] The choice of the overall constant "$-i$" conforms to the convention of [1, pp. 201–202]. Other sources may omit the negative sign; mathematical texts (unconcerned with physical applications) may also omit the "i."

Indeed, for $D=2$:

$$\mathscr{L}_{(2)} = -i\,\varepsilon_{ij}\,\delta^{jk}\,x^i\frac{\partial}{\partial x^k} \;=\; \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \;=\; \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} -i\frac{\partial}{\partial y} \\ i\frac{\partial}{\partial x} \end{bmatrix} \;=\; -\,i\Big[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big]. \tag{1.15}$$

Let's convert this into spherical coordinates:

$$= \begin{bmatrix} r\cos\phi & r\sin\phi \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial \phi}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \phi}{\partial y} \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \end{bmatrix}, \tag{1.16}$$

$$= i\,\Big\{[r\cos\phi]\Big[\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\Big] - [r\sin\phi]\Big[\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\Big]\Big\}, \tag{1.17}$$

$$= i\,\Big\{[r\cos\phi]\Big[\sin\phi\frac{\partial}{\partial r} + \Big(\frac{\cos\phi}{r}\Big)\frac{\partial}{\partial \phi}\Big] - [r\sin\phi]\Big[\cos\phi\frac{\partial}{\partial r} + \Big(-\frac{\sin\phi}{r}\Big)\frac{\partial}{\partial \phi}\Big]\Big\}, \tag{1.18}$$

$$= i(\cos^2\phi + \sin^2\phi)\frac{\partial}{\partial \phi} = i\frac{\partial}{\partial \phi}. \qquad \textit{(\ldots\textbf{Whew!})} \tag{1.19}$$

But, don't take my word for it: do the math! The Cartesian representation of the $D=3$ case is given in Eq. (3.7) below, while the spherical coordinate representation is [1, p. 202]:

$$\vec{\mathscr{L}}_{(3)} = -i(\vec{r}\times\vec{\nabla}) = i\Big[\,\hat{\mathrm{e}}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \;-\; \hat{\mathrm{e}}_\phi\frac{\partial}{\partial\theta}\,\Big] \tag{1.20}$$

which seems so much simpler… until one realizes that $\hat{\mathrm{e}}_\theta$ and $\hat{\mathrm{e}}_\phi$ are not constant, and hide some of the angular dependence:

$$\hat{\mathrm{e}}_\theta = \cos\theta\,\cos\phi\,\hat{\mathrm{e}}_x + \cos\theta\,\sin\phi\,\hat{\mathrm{e}}_y - \sin\phi\,\hat{\mathrm{e}}_z, \tag{1.21}$$

$$\hat{\mathrm{e}}_\phi = -\sin\phi\,\hat{\mathrm{e}}_x + \cos\phi\,\hat{\mathrm{e}}_y. \tag{1.22}$$

The spherical coordinate version it is *much* more complicated for $D>3$, and is not given here.

The Helmholz equation (1.1) therefore becomes:

$$\frac{1}{r^{D-1}}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}f(\vec{r}_{(D)})}{\mathrm{d}r}\Big)\Big] - \frac{1}{r^2}\mathscr{L}^2_{(D)}\,f(\vec{r}_{(D)}) + k^2 f(\vec{r}_{(D)}) = \;0, \tag{1.23}$$

or, after multiplying through by r^2,

$$\frac{1}{r^{D-3}}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}f(\vec{r}_{(D)})}{\mathrm{d}r}\Big)\Big] + k^2r^2 f(\vec{r}_{(D)}) = \;\mathscr{L}^2_{(D)}\,f(\vec{r}_{(D)}), \tag{1.24}$$

which easily separates the radial coordinate, r, from the angular ones. We look for $f(\vec{r}_{(D)})$ in the factorized form $f(\vec{r}_{(D)}) = R_Q(r)\,Y_Q(\theta_1,\cdots,\theta_{D-2},\phi)$ and proceed as usual:

$$\frac{1}{r^{D-3}}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}R_Q(r)}{\mathrm{d}r}\Big)\Big] + (kr)^2R_Q(r) = Q\,R_Q(r), \tag{1.25}$$

$$\mathscr{L}^2_{(D)}\,Y_Q(\theta_1,\cdots,\theta_{D-2},\phi) = Q\,Y_Q(\theta_1,\cdots,\theta_{D-2},\phi). \tag{1.26}$$

The general solution will then be of the form $f(\vec{r}_{(D)}) = \sum_Q R_Q(r)\,Y_Q(\theta_1,\cdots,\theta_{D-2},\phi)$, where the Q-sum is over all values of Q allowed by boundary conditions and other restrictions, such as periodicity in the ϕ-angle.

We now turn to analyze (1.25) in § 2, and will then return to (1.26) in § 3.

2 B L T → The Bessel Equations' Hierarchy

The radial equation may be rewritten in the following equivalent formats:

$$\frac{1}{r^{D-1}}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}R_Q(r)}{\mathrm{d}r}\Big)\Big]+k^2R_Q(r)-\frac{Q}{r^2}\,R_Q(r)=\;0, \tag{2.1a}$$

$$\frac{\mathrm{d}^2R_Q(r)}{\mathrm{d}r^2}+\frac{D-1}{r}\frac{\mathrm{d}R_Q(r)}{\mathrm{d}r}+k^2R_Q(r)-\frac{Q}{r^2}\,R_Q(r)=\;0, \tag{2.1b}$$

$$r^2\frac{\mathrm{d}^2R_Q(r)}{\mathrm{d}r^2}+(D-1)r\frac{\mathrm{d}R_Q(r)}{\mathrm{d}r}+(k\,r)^2R_Q(r)-Q\,R_Q(r)=\;0, \tag{2.1c}$$

$$\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}R_Q(r)}{\mathrm{d}r}\Big)\Big]+k^2r^{D-1}R_Q(r)-Q\,r^{D-3}\,R_Q(r)=\;0. \tag{2.1d}$$

It should be clear from the version (2.1c) that the change of variables $z=kr$ absorbes k in a rescaling of r, whereupon one frequently writes $R_Q(kr)$ in place of $R_Q(r)$. The last version, Eq. (2.1d) makes it clear that this differential equation may be identified as a Sturm-Liouville equation in *two very different ways*.

(I): In the first (& ***standard***) case, we fix Q, and write:

$$\mathcal{L}_Q=\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Big]-Q\,r^{D-3},\qquad p(r)=r^{D-1},\quad q(r)=-r^{D-3}, \tag{2.2}$$

$$\lambda=+k^2,\qquad w(r)=r^{D-1}, \tag{2.3}$$

from which it follows that the Bessel equation (2.1d) may be written as

$$\mathcal{L}_Q[R_Q(kr)]+k^2\,r^{D-1}\,R_Q(kr)=0, \tag{2.4}$$

which is a Sturm-Liouville equation and so implies the orthogonality condition

$$\int_a^b \mathrm{d}r\,r^{D-1}\;R_Q^*(kr)\,R_Q(k'r)\;=\;N_k\,\delta_{k,k'}, \tag{2.5}$$

provided limits a,b are chosen so that

$$\Big[R_Q^*(kr)\,r^{D-1}\frac{\mathrm{d}R_Q(k'r)}{\mathrm{d}r}\Big]_a^b\;=\;0,\qquad \forall k,k'. \tag{2.6}$$

(II): Alternatively (& ***non-standard***) , we may fix k instead, and write:

$$\mathcal{L}_k=\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Big]+k^2r^{D-1},\qquad p(r)=q(r)=r^{D-1}, \tag{2.7}$$

$$\lambda=-Q,\qquad w(r)=r^{D-3}, \tag{2.8}$$

from which it follows that the Bessel equation (2.1d) may be written as

$$\mathcal{L}_k[R_Q(kr)]-Q\,r^{D-3}\,R_Q(kr)=0, \tag{2.9}$$

which is a Sturm-Liouville equation and so implies the orthogonality condition

$$\int_a^b \mathrm{d}r\,r^{D-3}\;R_Q^*(kr)\,R_{Q'}(kr)\;=\;N_Q\,\delta_{Q,Q'}, \tag{2.10}$$

provided limits a, b are chosen so that

$$\Big[R^*_Q(kr)\,r^{D-1}\frac{\mathrm{d}R_{Q'}(kr)}{\mathrm{d}r}\Big]_a^b \;=\; 0, \qquad \forall Q, Q'. \tag{2.11}$$

It should be clear that Eq. (1.26) may—and in fact does—lead to a similar orthogonality condition, fixing Q as the eigenvalue of $\mathscr{L}^2_{(D)}$. For this reason, the second option, (2.7)–(2.11) is used rarely, if ever. In turn, the first option, (2.2)–(2.6), is in standard use.

The Bessel equations, derived above for arbitrary $D \geq 1$, thus form a semi-infinite sequence:

$$\text{(there's no } Q \text{ for } D=1) \qquad \frac{\mathrm{d}^2 R^{(1)}(kr)}{\mathrm{d}r^2} + k^2\, R^{(1)}(kr) = 0, \qquad D = 1\,; \tag{2.12}$$

$$\frac{1}{r}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}R^{(1)}_m(kr)}{\mathrm{d}r}\Big)\Big] + \Big[k^2 - \frac{m^2}{r^2}\Big]\,R^{(1)}_m(kr) = 0, \qquad D = 2\,; \tag{2.13}$$

$$\frac{1}{r^2}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\frac{\mathrm{d}R^{(1)}_\ell(kr)}{\mathrm{d}r}\Big)\Big] + \Big[k^2 - \frac{\ell(\ell{+}1)}{r^2}\Big]\,R^{(1)}_\ell(kr) = 0, \qquad D = 3\,; \tag{2.14}$$

and so on. We have used the standard notations for $Q = m^2$ in $D = 2$ and $Q = \ell(\ell{+}1)$ in $D = 3$; this follows from the nature of $\mathscr{L}^2_{(2)}$ and $\mathscr{L}^2_{(3)}$, and will be examined subsequently.

Note that $\mathscr{L}^2_{(1)} \equiv 0$ as there are no angular variables, so that there is no Q-term in (2.12); whence we wrote $\emptyset$ in place of Q. Also, this is the probably the best known of the equations in the sequence, being solved by a linear combination of $\sin(kr)$ and $\cos(kr)$, or alternatively, of e^{ikr} and e^{-ikr}. In a sense then, the 'cylindrical Bessel functions' of the first and second kind, $J_m(kr)$ and $N_m(kr)$, that solve the equation (2.13) and the 'spherical Bessel functions' of the first and second kind, $j_\ell(kr)$ and $n_\ell(kr)$, that solve the equation (2.14) are then generalizations of $\sin(kr)$ and $\cos(kr)$, respectively.

In fact, this relationship is quite solid. Consider substituting

$$R(r) = r^\alpha\, P(r), \tag{2.15}$$

$$\frac{\mathrm{d}R(r)}{\mathrm{d}r} = r^\alpha\,\Big[\frac{\mathrm{d}P(r)}{\mathrm{d}r} + \frac{\alpha}{r}P(r)\Big], \tag{2.16}$$

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} = r^\alpha\,\Big[\frac{\mathrm{d}^2P(r)}{\mathrm{d}r^2} + \frac{2\alpha}{r}\frac{\mathrm{d}P(r)}{\mathrm{d}r} + \frac{\alpha(\alpha{-}1)}{r^2}P(r)\Big], \tag{2.17}$$

into Eq. (1.25), we find

$$r^\alpha\,\Big[\frac{\mathrm{d}^2P(r)}{\mathrm{d}r^2} + \frac{2\alpha{+}D{-}1}{r}\frac{\mathrm{d}P(r)}{\mathrm{d}r} + \Big(k^2 - \frac{Q - \alpha(D+\alpha{-}2)}{r^2}\Big)P(r)\Big] = 0. \tag{2.18}$$

As $r^\alpha \to 0$ vanishes only at $r = 0$ if $\alpha > 0$ or at $r \to \infty$ if $\alpha < 0$, we obtain:

$$\frac{\mathrm{d}^2P(r)}{\mathrm{d}r^2} + \frac{2\alpha{+}D{-}1}{r}\frac{\mathrm{d}P(r)}{\mathrm{d}r} + \Big(k^2 - \frac{Q - \alpha(D+\alpha{-}2)}{r^2}\Big)P(r) = 0. \tag{2.19}$$

From this result, you can see how to pick α so as to hop from any D-dimensional Bessel equation to any other D'-dimensional one.

2.1 A Few Special Cases

Case A: Choose $\alpha = -\frac{D-1}{2}$, whereupon Eq. (2.19) turns into:

$$\frac{\mathrm{d}^2P}{\mathrm{d}r^2} + \Big[k^2 - \frac{Q + (D{-}1)(D{+}3)/4}{r^2}\Big]P \;=\; 0. \tag{2.20}$$

For $Q = -\frac{(D-1)(D-3)}{4}$, this simplifies further, into:

$$P''(r) + k^2 P(r) \;=\; 0, \qquad \Rightarrow \qquad P(r) = A\sin(kr+\varphi), \qquad A,\varphi = \mathit{const}. \tag{2.21}$$

Going back to Eq. (1.25), we find that its special case:

$$\frac{1}{r^{D-1}}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}R}{\mathrm{d}r}\Big)\Big] + \Big[k^2 - \frac{(D-1)(D-3)}{4r^2}\Big]\,R \;=\; 0, \tag{2.22}$$

is indeed solved by $R(r) = A\,r^{(1-D)/2}\,\sin(kr+\varphi)$.

Case B: On the other hand, selecting $\alpha = 1 - \frac{D}{2}$, Eq. (2.19) turns into:

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \Big[k^2 - \frac{4Q + (2-D)^2}{4r^2}\Big]P \;=\; 0, \tag{2.23}$$

which is solved by $P(r) = AJ_\mu(kr) + BN_\mu(kr)$, where $\mu = \pm\sqrt{Q + \frac{1}{4}(D-2)^2}$, and $J_\mu(kr)$ and $N_\mu(kr)$ are the cylindrical Bessel functions of the first and second kind, respectively. Therefore, Eq. (2.19) is solved by

$$R(r) = r^{\frac{2-D}{2}}\big[AJ_\mu(kr) + BN_\mu(kr)\big], \qquad \mu = \pm\sqrt{Q + \tfrac{1}{4}(D-2)^2}, \tag{2.24}$$

in terms of cylindrical Bessel functions.

Case C: Alternatively, with $\alpha = -\frac{D-3}{2}$, Eq. (2.19) turns into:

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + \Big[k^2 - \frac{4Q + (D-1)(D-3)}{4r^2}\Big]P \;=\; 0, \tag{2.25}$$

which is solved by $P(r) = a\,j_\ell(kr) + b\,n_\ell(kr)$, where $\ell = \pm\sqrt{Q + \frac{1}{4}(D-2)^2} - \frac{1}{2}$, so that Eq. (2.19) is also solved by

$$R(r) = r^{\frac{3-D}{2}}\big[a\,j_\ell(kr) + b\,n_\ell(kr)\big], \qquad \ell = \pm\sqrt{Q + \tfrac{1}{4}(D-2)^2} - \tfrac{1}{2}, \tag{2.26}$$

in terms of spherical Bessel functions. Clearly, we thus have the identity

$$\big[AJ_\mu(kr) + BN_\mu(kr)\big] \;=\; \sqrt{r}\big[a\,j_{\mu-\frac{1}{2}}(kr) + b\,n_{\mu-\frac{1}{2}}(kr)\big], \tag{2.27}$$

relating spherical Bessel functions to the cylindrical ones.

In fact, we have the general relationship:

$$R^{(D)}_\mu(kr) \;=\; \frac{1}{\sqrt{r}}\,R^{(D-1)}_{\mu+\frac{1}{2}}(kr), \qquad D = 2,3,4,\dots \tag{2.28}$$

— ⋆ —

Case $k = 0$: In this special case, Eq. (2.19) becomes homogeneous in r:

$$\frac{1}{r^{D-1}}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^{D-1}\frac{\mathrm{d}R_Q(r)}{\mathrm{d}r}\Big)\Big] - \frac{Q}{r^2}\,R_Q(r) \;=\; 0, \tag{2.29}$$

and so is solved by pure powers. Indeed, by substituting $R_Q(r) = r^\beta$, we obtain:

$$R_Q(r) \;=\; C_+\,r^{\beta_+} + C_-\,r^{\beta_-}, \qquad \beta_\pm = \frac{2-D}{2} \pm \sqrt{\tfrac{1}{4}(D-2)^2 + Q}. \tag{2.30}$$

Since the $k \to 0$ limiting case of the Bessel equations (2.19) are the homegeneous equations (2.29), this implies the $k \to 0$ asymptotic behavior:

$$J_\mu(kr) \sim r^\mu, \qquad N_\mu(kr) \sim r^{-\mu}, \qquad D = 2\,; \tag{2.31}$$

$$j_\ell(kr) \sim r^\ell, \qquad n_\ell(kr) \sim r^{-(\ell+1)}, \qquad D = 3\,; \tag{2.32}$$

and so on.

3 The Angular Part

When separating the radial part from the angular one, Eq. (1.1) produces Eqs. (1.25) on one hand, and the corresponding angular equations (1.26) on the other. The separating constant, Q, turns out to acquire the following values

$$D=1: \ Q\equiv 0, \quad D=2: \ Q=m^2, \quad D=3: \ Q=\ell(\ell{+}1), \quad \cdots \quad Q_{(D)}=n(n{+}D{-}1), \tag{3.1}$$

where m,ℓ,n are (*typically,* see below) integers or proper half-integers: $2m,2\ell,2n\in\mathbb{Z}$. We now turn to motivate this in dimensions $D=2,3$.

3.1 B L T→ Two-dimensional Space & Trigonomentry

Using Eq. (1.13b), we have that

$$\frac{\partial^2}{\partial\phi^2}Y(\phi)+m^2\,Y(\phi), \quad\Rightarrow\quad Y(\phi)=A\,e^{im\phi}=A\left(\cos(m\phi)+i\sin(m\phi)\right). \tag{3.2}$$

If there are no ϕ-boundaries and $\phi\in[0,2\pi]$ with $\phi\equiv\phi{+}2\pi$, one typically requires single-valued periodicity, $Y(\phi+2\pi)\stackrel{!}{=}Y(\phi)$. This implies:

$$\left(A\,e^{im(\phi+2\pi)}=A\,e^{2\pi i\,m}\,e^{im\phi}\right)\stackrel{!}{=}A\,e^{im\phi} \ : \qquad e^{2\pi i\,m}\stackrel{!}{=}1 \quad\Rightarrow\quad m\in\mathbb{Z}, \tag{3.3}$$

However, in quantum mechanics we also need those double-valued functions (*spinors*) which satisfy $Y(\phi+2\pi)=-Y(\phi)$, in which case

$$\begin{aligned} A\,\sin\left(m(\phi+2\pi)+\varphi\right) &= A\sin(m\phi+\varphi)\cos(2m\pi)+A\cos(m\phi+\varphi)\sin(2m\pi),\\ &=(-1)^{2m}A\sin(m\phi+\varphi)=-A\,\sin(m\phi+\varphi) \ : \qquad m\in(\mathbb{Z}+\tfrac{1}{2}), \end{aligned} \tag{3.4}$$

Rotations in $D{=}2$-dimensional space form the *SO*(2) $\simeq U(1)$ algebra, and this is seen to distinguish between the tensorial representations of the *SO*(2) group, with $m\in\mathbb{Z}$, and the spinorial representations of *Spin*(2), the double-cover of *SO*(2): being double-valued under a 2π rotation, spinors are not *faithful*representations of *SO*(2), but only of *Spin*(2). The 1st order differential operator $\mathscr{L}_{(2)}$ *generates* the *SO*(2) group, the elements of which can be written as $g_\alpha:=\exp\{i\alpha\mathscr{L}_{(2)}\}$, with $\alpha\equiv\alpha+2\pi$ being a real angle, parametrizing the rotations in $D=2$-dimensional space.

We will see that a suitable generalization of these facts prevails for higher D also.

3.2 B L T→ Three-dimensional Space & Spherical Harmonics

In $D=3$-dimensional space, the partial differential equation (1.26) may be separated into the system:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\,\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left(Q-\frac{m^2}{\sin^2\theta}\right)\Theta=0, \tag{3.5}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+m^2\Phi=0, \tag{3.6}$$

where the first is the associated Legendre equation, and the second the trigonometric equation. Now, Eq. (1.14) dictates that $\mathscr{L}^2_{(3)}$ be the magnitude-square of a vector differential operator. This operator is most easily written in the Cartesian coordinate system, where $g^{kl}=\delta^{kl}$, so

$$\mathscr{L}_x:=i\Big(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\Big), \quad \mathscr{L}_y:=i\Big(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\Big), \quad \mathscr{L}_z:=i\Big(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\Big). \tag{3.7}$$

Note that all three $\mathscr{L}_i$'s are homogeneous, of degree 0, that is, invariant under homoteties $(x, y, z) \to (\lambda x, \lambda y, \lambda z)$, $\lambda \in \mathbb{R}^*$. This implies that all three $\mathscr{L}_i$'s are independent of r: they are functions of θ, ϕ, and 1st order differential operators with respect to θ, ϕ only[3]. Note that the definitions (3.7) ensure that the $\mathscr{L}_i$'s are all Hermitian as differential operators:

$$\int_V \mathrm{d}^3\vec{r}\, f^*(\vec{r}) \left[\mathscr{L}_i\, g(\vec{r})\right] = + \int_V \mathrm{d}^3\vec{r} \left[\mathscr{L}_i\, f(\vec{r})\right]^* g(\vec{r}), \tag{3.8}$$

provided $f(\vec{r}), g(\vec{r})$ vanish at ∂V.

The Cartesian representation (3.7) is, however, by far the easiest one to prove by direct calculations that

$$\left[\mathscr{L}_j\,,\,\mathscr{L}_k\right] = i\,\varepsilon_{jkl}\,\mathscr{L}_l. \tag{3.9}$$

That is, besides spanning a 3-dimensional vector space, the three $\mathscr{L}_i$'s in fact span a 3-dimensional algebra (vector space equipped with a closed product), called the *angular momentum algebra*, or the algebra of the *SO*(3) group. The relationship with the *SO*(3) group means that the group elements of this *SO*(3) may be written as $g_{\vec{\alpha}} := \exp\{i\alpha^j \mathscr{L}_j\}$, where the three real angles, $\alpha^1, \alpha^2, \alpha^3$ (formally assembled into a 3-vector, $\vec{\alpha}$, as if its Cartesian coordinates), parametrize the rotations in $d = 3$-dimensional space—such as the Euler angles, familiar from the classical mechanics of rigid bodies. Most of the details of the algebra (3.9) have been discussed in § 4.3 of Ref. [1], but we recapture here the salient points.

Firstly, note that if two operators, $\hat{A}$ and $\hat{B}$ are to have a simultaneous system of eigenvectors

$$\begin{cases} \hat{A}\,|a,b\rangle &=& a\,|a,b\rangle\,, \\ \hat{B}\,|a,b\rangle &=& b\,|a,b\rangle\,, \end{cases} \tag{3.10}$$

they better commute, since

$$\begin{cases} \hat{B}\hat{A}\,|a,b\rangle &=& \hat{B}a\,|a,b\rangle &=& a\hat{B}\,|a,b\rangle &=& ab\,|a,b\rangle\,, \\ \hat{A}\hat{B}\,|a,b\rangle &=& \hat{A}b\,|a,b\rangle &=& b\hat{A}\,|a,b\rangle &=& ab\,|a,b\rangle\,, \end{cases} \tag{3.11}$$

the difference of which yields

$$\left[\hat{A}\,,\,\hat{B}\right]|a,b\rangle = 0. \tag{3.12}$$

Since no two of the three $\mathscr{L}_i$'s commute, the eigenvectors of any one of the $\mathscr{L}_i$'s *cannot* be eigenvectors also of either of the other two. However, an elementary iteration of Eq. (3.9) implies that

$$\left[\mathscr{L}_i\,,\,\mathscr{L}^2\right] = 0, \qquad i = x, y, z, \qquad \mathscr{L}^2 := \mathscr{L}_x^2 + \mathscr{L}_y^2 + \mathscr{L}_z^2. \tag{3.13}$$

The geometrical interpretation of this is "obvious": (**1**) the $\mathscr{L}_i$'s generate rotations, (**2**) $\mathscr{L}^2$ is the *magnitude*-square of the vector $\vec{\mathscr{L}}$, and (**3**) magnitudes of vectors are invariant under rotations. The algebraic consequence, however, is that $\mathscr{L}^2$ together with any one of $\mathscr{L}_i$ can serve—and indeed forms the *maximal commuting set* of operators: it turn out that, in dimension $d = 3$, no operator can be constructed from the three $\mathscr{L}_i$ that would commute with both $\mathscr{L}^2$ and one chosen from among $\mathscr{L}_i$, say $\mathscr{L}_z$, and not be a function $f(\mathscr{L}^2, \mathscr{L}_z)$.

Following the standard approach, we select $\mathscr{L}_3 = \mathscr{L}_z$ to pair with $\mathscr{L}^2$, and define the basis[4] of their simultaneous eigenvectors to satisfy

$$\begin{cases} \mathscr{L}^2\;|Q,m\rangle &=& Q\;|Q,m\rangle\,, \\ \mathscr{L}_3\;|Q,m\rangle &=& m\;|Q,m\rangle\,. \end{cases} \tag{3.14}$$

[3] But, don't take my word for it: follow through the change of coordinates yourself!

[4] A basis is a *maximal* and *complete* collection of elements of a vector space, which is also *orthonormalizable* with respect to a scalar product if such is defined.

We then combine $\mathscr{L}_x$ with $\mathscr{L}_y$ into

$$\mathscr{L}_\pm := \mathscr{L}_x \pm i\mathscr{L}_y, \qquad \big(\mathscr{L}_+\big)^\dagger = \mathscr{L}_-, \tag{3.15}$$

and explore their action on $|Q,m\rangle$. To that end, we first rewrite Eq. (3.9) and Eq. (3.13) as

$$\big[\,\mathscr{L}_3, \mathscr{L}_\pm\,\big] = \pm\mathscr{L}_\pm, \qquad\qquad \big[\,\mathscr{L}_+, \mathscr{L}_-\,\big] = 2\mathscr{L}_3, \tag{3.16}$$

$$\big[\,\mathscr{L}^{\,2}, \mathscr{L}_i\,\big] = 0, \quad i = +,-,3, \qquad\qquad \mathscr{L}^{\,2} = \mathscr{L}_\mp\mathscr{L}_\pm + \mathscr{L}_3^{\,2} \pm \mathscr{L}_3. \tag{3.17}$$

Since $\mathscr{L}_i, \mathscr{L}^2$ are differential operators with respect to the angles θ,ϕ, the vectors $|Q,m\rangle$ are in fact functions $Y_{Q,m}(\theta,\phi)$ of those angles. For such vectors, we introduce the scalar ("dot") product

$$\langle Q',m'|Q,m\rangle := \int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\,\big[Y_{Q',m'}(\theta,\phi)\big]^* \, Y_{Q,m}(\theta,\phi), \tag{3.18}$$

and use Gram-Schmidt orthogonalization [1, § 5.2] to organize these functions into an orthogonal collection, — which we also normalize so that:

$$\langle Q',m'|Q,m\rangle = \delta_{Q',Q}\,\delta_{m',m}. \tag{3.19}$$

Abbreviating $\langle Q,m|\cdots|Q,m\rangle$ as $\langle\cdots\rangle$, we have that

$$Q = \langle\mathscr{L}^{\,2}\rangle = \langle Q,m|\;\mathscr{L}^{\,2}\,|Q,m\rangle = \langle Q,m|\;Q\,|Q,m\rangle = Q\,\big(\langle Q,m|Q,m\rangle \;=\; 1\big), \tag{3.20}$$

$$= \langle\mathscr{L}_x^2 + \mathscr{L}_y^2 + \mathscr{L}_z^2\rangle = \langle\mathscr{L}_x^2\rangle + \langle\mathscr{L}_y^2\rangle + \langle\mathscr{L}_z^2\rangle, \tag{3.21}$$

$$= \big\|\mathscr{L}_x\,|Q,m\rangle\,\big\|^2 + \big\|\mathscr{L}_x\,|Q,m\rangle\,\big\|^2 + m^2 \geq m^2, \tag{3.22}$$

$$= \langle\mathscr{L}_\mp\mathscr{L}_\pm\rangle + \langle\mathscr{L}_3^{\,2}\rangle \pm \langle\mathscr{L}_3\rangle, \tag{3.23}$$

$$= \big\|\mathscr{L}_\pm\,|Q,m\rangle\,\big\|^2 + m^2 \pm m = \big\|\mathscr{L}_\pm\,|Q,m\rangle\,\big\|^2 + m(m{\pm}1). \tag{3.24}$$

Since $\big\|\mathscr{L}_\pm\,|Q,m\rangle\,\big\|^2 \geq 0$, we have obtained that

$$Q \geq\; m(m{\pm}1). \tag{3.25}$$

Next, we inquire whether $\mathscr{L}_\pm\,|Q,m\rangle$ is an element of the basis (3.14). To that end, we need to see if the $\mathscr{L}_\pm\,|Q,m\rangle$'s are themselves eigenvectors of $\mathscr{L}^{\,2}$ and $\mathscr{L}_3$:

$$\begin{aligned}\mathscr{L}^{\,2}\,\big(\mathscr{L}_\pm\,|Q,m\rangle\,\big) &= \underbrace{\big[\,\mathscr{L}^{\,2}\,,\,\mathscr{L}_\pm\,\big]}_{=0} + \mathscr{L}_\pm\mathscr{L}^{\,2}\,|Q,m\rangle = \mathscr{L}_\pm\,Q\,|Q,m\rangle \\ &= Q\,\big(\mathscr{L}_\pm\,|Q,m\rangle\,\big),\end{aligned} \tag{3.26}$$

$$\begin{aligned}\mathscr{L}_3\,\big(\mathscr{L}_\pm\,|Q,m\rangle\,\big) &= \underbrace{\big[\,\mathscr{L}_3\,,\,\mathscr{L}_\pm\,\big]}_{=\pm\mathscr{L}_\pm} + \mathscr{L}_\pm\mathscr{L}_3\,|Q,m\rangle = \big(\pm\,\mathscr{L}_\pm + \mathscr{L}_\pm\,m\big)\,|Q,m\rangle \\ &=\; (m\pm1)\,\big(\mathscr{L}_\pm\,|Q,m\rangle\,\big),\end{aligned} \tag{3.27}$$

Thus, $\mathscr{L}^2$ does not change $|Q,m\rangle$, whereas $\mathscr{L}_+$ raises (and $\mathscr{L}_-$ lowers) the eigenvalue m in $|Q,m\rangle$. Therefore, it must be that

$$\mathscr{L}_\pm\,|Q,m\rangle \propto |Q,m{\pm}1\rangle\,, \qquad \big(\mathscr{L}_\pm\,|Q,m\rangle\,\big) = N_\pm(Q,m)\,|Q,m{\pm}1\rangle\,. \tag{3.28}$$

Now, from Eq. (3.25), it follows that:

$$m > 0\;; \qquad Q\begin{cases}\geq m(m{+}1),\\ \geq m(m{-}1),\end{cases} \qquad \Rightarrow Q \geq m(m+1), \tag{3.29}$$

$$m < 0\;; \qquad Q\begin{cases}\geq -|m|(-|m|{+}1),\\ \geq -|m|(-|m|{-}1),\end{cases} \qquad \Rightarrow Q \geq -|m|(-|m|-1) = |m|(|m|+1), \tag{3.30}$$

so that the last inequality (3.30) in fact applies to both $m < 0$ and $m > 0$: $Q \geq |m|(|m|{+}1)$.

So, denote $\ell \stackrel{\text{def}}{=} \max(|m|)$, and consider applying $\mathscr{L}_+$ on $|Q,\ell\rangle$:

$$\mathscr{L}_+\,|Q,\ell\rangle = N_+\,|Q,\ell{+}1\rangle \equiv 0, \quad \text{since} \quad m \leq \big(\ell \stackrel{\text{def}}{=} \max(|m|)\big). \tag{3.31}$$

Applying $\mathscr{L}_-$ to this equation, we obtain:

$$0 = \mathscr{L}_-\mathscr{L}_+\,|Q,\ell\rangle = \big[\mathscr{L}^{\,2} - \mathscr{L}_3^{\,2} - \mathscr{L}_3\big]\,|Q,\ell\rangle = \big[Q - \ell^2 - \ell\big]\,|Q,\ell\rangle = [Q - \ell(\ell+1)]\,|Q,\ell\rangle\,. \tag{3.32}$$

Avoiding the trivial solution ($|Q,\ell\rangle = 0$) implies that $Q = \ell(\ell{+}1)$.

Hereafter, we rename $|Q,m\rangle \to |\ell,m\rangle$.

Since $\mathscr{L}_\pm, \mathscr{L}_3, \mathscr{L}^{\,2}$ are the *only* operators that operate on the $|\ell,m\rangle$, it follows that iterations of Eq. (3.28) *must* exhaust the possible $|\ell,m\rangle$'s for any fixed ℓ. Therefore,

$$-\ell \leq m \leq +\ell, \qquad \ell := \max\big(|m|\big). \tag{3.33}$$

Furthermore, since iterations of Eq. (3.28) provide the *only* way to change m, which happens in unit increments, it follows that

$$\triangle m \in \mathbb{Z}. \tag{3.34}$$

Since $\ell \stackrel{\text{def}}{=} \max(|m|)$, the allowed values of m must range, in unit increments, between $-\ell$ and ℓ — starting from one and reaching the other, we have that

$$\ell - (-\ell) \in \mathbb{Z}, \quad \Rightarrow \quad 2\ell \in \mathbb{Z}. \tag{3.35}$$

That is, we have just proven that ℓ can only be an integer or a proper half-integer. Also, Eqs. (3.33) and (3.34) imply that m takes on $2\ell{+}1$ values, for any fixed ℓ.

This is the announced generalization of the case in $D = 2$-dimensional space.

We can now also determine (the magnitude of) the normalization constants $N_\pm(Q,m)$:

$$\begin{aligned}|N_+|^2 &= \big\|N_+\,|\ell,m+1\rangle\,\big\| = \big\|\mathscr{L}_+\,|\ell,m\rangle\,\big\|^2, &(3.36)\\ &= \langle\mathscr{L}_-\mathscr{L}_+\rangle = \langle\mathscr{L}^{\,2} - \mathscr{L}_3^{\,2} - \mathscr{L}_3\rangle = \ell(\ell{+}1) - m(m{+}1)\;; &(3.37)\\ N_+ &= \sqrt{\ell(\ell{+}1) - m(m{+}1)}. &(3.38)\end{aligned}$$

By the same token, considering however $\|\mathscr{L}_-\,|\ell,m\rangle\,\|^2$ instead:

$$N_- = \sqrt{\ell(\ell{+}1) - m(m{-}1)}. \tag{3.39}$$

Therefore, for each $2\ell \in \mathbb{Z}$, we have that the vector space spanned by the eigenbasis

$$\big\{\ |\ell,m\rangle : m = -\ell, -(\ell{-}1), \cdots, (\ell{-}1), \ell\,\big\} \tag{3.40}$$

is the $(2\ell{+}1)$-dimensional representation[5] of *Spin*(3) if $\ell \in \mathbb{Z}$, or *Spin*(3), if $\ell \in \mathbb{Z}{+}\frac{1}{2}$. That is to say, given a choice of $\ell \in \frac{1}{2}\mathbb{Z}$, there exists a vector space

$$\mathcal{V}_\ell := \{r_{\ell,m}\,|\ell,m\rangle\,,\ (r_{\ell,-\ell},\cdots,r_{\ell,+\ell}) \in \mathbb{R}^{2\ell+1}\}, \tag{3.41}$$

such that *SO*(3)-rotations act by:

$$\mathscr{L}^2\,|\ell,m\rangle = \ell(\ell{+}1)\,|\ell,m\rangle\,, \tag{3.42}$$

$$\mathscr{L}_x\,|\ell,m\rangle = \tfrac{1}{2}\sqrt{\ell(\ell{+}1)-m(m{+}1)}\,|\ell,m{+}1\rangle + \tfrac{1}{2}\sqrt{\ell(\ell{+}1)-m(m{-}1)}\,|\ell,m{-}1\rangle\,, \tag{3.43}$$

$$\mathscr{L}_y\,|\ell,m\rangle = \tfrac{1}{2i}\sqrt{\ell(\ell{+}1)-m(m{+}1)}\,|\ell,m{+}1\rangle - \tfrac{1}{2i}\sqrt{\ell(\ell{+}1)-m(m{-}1)}\,|\ell,m{-}1\rangle\,, \tag{3.44}$$

$$\mathscr{L}_z\,|\ell,m\rangle = m\,|\ell,m\rangle\,. \tag{3.45}$$

Thus, each $\mathcal{V}_\ell \simeq \mathbb{R}^{2\ell+1}$ is a $(2\ell{+}1)$-dimensional vector space, for all $\ell \in \frac{1}{2}\mathbb{Z}$. It may be parametrized by the $2\ell{+}1$ components "$|\ell,m\rangle$," with $|m| \leq \ell$, and is a $(2\ell{+}1)$-dimensional representation of *Spin*(3).

I trust this is clearly easier than trying to solve Eq. (3.5), or determining how

$$\vec{\mathscr{L}} = i\hat{\mathrm{e}}_\theta\,\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} - i\hat{\mathrm{e}}_\phi\,\frac{\partial}{\partial\phi}, \tag{3.46}$$

$$\mathscr{L}_\pm = \pm e^{\pm i\phi}\Big(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\Big) \tag{3.47}$$

act on the spherical harmonics:

$$Y_\ell^m(\theta,\phi) := (-1)^m\sqrt{\frac{2\ell{+}1}{4\pi}\frac{(\ell{-}m)!}{(\ell{+}m)!}}\,P_\ell^m(\cos\theta)\,e^{im\phi}, \tag{3.48}$$

and how $\mathscr{L}_\pm$ reconstruct all the $Y_\ell^m(\theta,\phi)$'s from $Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$. Clearly (I should hope), there is a perfect, 1–1 correspondence between the spherical harmonics and the formal eigenvectors: $Y_\ell^m(\theta,\phi) \xleftrightarrow{1-1} |\ell,m\rangle$.

3.3 Higher Dimensions

Now, Eq. (3.5) is, for $m=0$ and with $\xi = \cos\theta_1$, becomes

$$(1-\xi^2)\frac{\mathrm{d}^2\Theta_1}{\mathrm{d}\xi^{\,2}} - (2D{-}1)\,\xi\,\frac{\mathrm{d}\Theta_1}{\mathrm{d}\xi} + \ell(\ell+D{-}2)\Theta_1 = 0, \tag{3.49}$$

known as the Gegenbauer differential equation. As it turns out, this equation is solved in terms of the usual associated Legendre polynomials (of both the 1st and the 2nd kind) [2]:

$$\Theta_1(\xi) = (\xi^2-1)^{\frac{\mu}{2}}\,\big[C_1\,P_\nu^\mu(\xi) + C_2\,Q_\nu^\mu(\xi)\big], \qquad \nu := \ell + \tfrac{D-3}{2}, \quad \mu := \tfrac{3-D}{2}, \tag{3.50}$$

just as the higher dimensional Bessel equations are all solved in terms of the cylindrical ones (2.24).

This hierarchy is related to the fact that *SO*(D)—or, more accurately its algebra, $\mathfrak{so}(D)$—has $\lfloor\frac{D+1}{2}\rfloor$ algebraically independent *Casimir operators*, for $D \leq 2$: mutually commuting operators constructed from

[5] The groups *SO*(3) and *Spin*(3) share the same algebra, and the $\ell \in \mathbb{Z}$ representations (3.40). However, when $\ell \in \mathbb{Z}{+}\frac{1}{2}$, a rotation by 2π is not equivalent to the identity: $\exp\{i\varphi\mathscr{L}_z\}\,|\frac{1}{2},\pm\frac{1}{2}\rangle = \exp\{\pm i\varphi/2\}\,|\frac{1}{2},\pm\frac{1}{2}\rangle$, so $\exp\{2i\pi\mathscr{L}_z\}\,|\frac{1}{2},\pm\frac{1}{2}\rangle = \exp\{\pm i\pi\}\,|\frac{1}{2},\pm\frac{1}{2}\rangle = -\,|\frac{1}{2},\pm\frac{1}{2}\rangle$: spinors return to themselves only upon a 4π rotation. The group *Spin*(3) is then *defined* to be a double-cover of *SO*(3), precisely as the 2-leaved Riemann sheet of values of $\sqrt{z}$ had to be defined for the $\mathbb{C}$-valued $\sqrt{\ }$-function of $z \in \mathbb{C}$.

the generators of the group. For $n=1$, $\mathfrak{so}(1)=\emptyset$, and $SO(1)=\mathbb{Z}_2$ is a discrete group, so that no $\mathscr{L}$'s exist. For $d>3$, $\mathscr{L}$ is a tensor operator, of rank $(D-2)$. Then, $\mathscr{L}^2 := \|\mathscr{L}\|^2$ is still one of the Casimir operators, but a specific choice of the remaining $\lfloor\frac{D-1}{2}\rfloor$ ones is less obvious. Correspondingly, there is a varied choice of Casimir eigenvectors, $|\ell,\nu_i,\cdots,\nu_n,m\rangle$, where $n=\lfloor\frac{D-3}{2}\rfloor$, accompanied by correspondingly various relations between the eigenvalues $\ell,\nu_i,\cdots,\nu_n,m$.

Moreover, the choice of ℓ (and so, ultimately the dimension of the basis) no longer determines the basis $|\ell,\nu_i,\cdots,\nu_n,m\rangle$ up to linear transformations, and so no longer determines uniquely the representation of $SO(D)$: for any $d\geq 4$, there do exist choices of ℓ with different bases $|\ell,\nu_i,\cdots,\nu_n,m\rangle$, *i.e.*, same-dimensional but different representations of $SO(D)$.

Finaly, with growing D, the choices of ℓ become increasingly more sparse: whereas $SO(3)$ has a representation for every $(2\ell{+}1)\in\mathbb{Z}_{\geq 0}$, this is not true for $SO(D)$ with $D\geq 4$.

Consequently, different methods seem preferable—and have been used—when trying to list all the possible representations of $SO(D)$, for $D\geq 4$. Most often, however, they do use the obvious chain of imbeddings $SO(D)\supset SO(D{-}1)\supset\cdots\supset SO(3)\supset SO(2)$.

4 BLT → Legendre Equation

Consider again

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\sin\theta\,\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\Big)+\Big(\ell(\ell{+}1)-\frac{m^2}{\sin^2\theta}\Big)\Theta=0, \tag{4.1}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+m^2\Phi=0, \tag{4.2}$$

and change variables: $x:=\cos\theta$, so $\frac{\mathrm{d}}{\mathrm{d}\theta}=\frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}}{\mathrm{d}x}=-\sin\theta\frac{\mathrm{d}}{\mathrm{d}x}=-\sqrt{1-\cos^2\theta}\frac{\mathrm{d}}{\mathrm{d}x}=-\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}$ in (4.1):

$$\frac{1}{\sqrt{1-x^2}}\Big(-\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\Big)\Big(\sqrt{1-x^2}\Big(-\sqrt{1-x^2}\frac{\mathrm{d}\Theta}{\mathrm{d}x}\Big)\Big)+\Big(\ell(\ell{+}1)-\frac{m^2}{1-x^2}\Big)\Theta=0, \tag{4.3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big((1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\Big)+\Big(\ell(\ell{+}1)-\frac{m^2}{1-x^2}\Big)\Theta=0, \tag{4.3b}$$

$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2}-2x\frac{\mathrm{d}\Theta}{\mathrm{d}x}+\Big(\ell(\ell{+}1)-\frac{m^2}{1-x^2}\Big)\Theta=0. \tag{4.3c}$$

Clearing the denominators, we see that this can be solved by the method of series, but we will not do so here. Instead, we digress:

4.1 Generating Function

The Green's function for the Laplacian in 3-dimensional space is $\frac{1}{|\vec r-\vec r'|}$, which motivates us to consider $|\vec a-\vec b|=a^2+b^2-2ab\cos\theta$, where $a=|\vec a|$, $b=|\vec b|$ and θ is the angle between $\vec a$ and $\vec b$. Assume that $|\vec a|>|\vec b|$, so that $t:=\frac{b}{a}<1$, and we can write

$$\frac{1}{|\vec a-\vec b|}=\frac{1}{|\vec a|}\,g_0(x,t),\quad g_0(x,t):=\big(1+t^2-2xt\big)^{-1/2},\quad x=\cos\theta,\ \ t=\tfrac{|\vec b|}{|\vec a|}. \tag{4.4}$$

Since we are often concerned with derivatives of $\frac{1}{|\vec r-\vec r'|}$, we start consider

$$g_m(x,t):=\big(1+t^2-2xt\big)^{m-1/2},\quad m\in\mathbb{Z}. \tag{4.5}$$

Treating $t(t-2x)$ as a small parameter, we have

$$g_m(x,t) := \big(1+t^2-2xt\big)^{m-1/2} = \sum_{r=0}^{\infty}\binom{m-\frac{1}{2}}{r}(t^2-2xt)^r, \tag{4.6}$$

$$= \sum_{r=0}^{\infty}\binom{m-\frac{1}{2}}{r}\sum_{s=0}^{r}\binom{r}{s}(-1)^s t^{2(r-s)}(2xt)^s = \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\binom{m-\frac{1}{2}}{r}\binom{r}{s}t^{2r-s}\,(2x)^s. \tag{4.7}$$

Substituting $r=(\ell+s)/2$ and using that therefore $s=2r-\ell\leqslant r$ implies that $r\leqslant\ell$ for the upper limit of the second sum, this becomes

$$= \sum_{\ell=0}^{\infty}\bigg[\underbrace{\sum_{s=0}^{\ell}(-1)^s\binom{m-\frac{1}{2}}{(\frac{\ell+s}{2})}\binom{(\frac{\ell+s}{2})}{s}(2x)^s}_{P_\ell^m(x)}\bigg]t^\ell, \tag{4.8}$$

and are called the associated Legendre polynomials.

The associated Legendre polynomials may also be calculated from Rodrigues' derivative formula:

$$P_\ell^m(x) := (-1)^m\frac{(1-x^2)^{m/2}}{2^\ell\,\ell!}\bigg(\Big[\frac{\mathrm{d}}{\mathrm{d}x}\Big]^{\ell+m}(x^2-1)^\ell\bigg).$$

The following are some of the lowest-ℓ, m associated Legendre polynomials

$$P_0^0(x) = \frac{(1-x^2)^{0/2}}{2^0\,0!}\bigg(\Big[\frac{\mathrm{d}}{\mathrm{d}x}\Big]^{0+0}(x^2-1)^0\bigg) = 1;$$

$$P_1^0(x) = \frac{(1-x^2)^{0/2}}{2^1\,1!}\bigg(\Big[\frac{\mathrm{d}}{\mathrm{d}x}\Big]^{1+0}(x^2-1)^1\bigg) = \frac{1}{2}\Big(\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\Big) = \frac{1}{2}(2x) = x = \cos\theta;$$

$$P_1^1(x) = -\frac{(1-x^2)^{1/2}}{2^1\,1!}\bigg(\Big[\frac{\mathrm{d}}{\mathrm{d}x}\Big]^{1+1}(x^2-1)^1\bigg) = -\frac{\sqrt{1-x^2}}{2}\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)\Big) = -\frac{1}{2}\sqrt{1-\cos^2\theta}\,(2) = -\sin\theta;$$

$$P_1^{-1}(x) = -\frac{(1-x^2)^{-1/2}}{2^1\,1!}\bigg(\Big[\frac{\mathrm{d}}{\mathrm{d}x}\Big]^{1-1}(x^2-1)^1\bigg) = -\frac{1}{2\sqrt{1-x^2}}(x^2-1) = +\frac{1}{2}\sqrt{1-\cos^2\theta}\,(2) = \sin\theta.$$

Thus: with θ the azimuthal angle in spherical coordinates, the Cartesian coordinates are:

$$z = r\,P_1^0(\cos\theta),\quad y = \pm r\,P_1^{\mp1}(\cos\theta)\sin\phi,\quad x = \pm r\,P_1^{\mp1}(\cos\theta)\cos\phi,$$
$$\text{so that}\quad x\pm iy = \mp r\,P_1^{\pm1}(\cos\theta)\,e^{\pm i\phi}.$$

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*. Academic Press, 7 ed., 2012.

[2] `http://mathworld.wolfram.com/GegenbauerDifferentialEquation.html`

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