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Don't Panic !

Mathematical Methods I

Midterm 2: 2014, Nov. 12.

Solution Soliloquy

As this solution set is also used to to discuss the computations in a hopefully pedagogical manner, it is considerably longer, more varied and more detailed than was expected of the student solutions.

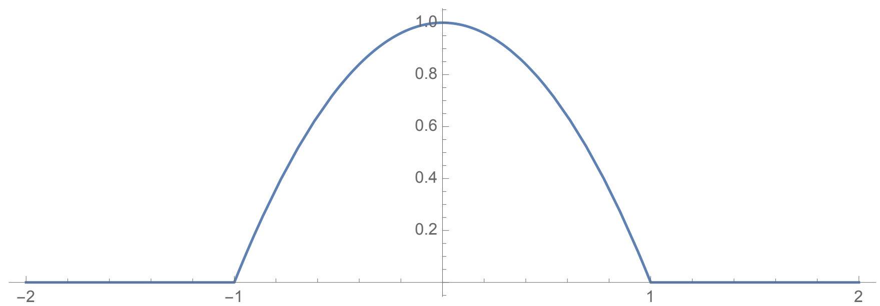
1. Consider the function

$$f(x) := \begin{cases} 1-x^2, & \text{for} \quad x \in [-1,1], \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

[5 pt] ***a.*** Sketch the function and determine k_n so $f(x)$ could be represented, within $x \in [-1,1]$, by the Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(k_n x) + b_n \sin(k_n x)\big]$.

Solution

A sketch of the function $f(x)$ is shown to the right. The key feature to observe is that the function is symmetric with respect to the reflection $x \to -x$: $f(-x) = f(x)$. Also, the obvious boundary conditions are $f(-1) = 0 = f(1)$.



Since the function $f(x)$ should be represented *within* $x \in [-1,1]$, we must choose k_n so that the summands $\cos(k_n x)$ and $\sin(k_n x)$ have an integral multiple of the full period (2π) between $x = \pm 1$. That is, we require (we will return to this issue at the end of this problem):

$$\Big(\big[k_n x\big]_{-1}^{1} = \big[k_n(1) - k_n(-1)\big] = 2k_n \Big) \overset{!}{=} n{\cdot}2\pi, \qquad \Rightarrow \qquad k_n \overset{!}{=} n\pi. \tag{2}$$

That is, we seek a Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(n\pi x) + b_n \sin(n\pi x)\big]. \tag{3}$$

[10+10 pt] ***b.*** Compute a_n and b_n in this Fourier series.

Solution

Since $f(-x) = f(x)$ and $\sin(n\pi(-x)) = -\sin(n\pi x)$, we immediately conclude that $b_n = 0$.

Next, we use the standard formula for the a_n coefficients, treating the $n = 0$ case separately. The integrals are then all normalized by half of the span $x \in [-1,1]$, *i.e.*, by $\frac{1}{2} \cdot 2 = 1$:

$$a_0 = \int_{-1}^{1} \mathrm{d}x \; f(x) = \int_{-1}^{1} \mathrm{d}x \; (1 - x^2) = \Big[x - \frac{x^3}{3}\Big]_{-1}^{1} = \big(1 - (-1)\big) - \Big(\frac{1^3}{3} - \frac{(-1)^3}{3}\Big) = 2 - \frac{2}{3} = \frac{4}{3}.$$

For the remaining coefficients, we will need the integrals

$$a_n := \int_{-1}^{1} \mathrm{d}x \;\; \cos(n\pi x) f(x) = \int_{-1}^{1} \mathrm{d}x \;\; \cos(n\pi x)(1-x^2), \qquad \text{change:} \; \begin{cases} \phi = \pi x, \\ x = \phi/\pi, \end{cases}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{d}\phi \;\; \cos(n\phi)\Big(1 - \big(\tfrac{\phi}{\pi}\big)^2\Big) = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{d}\phi \;\; \cos(n\phi) - \frac{1}{\pi^3} \int_{-\pi}^{\pi} \mathrm{d}\phi \;\; \cos(n\phi)\, \phi^2,$$

$$= \frac{1}{\pi}\underbrace{\left[\frac{\sin(n\phi)}{\pi}\right]_{-\pi}^{\pi}}_{=0} - \frac{1}{\pi^3}\int_{-\pi}^{\pi} \mathrm{d}\phi\ \cos(n\phi)\,\phi^2,$$

where we evaluate the remaining integral integrating by parts:

$$= -\frac{1}{\pi^3}\Bigg\{\underbrace{\left[\Big(\frac{\sin(n\phi)}{n}\Big)\phi^2\right]_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} \mathrm{d}\phi\ \Big(\frac{\sin(n\phi)}{n}\Big)(2\phi)\Bigg\} = +\frac{2}{n\pi^3}\int_{-\pi}^{\pi} \mathrm{d}\phi\ \sin(n\phi)\,\phi,$$

$$= \frac{2}{n\pi^3}\Bigg\{\left[\Big(\frac{-\cos(n\phi)}{n}\Big)\phi\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \mathrm{d}\phi\ \Big(\frac{-\cos(n\phi)}{n}\Big)(1)\Bigg\},$$

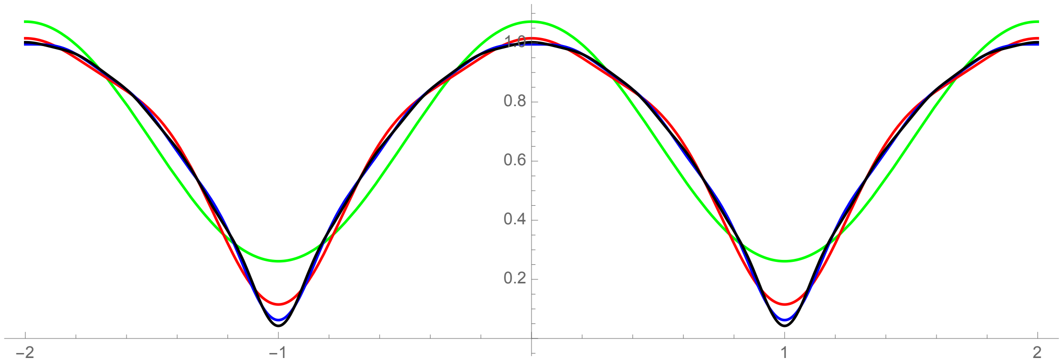
$$= \frac{2}{n\pi^3}\Bigg\{\left[\Big(\frac{-(-1)^n}{n}\Big)\pi - \Big(\frac{-(-1)^n}{n}\Big)(-\pi)\right] + \frac{1}{n}\int_{-\pi}^{\pi} \mathrm{d}\phi\ \cos(n\phi)\Bigg\},$$

$$= \frac{2}{n\pi^3}\Bigg\{-\frac{2\pi(-1)^n}{n} + \frac{1}{n}\underbrace{\left[\frac{\sin(n\phi)}{n}\right]_{-\pi}^{\pi}}_{=0}\Bigg\} = -\frac{4(-1)^n}{n^2\pi^2} \tag{4}$$

Thus we have obtained

$$f(x) := \left\{\begin{array}{ll} 1-x^2, & \text{for } \ x\in[-1,1] \\ 0 & \text{otherwise} \end{array}\right\} = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x), \qquad |x|<1. \tag{5}$$

The illustration below shows how the Fourier series quickly approximates the true function $f(x)$, by plotting the series truncated to $n \leqslant 1$ (green), $n \leqslant 3$ (red), $n \leqslant 6$ (blue) and $n \leqslant 9$ (black):



The plot of the Fourier series has been extended beyond $x \in [-1,1]$ to showcase the periodicity of the Fourier series outside the required span/interval, $x \in [-1,1]$. That is, the period of the Fourier representation of $f(x)$ is evidently 2, while the period of the whole collection $\{\cos(n\phi), \sin(n\phi),\ n = 1,2,3\dots\}$ of functions is 2π. The rescaling of the length of the period from 2 to 2π is accomplished by the result (2). Since the region of interest was only $x \in [-1,1]$, the values $x = \pm 1$ are the boundaries of the application region; that the Fourier series differs from the original function *outside* those boundaries is not relevant.

[15 pt] ***c.*** Verify if the term-by-term $\frac{\mathrm{d}}{\mathrm{d}x}$-derivative of this Fourier series represents $\frac{\mathrm{d}f}{\mathrm{d}x}$ within $x \in [-1,1]$.

Solution

On one hand, $\frac{\mathrm{d}f}{\mathrm{d}x} = -2x$, whereas the term-by-term derivative of the Fourier representation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)\Big) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x). \tag{6}$$

To check that they agree, we determine the coefficients for the Fourier series of $f'(x) = -2x$:

$$a_0 = \int_{-1}^{1} \mathrm{d}x\ (-2x) = 0 \qquad \text{and} \qquad a_n = \int_{-1}^{1} \mathrm{d}x\ \cos(n\pi x)(-2x) = 0; \tag{7}$$

$$\begin{aligned} b_n &= \int_{-1}^{1} \mathrm{d}x\ \sin(n\pi x)(-2x) = \Big[\Big(-\frac{\cos(n\pi x)}{n\pi}\Big)(-2x)\Big]_{-1}^{1} - \int_{-1}^{1} \mathrm{d}x\ \Big(-\frac{\cos(n\pi x)}{n\pi}\Big)(-2), \\ &= \frac{4(-1)^n}{n\pi}, \quad \text{which perfectly agrees with (6).} \quad \checkmark \end{aligned} \tag{8}$$

We return now to the issue of determining k_n in part ***a***.

As indicated, k_n is supposed to depend on n, *and* all Fourier series use sequences of trigonometric functions constructed by making the argument of the trigonometric functions be an integral (n-fold) multiple of a suitably rescaling of the argument of the function to be represented. For example, [1, Eqs. (19.10–12)] represents a function $f(x)$ for $x \in [-L, L]$ and uses

$$\cos\big(n\tfrac{\pi x}{L}\big) \quad \text{and} \quad \sin\big(n\tfrac{\pi x}{L}\big). \tag{9}$$

Each of these functions complete an integral (n) number of full periods between $-L$ and L. (Sketch a few of them, for a few small values of n, to convince yourself of this fact if it isn't obvious.) Rather than providing a general rationale for this requirement, let's explore what happens when we *do not* restrict k_n other than writing $k_n = nk$ where k is left undetermined other than fixing $k > 0$, but we do set $L = 1$. That is to say, let us explore

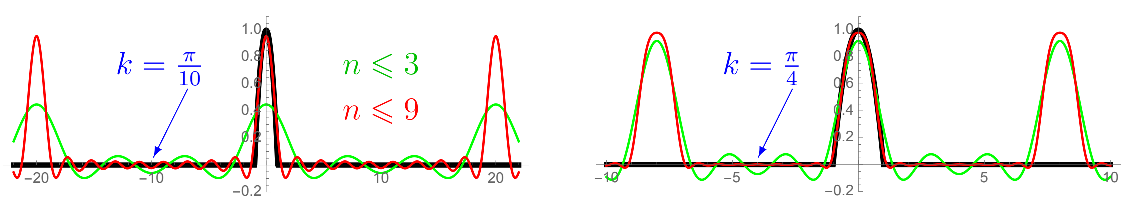
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[A_n \cos(nkx) + B_n \sin(nkx)\big], \qquad a_0 = \int_{-1}^{1} \mathrm{d}x\ f(x), \tag{10}$$

$$A_n = \int_{-1}^{1} \mathrm{d}x\ \cos(nkx)\, f(x) \qquad \text{and} \qquad B_n = \int_{-1}^{1} \mathrm{d}x\ \sin(nkx)\, f(x), \tag{11}$$

for the particular function at hand (1). It is straightforward that $B_n = 0$ since $\sin(nkx)$ are odd (antisymmetric) and the function (1) is symmetric; their product must be an antisymmetric function, whereby the integral in symmetric limits must vanish. On the other hand, integration by parts readily produces

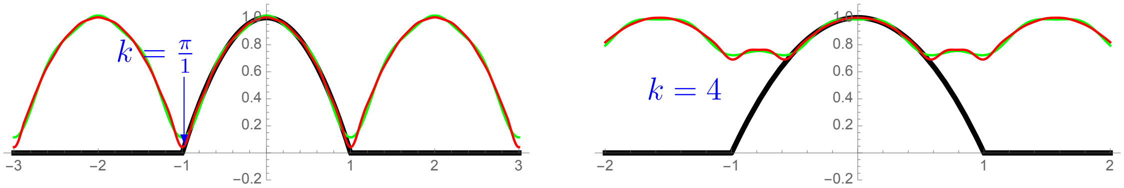
$$A_n = \int_{-1}^{1} \mathrm{d}x\ \cos(nkx)\,(1-x^2) = 4\frac{\sin(kn) - kn\cos(kn)}{k^3 n^3}, \qquad n = 1, 2, 3 \ldots \tag{12}$$

Without having specified k, these remain functions of this continuous variable. Again, rather than providing a general argument about a mandatory value for k, here are a couple of plots:



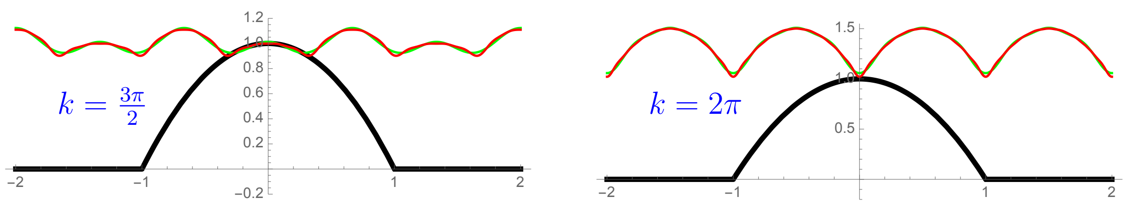
(13a)

Note that the period of this Fourer series is $[-\frac{\pi}{k}, +\frac{\pi}{k}]$.



(13b)

As soon as $k > \pi$, not only is the period $[-\frac{\pi}{k}, +\frac{\pi}{k}]$ shorter than $[-1, 1]$, but the Fourier series looks nothing like the original function, plotted in thick black ink.



(13c)

So, while $k > \pi$ is unacceptable (the Fourier series looks substantially different from the original function), the values $k \leqslant \pi$ are acceptable for representing the function in the span $x \in [-\frac{\pi}{k}, +\frac{\pi}{k}]$. Within the $x \in [-1, 1]$ interval, the choice $k = \pi$ is optimal, since already the first four terms ($n \leqslant 3$, plotted in green ink) represent the original function very well.

In addition, for $k = \pi$, the formula (12) simplifies to (4), which is at once both much simpler and also a much faster-converging series: The blue, $n \leqslant 9$ truncation of (4) for $k = \pi$ is already remarkably close, while the corresponding truncations shown (13a) for $k = 10$ (left-hand side) and $k = 4$ (right-hand side) are clearly considerably worse within the region of interest, $x \in [-1, +1]$.

Thus—as the problem was given—it is not strictly *wrong* to not fix $k = \pi$; however, it must be at least stipulated that $k \leqslant \pi$, with $k = \pi$ being the optimal choice.

In turn, if—contrary to the statement of this problem—we were interested in representing the function *outside* $x \in [-1, +1]$, larger and larger values of k should be used. We will return to this type of problem in Math Methods II (PHYS-217).

2. Given the generating function $g(x, t) = e^{x(x+t)} = \sum_{n=0}^{\infty} A_n(x)\, t^n$,

[6 pt] ***a.*** Compute the series representation of $A_n(x)$ by expanding the generating function.

Solution

Writing $e^{x(x+t)} = e^{x^2} e^{xt}$ and expanding the two exponentials, we have

$$g(x,t) = e^{x(x+t)} = e^{x^2} e^{xt} = \sum_{j=0}^{\infty} \frac{(x^2)^j}{j!} \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Big[\underbrace{\sum_{j=0}^{\infty} \frac{x^{2j+n}}{j!\, n!}}_{A_n(x)} \Big] t^n,$$

$$A_n(x) = \frac{1}{n!} \sum_{j=0}^{\infty} \frac{x^{2j+n}}{j!}, \quad = \frac{x^n}{n!} \sum_{j=0}^{\infty} \frac{x^{2j}}{j!} = \frac{x^n}{n!} e^{x^2}. \tag{14}$$

In this case, the series is simple enough to re-sum the result; this is typically not the case.

[6 pt] **b.** Compute the $\frac{\partial}{\partial t}$-derivative of $g(x,t)=\sum_{n=0}^{\infty}A_n(x)\,t^n$ and the resulting recurrence relation.

Solution

We calculate the derivative separately from $g(x,t)=e^{x(x+t)}$ and from $g(x,t)=\sum_{n=0}^{\infty}A_n(x)\,t^n$, and equate the results:

$$\frac{\partial g}{\partial t}=\frac{\partial}{\partial t}e^{x(x+t)}=x\,e^{x(x+t)}=x\sum_{n=0}^{\infty}A_n(x)\,t^n. \tag{15a}$$

$$=\frac{\partial}{\partial t}\sum_{n=0}^{\infty}A_n(x)\,t^n=\underbrace{\sum_{n=0}^{\infty}A_n(x)\,n\,t^{n-1}}_{n\mapsto n+1}=\sum_{n=-1}^{\infty}(n{+}1)A_{n+1}(x)\,t^n. \tag{15b}$$

Note that this last series has the "extra term" at $n=-1$, but that this term in fact vanishes: $((-1){+}1)A_{(-1)+1}(x)\,t^{-1}=0\,A_0(x)\,t^{-1}$. Equating (15a) with (15b) therefore produces

$$0=\sum_{n=0}^{\infty}\Big\{x\,A_n(x)-(n{+}1)A_{n+1}(x)\Big\}t^n \qquad\Rightarrow\qquad \boxed{A_{n+1}(x)=\frac{x}{n{+}1}A_n(x)}. \tag{15c}$$

[6 pt] **c.** Compute the $\frac{\partial}{\partial x}$-derivative of $g(x,t)=\sum_{n=0}^{\infty}A_n(x)\,t^n$ and the resulting recurrence relation.

Solution

We again calculate the derivative separately from $g(x,t)=e^{x(x+t)}$ and from $g(x,t)=\sum_{n=0}^{\infty}A_n(x)\,t^n$, and equate the results:

$$\frac{\partial g}{\partial x}=\frac{\partial}{\partial x}e^{x(x+t)}=(2x{+}t)\,e^{x(x+t)}=(2x{+}t)\sum_{n=0}^{\infty}A_n(x)\,t^n,$$

$$=2x\sum_{n=0}^{\infty}A_n(x)\,t^n+\underbrace{\sum_{n=0}^{\infty}A_n(x)\,t^{n+1}}_{n\mapsto n-1}=2x\sum_{n=0}^{\infty}A_n(x)\,t^n+\sum_{n=1}^{\infty}A_{n-1}(x)\,t^n. \tag{16a}$$

$$=\frac{\partial}{\partial x}\sum_{n=0}^{\infty}A_n(x)\,t^n=\sum_{n=0}^{\infty}A_n'(x)\,t^n. \tag{16b}$$

The second of the series in (16a) misses the $n=0$ term. Equating (16a) with (16b) therefore produces

$$0=\Big\{2xA_0(x)-A_n'(x)\Big\}t^0+\sum_{n=1}^{\infty}\Big\{2x\,A_n(x)+A_{n-1}(x)-A_n'(x)\Big\}t^n. \tag{16c}$$

The initial, simpler term separately implies:

$$A_0'(x)=2xA_0(x), \tag{16d}$$

which is in fact easy to solve:

$$\frac{\mathrm{d}A_0(x)}{\mathrm{d}x}=2x\,A_0 \quad\Rightarrow\quad \frac{\mathrm{d}A_0(x)}{A_0(x)}=2x\,\mathrm{d}x \quad\Rightarrow\quad A_0(x)=a_0\,e^{x^2}. \tag{16e}$$

Given this solution and the *algebraic* recursion (15c), we in fact can compute the whole infinite sequence of functions $A_n(x)$, one-by-one. (Of course, (14) already gives each $A_n(x)$, albeit in power-series form.) Nevertheless, we are sort out also the generic ($n \geqslant 1$) recursion relation by comparing the generic ($n \geqslant 1$) terms in (16c):

$$\boxed{A_n'(x) = 2x A_n(x) + A_{n-1}(x)}. \tag{16f}$$

[7 pt] ***d.*** Combine the two recursion relations and obtain a differential equation satisfied by $A_n(x)$.

Solution

From (15c) it follows, by shifting $n \mapsto n-1$ and solving for $A_{n-1}(x)$, that $A_{n-1}(x) = \frac{n}{x} A_n(x)$. Substituting this into (16f), we obtain:

$$A_n'(x) = 2x A_n(x) + \frac{n}{x} A_n(x) \qquad i.e. \qquad x A_n'(x) - (2x^2 + n) A_n(x) = 0. \tag{17}$$

This is the required differential equation. For this simple case, it turned out to be a 1st order equation; typically, this is not the case.

[10 pt] ***e.*** Solve the differential equation from part ***d***, and compare with your result from part ***a***.

Solution

The equation (17) is in fact easy to solve:

$$x A_n'(x) = (2x^2 + n) A_n(x) \qquad \Rightarrow \qquad \frac{\mathrm{d}A_n(x)}{A_n(x)} = \Big(2x + \frac{n}{x}\Big)\mathrm{d}x, \qquad n \geqslant 1, \tag{18}$$

which we integrate straightforwardly to obtain:

$$\ln\big(A_n(x)\big) = x^2 + n\,\ln(x) + \ln(C) \qquad \Rightarrow \qquad A_n(x) = C\, e^{x^2 + n\ln(x)} = C\, e^{x^2} x^n, \quad n \geqslant 1. \tag{19}$$

In fact, the $n \to 0$ "limit" of this reproduces the $A_0(x)$ solution (16e). Nothing would be remiss if it did not; the two branches of the differential equations, the special case (16d) and the generic ($n \geqslant 1$) cases (18), *jointly* determine all the coefficient functions $A_n(x)$, $n \geqslant 0$. The fact that we can even easily solve these equations is a consequence of the simple generating function, and it does not happen in general.

Comparing (19) with (14) above, we see that the *choice* of the normalization constant $C_n = \frac{1}{n!}$—and so, in fact, $C_0 = a_0$ from (16e)—makes the two solutions perfectly equal. Again, it does not typically happen that we can so completely determine the coefficient functions $A_n(x)$ so easily.

3. The vertical displacement $h(\rho, \phi, t)$ of a circular drumhead of radius a satisfies the wave equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial h}{\partial\rho}\Big) + \frac{1}{\rho^2}\frac{\partial^2 h}{\partial\phi^2} - \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = 0. \tag{20}$$

[15 pt] ***a.*** Writing $h(\rho, \phi, t) = P(\rho)\Phi(\phi)T(t)$, separate this partial differential equation into a coupled system of three ordinary differential equations, one for each of $P(\rho)$, $\Phi(\phi)$ and $T(t)$.

Solution

Substituting the expected form $h(\rho,\phi,t) = P(\rho)\Phi(\phi)T(t)$ directly in the equation and dividing through by $h(\rho,\phi,t) = P(\rho)\Phi(\phi)T(t)$, we have

$$\frac{1}{\rho\,P(\rho)}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big) + \frac{1}{\rho^2\,\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} - \frac{1}{v^2\,T(t)}\frac{\partial^2 T(t)}{\partial t^2} = 0. \tag{21}$$

Rearranging, we have

$$\underbrace{\frac{1}{\rho\,P(\rho)}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big) + \frac{1}{\rho^2\,\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}}_{\text{manifestly } t\text{-independent}} \quad = \quad \underbrace{\frac{1}{v^2\,T(t)}\frac{\partial^2 T(t)}{\partial t^2}}_{\text{manifestly } \rho,\phi\text{-independent}} \quad . \tag{22}$$

The left-hand side quantity must be equal to the right-hand side quantity, which is a quantity that is independent of t (owing to the left-hand side) and also ρ and ϕ (owing to the right-hand side). Therefore, this quantity can depend on none of these three independent variables and so must in fact be a constant; call it $-K^2$.

Equating the right-hand side of (22) with $-K^2$ produces

$$\frac{1}{v^2\,T(t)}\frac{\partial^2 T(t)}{\partial t^2} = -K^2 \quad \Rightarrow \quad \boxed{\frac{\partial^2 T(t)}{\partial t^2} + (vK)^2 T(t) = 0.} \tag{23}$$

Equating the left-hand side of (22) with $-K^2$ produces:

$$\frac{1}{\rho\,P(\rho)}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big) + \frac{1}{\rho^2\,\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = -K^2,$$

$$i.e. \qquad \underbrace{\frac{\rho}{P(\rho)}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big) + \rho^2 K^2}_{\text{manifestly } \phi,t\text{-independent}} \quad = \quad \underbrace{-\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}}_{\text{manifestly } \rho,t\text{-independent}} \quad . \tag{24}$$

Again, the left-hand side quantity must be equal to the right-hand side quantity, which is a quantity that is independent of ϕ and t (owing to the left-hand side) and also ρ and t (owing to the right-hand side). Therefore, this quantity can depend on none of these three independent variables and so must in fact be a constant; call it M^2.

Equating—separately—the left-hand side and the right-hand side of (24) with M^2 produces two separate ordinary differential equations,

$$\boxed{\frac{\rho}{P(\rho)}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big) + \rho^2 K^2 = M^2}, \tag{25a}$$

$$\boxed{-\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = M^2}. \tag{25b}$$

Rearranging the terms in these equations and listing them together with (23), we obtain:

$$\rho^2\frac{\partial^2 P(\rho)}{\partial\rho^2} + \rho\frac{\partial P(\rho)}{\partial\rho} + \big[(K\rho)^2 - M^2\big]P(\rho) = 0, \tag{26a}$$

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + M^2\Phi(\phi) = 0, \tag{26b}$$
$$\frac{\partial^2 T(t)}{\partial t^2} + (vK)^2 T(t) = 0. \tag{26c}$$

[10 pt] ***b.*** Using $\Phi(\phi) = e^{im\phi}$ and $T(t) = e^{i\omega t}$, determine the general solution for $P(\rho)$ such that it satisfies the obvious boundary condition, $h(a, \phi, t) = 0$: the drumhead is clamped down at the rim.

Solution

Substituting $\Phi(\phi) = e^{im\phi}$ in (26b), we see that $M = m$, and substituting $T(t) = e^{i\omega t}$ in (26c), we see that $vK = \omega$, *i.e.*, $K = \frac{\omega}{v}$. Since (26a) is the Bessel equation, it is solved by the linear combination of the Bessel function $J_m(K\rho)$ and the Neumann function[1] $N_m(K\rho)$.

Incidentally, even without completing the separation of variables as done in the previous part, substituting $\Phi(\phi) = e^{im\phi}$ and $T(t) = e^{i\omega t}$, *i.e.*, $h(\rho, \phi, t) = P(\rho)\, e^{im\phi}\, e^{i\omega t}$ into (20) produces

$$0 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big)\, e^{im\phi}\, e^{i\omega t} + P(\rho)\,\frac{1}{\rho^2}\frac{\partial^2 e^{im\phi}}{\partial\phi^2}\, e^{i\omega t} - P(\rho)\, e^{im\phi}\frac{1}{v^2}\frac{\partial^2 e^{i\omega t}}{\partial t^2}, \tag{27}$$
$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big)\, e^{im\phi}\, e^{i\omega t} + P(\rho)\,\Big(-\frac{m^2}{\rho^2}e^{im\phi}\Big)\, e^{i\omega t} - P(\rho)\, e^{im\phi}\frac{1}{v^2}\Big(-\omega^2 e^{i\omega t}\Big), \tag{28}$$
$$= \frac{1}{\rho^2}\Big\{\rho\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial P(\rho)}{\partial\rho}\Big) - m^2 P(\rho) + \Big(\frac{\omega}{v}\rho\Big)^2 P(\rho)\Big\}\, e^{im\phi}\, e^{i\omega t}. \tag{29}$$

Since neither $e^{im\phi}$ nor $e^{i\omega t}$ vanishes for all ϕ, t, the quantity in the braces must vanish, which is identical to (26a) upon identifying $M = m$ and $K = \omega/v$.

As pointed out in class, $\lim_{\rho\to 0} N_m(K\rho) = \infty$, which is impossible in representing the (*obviously finite*) vertical displacement of the drumhead. Therefore, the ρ-factor is reduced to the Bessel function $J_m(K\rho)$. In addition, other than serving to separate the partial differential equation (20) into the system (26) of ordinary differential equations, there was—*initially*—no restriction on the values of K and M. In fact, even the signs of $-K^2$ in (23) and $+M^2$ in (25) were chosen "for subsequent convenience"—so that the equations in the system (26) would turn out more easily recognizable.

The *general* solution therefore has to be formed as a linear combination of the solutions obtained with various permissible values of $M = m$ and $K = \frac{\omega}{v}$:

$$h(\rho, \phi, t) = \sum_{K,m} c_{K,m}\; J_m(K\rho)\, e^{im\phi}\, e^{i\omega t}, \tag{30}$$

except that we have not yet determined the values over which K and m are to be summed.

***m*:** Picking any point on the drumhead, $h(\rho, \phi, t)$ represents its vertical displacement. Therefore, $h(\rho, \phi{+}2\pi, t)$ must represent the same value, implying that $e^{im\phi} = e^{im(\phi+2\pi)} = e^{im\phi}e^{2m\pi i}$. That is, m must be integers.

[1] Arfken, Webber and Harris denote this same function $Y_m(K\rho)$.

***K*:** The coefficient in the argument of $J_m(K\rho)$ is to be determined so that $h(a,\phi,t)=0$: there is no displacement at the rim of the drum, where the drumhead is clamped down. This enforces $J_m(Ka)=0$, which implies that for each possible choice of K, Ka must equal a zero of the m^{th} Bessel function. As discussed in class, the zeros of the Bessel function are not integral multiples of any particular unit, but can be enumerated and are tabulated [1, Table 14.1]; write α_{mi} for the i^{th} zero of $J_m(x)$. We thus set $Ka=\alpha_{m,i}$ and we sum over all permissible values of K by summing over i, which enumerate the requisite zero-locations.

Recalling that $K=\frac{\omega}{v}$, so $\omega=vK$, and that v is the speed of sound in the given drumhead (and so a fixed constant), we have:

$$h(\rho,\phi,t)=\sum_{m=-\infty}^{\infty}\sum_{i=1}^{\infty}c_{i,m}\;J_m\Big(\alpha_{m,i}\frac{\rho}{a}\Big)\,e^{im\phi}\,e^{i\omega_{m,i}t},\qquad \omega_{m,i}=\tfrac{v}{a}\alpha_{m,i}, \tag{31}$$

[5 pt] ***c.*** Using Table 14.1, compute the five lowest values of the frequency ω and show them not to be integral multiples of a unit: this is why drums generally do not produce tones of well-defined pitch.

Solution

From (31), the frequencies $\omega_{m,i}$ supported by the drumhead are $\frac{v}{a}$-multiples of the zeros of the Bessel functions. From [1, Table 14.1], the lowest-valued five (*limited to five significant figures*) are:

$$\alpha_{0,1}=2.4048,\quad \alpha_{1,1}=3.8317,\quad \alpha_{2,1}=5.1356,\quad \alpha_{0,2}=5.5201,\quad \alpha_{3,1}=6.3802 \tag{32}$$

The lowest five frequencies are then $\frac{v}{a}$-multiples of these numbers, and are clearly not integral multiples of a fixed unit. The limited decimal expansion of course does not prove this, but is nevertheless quite manifest from the listing [1, Table 14.1]. For what it is worth, *Mathematica* can produce each (finitely-positioned) zero of each (finite-index) Bessel function to any desired number of decimals; here are the above numbers to 50 significant figures (a frivolous show-off):

$$\alpha_{0,1}=2.404\,825\,557\,695\,772\,768\,621\,631\,879\,326\,454\,643\,124\,244\,909\,146\,0\ldots, \tag{33a}$$
$$\alpha_{1,1}=3.831\,705\,970\,207\,512\,315\,614\,435\,886\,308\,160\,766\,564\,545\,274\,287\,8\ldots, \tag{33b}$$
$$\alpha_{2,1}=5.135\,622\,301\,840\,682\,556\,301\,401\,690\,137\,765\,456\,973\,772\,347\,500\,5\ldots, \tag{33c}$$
$$\alpha_{0,2}=5.520\,078\,110\,286\,310\,649\,596\,604\,112\,813\,027\,425\,221\,865\,478\,782\,9\ldots, \tag{33d}$$
$$\alpha_{3,1}=6.380\,161\,895\,923\,983\,506\,236\,614\,641\,942\,703\,305\,326\,303\,691\,903\,1\ldots \tag{33e}$$

Also, nothing in this problem determines the coefficients $c_{i,m}$; they could be determined from an initial condition, such as stating that drumhead is initially displaced to a specific shape ($h(\rho,\phi,0)\stackrel{!}{=}H(\rho,\phi)$, say), or that it receives an initial *change* in the displacement ($\big[\frac{\partial h}{\partial t}\big]_{t=0}\stackrel{!}{=}V(\rho,\phi)$, say). But, that's for another problem.

4. For any function $F(\theta,\phi)=\sum_{\ell,m}a_\ell{}^m\,Y_\ell^m(\theta,\phi)$,

[5 pt] ***a.*** prove that

$$\int_0^\pi\sin(\theta)\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\;\big|F(\theta,\phi)\big|^2=\sum_{\ell,m}|a_{\ell,m}|^2. \tag{34}$$

Solution

Substituting straightforwardly,

$$\begin{aligned}\int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi\,\big|F(\theta,\phi)\big|^2 &= \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi\,\Big|\sum_{\ell,m} a_\ell{}^m\,Y_\ell^m(\theta,\phi)\Big|^2,\\ &= \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi\,\Big[\sum_{\ell,m} a_\ell{}^m\,Y_\ell^m(\theta,\phi)\Big]^*\Big[\sum_{\lambda,\mu} a_\lambda{}^\mu\,Y_\lambda^\mu(\theta,\phi)\Big],\\ &= \sum_{\ell,\lambda,m,\mu}(a_\ell{}^m)^* a_\lambda{}^\mu \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi\,\Big[Y_\ell^m(\theta,\phi)\Big]^* Y_\lambda^\mu(\theta,\phi),\\ &= \sum_{\ell,\lambda,m,\mu}(a_\ell{}^m)^* a_\lambda{}^\mu\;\delta_{\ell,\lambda}\,\delta_{m,\mu} = \sum_{\ell,m}(a_\ell{}^m)^* a_\ell{}^m = \sum_{\ell,m}\big|a_\ell{}^m\big|^2. \qquad \checkmark \end{aligned} \tag{35}$$

[10 pt] ***b.*** Rewriting the integrand in terms of $Y_\ell^m(\theta,\phi)$ and using (34), calculate

$$\int_0^\pi \mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi\;\sin^3(\theta)\Big[\sin^2(\theta)\Big(5\sqrt{3}\cos(\phi)-\sqrt{5}\cos(3\phi)\Big)-4\sqrt{3}\cos(\phi)\Big]^2. \tag{36}$$

Hint: rewrite the integrand in (37) akin to that in (34), using that $\cos(x)=(e^{ix}+e^{-ix})/2$; expand the resulting expression for $F(\theta,\phi)$ and identify terms with the $Y_\ell^m(\theta,\phi)$ starting with the highest powers of $\sin(\theta)$ and/or $\cos(\theta)$.

Solution

Before we follow the hint, we note that a factor of $\sin(\theta)$ needs to be moved into the integration measure, while the remaining $\sin^2(\theta)$ may be moved inside the square brackets, leaving us with the integral

$$\int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi\;\Big[\sin^3(\theta)\Big(5\sqrt{3}\cos(\phi)-\sqrt{5}\cos(3\phi)\Big)-4\sqrt{3}\sin(\theta)\cos(\phi)\Big]^2, \tag{37}$$

so that

$$F(\theta,\phi)=\sin^3(\theta)\Big(5\sqrt{3}\cos(\phi)-\sqrt{5}\cos(3\phi)\Big)-4\sqrt{3}\sin(\theta)\cos(\phi) \tag{38}$$

is a real function and the desired integral in fact is exactly of the form (35). It remains then to express $F(\theta,\phi)$ as a concrete linear combination of spherical harmonics, *i.e.*, to determine the coefficients in the so-called Laplace expansion, $F(\theta,\phi)=\sum_{\ell,m} a_\ell{}^m\,Y_\ell^m(\theta,\phi)$. With these coefficients, $a_\ell{}^m$ ascertained, the value of the integral is $\sum_{\ell,m}|a_\ell{}^m|^2$.

We proceed following the hint,

$$\begin{aligned} F(\theta,\phi) &= \tfrac{1}{2}\sin^3(\theta)\Big(5\sqrt{3}(e^{\phi}+e^{-\phi})-\sqrt{5}(e^{3i\phi}+e^{-3i\phi})\Big)-2\sqrt{3}\sin(\theta)(e^{i\phi}+e^{-i\phi}),\\ &= \underbrace{\tfrac{5\sqrt{3}}{2}\sin^3(\theta)e^{\phi}-\tfrac{\sqrt{5}}{2}\sin^3(\theta)e^{3i\phi}-2\sqrt{3}\sin(\theta)e^{i\phi}}_{=f(\theta,\phi)}\;+\;\text{complex conjugate, } f^*(\theta,\phi). \end{aligned} \tag{39}$$

Reordering a little, we have:

$$f(\theta,\phi)=-\tfrac{\sqrt{5}}{2}\sin^3(\theta)e^{3i\phi}+\tfrac{5\sqrt{3}}{2}\sin^3(\theta)e^{\phi}-2\sqrt{3}\sin(\theta)e^{i\phi}. \tag{40}$$

The leading term, $\sin^3(\theta)\,e^{3i\phi}$, is unambiguously identified as being equal to:

$$\boxed{\sin^3(\theta)\,e^{3i\phi}=-\sqrt{\frac{64\pi}{35}}\,Y_3^3(\theta,\phi)}. \tag{41}$$

The next term, $\sin^3(\theta)\, e^{i\phi}$, does not appear in the given listing of the spherical harmonics as-is. However, we note that $e^{i\phi}$ does appear within $Y_3^1(\theta,\phi)$ and $Y_1^1(\theta,\phi)$, the former of which has $\sin(\theta)$ multiplied by a quadratic polynomial in $\cos(\theta)$. We therefore write:

$$\begin{aligned} f(\theta,\phi) &= -\frac{\sqrt{5}}{2}\Big[-\sqrt{\frac{64\pi}{35}}Y_3^3(\theta,\phi)\Big] + \frac{5\sqrt{3}}{2}\Big(\big(1-\cos^2(\theta)\big)\sin(\theta)\Big)e^{\phi} - 2\sqrt{3}\sin(\theta)e^{i\phi}, \\ &= \sqrt{\frac{16\pi}{7}}Y_3^3(\theta,\phi) - \frac{5\sqrt{3}}{2}\cos^2(\theta)\sin(\theta)\, e^{\phi} + \frac{\sqrt{3}}{2}\sin(\theta)\, e^{\phi}. \end{aligned} \tag{42}$$

Consulting the listing of spherical harmonics again, we see that the function $\cos^2(\theta)\sin(\theta)\, e^{i\phi}$ appears as one of the two terms in $Y_1^1(\theta,\phi)$ while its other term involves $\sin(\theta)\, e^{i\phi}$—which also appears in $Y_1^1(\theta,\phi)$. Thus, we may write:

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin(\theta)\, e^{i\phi} \qquad \Rightarrow \qquad \boxed{\sin(\theta)\, e^{i\phi} = -\sqrt{\frac{8\pi}{3}}\, Y_1^1(\theta,\phi)}, \tag{43}$$

and thus

$$\begin{aligned} Y_3^1(\theta,\phi) &= -\sqrt{\frac{7}{48\pi}}\Big(\frac{15}{2}\cos^2(\theta) - \frac{3}{2}\Big)\sin(\theta)\, e^{i\phi} = -\sqrt{\frac{525}{64\pi}}\cos^2(\theta)\sin(\theta)\, e^{i\phi} + \sqrt{\frac{21}{64\pi}}\sin(\theta)\, e^{i\phi}, \\ &= -\sqrt{\frac{525}{64\pi}}\cos^2(\theta)\sin(\theta)\, e^{i\phi} + \sqrt{\frac{21}{64\pi}}\Big(-\sqrt{\frac{8\pi}{3}}\, Y_1^1(\theta,\phi)\Big), \\ &= -\sqrt{\frac{525}{64\pi}}\cos^2(\theta)\sin(\theta)\, e^{i\phi} - \sqrt{\frac{7}{8}}\, Y_1^1(\theta,\phi). \end{aligned} \tag{44}$$

Rearranging terms, we have:

$$\boxed{-\cos^2(\theta)\sin(\theta)\, e^{i\phi} = \sqrt{\frac{64\pi}{525}}\, Y_3^1(\theta,\phi) + \sqrt{\frac{8\pi}{75}}\, Y_1^1(\theta,\phi)} \tag{45}$$

Substituting these into (42), we have

$$\begin{aligned} f(\theta,\phi) &= \sqrt{\frac{16\pi}{7}}Y_3^3(\theta,\phi) + \frac{5\sqrt{3}}{2}\Big[\sqrt{\frac{64\pi}{525}}\, Y_3^1(\theta,\phi) + \sqrt{\frac{8\pi}{75}}\, Y_1^1(\theta,\phi)\Big] + \frac{\sqrt{3}}{2}\Big[-\sqrt{\frac{8\pi}{3}}\, Y_1^1(\theta,\phi)\Big], \\ &= \sqrt{\frac{16\pi}{7}}Y_3^3(\theta,\phi) + \sqrt{\frac{16\pi}{7}}\, Y_3^1(\theta,\phi) + \sqrt{2\pi}\, Y_1^1(\theta,\phi) - \sqrt{2\pi}\, Y_1^1(\theta,\phi), \\ &= \sqrt{\frac{16\pi}{7}}\Big(Y_3^3(\theta,\phi) + Y_3^1(\theta,\phi)\Big). \end{aligned} \tag{46}$$

Therefore,

$$\begin{aligned} F(\theta,\phi) &= \sqrt{\frac{16\pi}{7}}\Big(Y_3^3(\theta,\phi) + \big(Y_3^3(\theta,\phi)\big)^* + Y_3^1(\theta,\phi) + \big(Y_3^1(\theta,\phi)\big)^*\Big), \\ \big[F(\theta,\phi)\big]^2 &= \frac{16\pi}{7}\Big(2\big|Y_3^3(\theta,\phi)\big|^2 + 2\big|Y_3^1(\theta,\phi)\big|^2 \;+\; \text{mixed terms}\Big), \end{aligned}$$

and

$$\begin{aligned} &\int_0^{\pi}\sin(\theta)\, d\theta \int_0^{2\pi} d\phi\; \Big[\sin^3(\theta)\Big(5\sqrt{3}\cos(\phi) - \sqrt{5}\cos(3\phi)\Big) - 4\sqrt{3}\sin(\theta)\cos(\phi)\Big]^2 \\ &\quad = 4\cdot\frac{16\pi}{7} = \frac{64\pi}{7}. \end{aligned} \tag{47}$$

The procedure (37)–(47) may indeed be comparable to a direct evaluation of the integral (37). However, if several such integrals are required, one first computes those results of the type (41), (43) and (45) as will be needed in the considered integrals, re-expresses all desired integrals in terms of the ${Y_\ell}^m(\theta, \phi)$'s, and then uses Parseval's identity (35).

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*. Academic Press, 7 ed., 2012.