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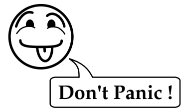

Mathematical Methods I

Midterm 1: 2019, Sep. 23.

Solutions – T. Hübsch

This is an "open Textbook (Arfken, Webber & Harris), open lecture notes" in-class exam. For full credit, show all your work. Hand in the solutions completed in class stapled to the question sheet; complete the *rest* of the Exam and hand it in **by Monday, 09/30/19, 2:00 pm**, for 2/3 of the indicated credit. ***Budget your time:*** *do first what you are sure you know how; use shortcuts, but be prepared to explain them afterwards.* The solution sketches provided here are considerably more complete than is expected of a student in the test.

1. Determine the convergence/divergence properties (absolute? conditional? uniform? range/radius of convergence?—as appropriate) for the following series:

[10 pt] ***a.*** $\sum_{k=0}^{\infty} \frac{k^2\,2^k}{(k+2)!}$.

Solution

Apply the ratio test:

$$1 \overset{!}{>} \lim_{k\to\infty}\left|\frac{\frac{(k+1)^2\,2^{k+1}}{(k+3)!}}{\frac{k^2\,2^k}{(k+2)!}}\right| = \lim_{k\to\infty}\left|\frac{\frac{(k+1)^2\,\cancel{2^k}\cdot 2^1}{(k+3)\cdot\cancel{(k+2)!}}}{\frac{k^2\,\cancel{2^k}}{\cancel{(k+2)!}}}\right| = \lim_{k\to\infty}\left|\frac{(k\cancel{+1})^2\cdot 2}{k^2\cdot(k\cancel{+3})}\right| = 2\lim_{k\to\infty}\left|\frac{k^2}{k^3}\right| = 2\lim_{k\to\infty}\left|\frac{1}{k}\right| = 0. \tag{1}$$

Since $1 > 0$, the ratio test implies that the series converges.

[10 pt] ***b.*** $\sum_{k=0}^{\infty}(-1)^k \frac{1}{1+k^2}$.

Solution

Applying the ratio test (and ignoring that the series alternates):

$$1 \overset{?}{>} \lim_{k\to\infty}\left|\frac{(-1)^{k+1}\frac{1}{1+(k+1)^2}}{(-1)^k\frac{1}{1+k^2}}\right| = \lim_{k\to\infty}\left|\frac{\frac{1}{1+(k\cancel{+1})^2}}{\frac{1}{1+k^2}}\right| = \lim_{k\to\infty}\left|\frac{\frac{1}{1+k^2}}{\frac{1}{1+k^2}}\right| = 1, \tag{2}$$

and the ratio test fails. In turn, (**1**) $\frac{1}{1+k^2} > \frac{1}{1+(k+1)^2}$ are monotonically decreasing for all $k \geqslant 0$, and also (**2**) $\lim_{k\to\infty}\frac{1}{1+k^2} = 0$. Therefore, Leibnitz' criterion is satisfied, implying that the series does converges *conditionally.*

In fact, the integral test can prove much more: We *do* know that

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \big[\arctan(x)\big]_0^\infty = \big[\tfrac{\pi}{2} - 0\big] = \tfrac{\pi}{2}. \tag{3}$$

According to the integral test then,

$$\tfrac{\pi}{2} = \int_0^\infty \frac{\mathrm{d}x}{1+x^2} \leqslant \sum_{k=0}^{\infty}\frac{1}{1+k^2} \leqslant \Big[(-1)^k\frac{1}{1+k^2}\Big]_{k=0} + \int_0^\infty \frac{\mathrm{d}x}{1+x^2} = 1 + \tfrac{\pi}{2}. \tag{4}$$

So, since even the non-alternating series $\sum_{k=0}^{\infty}\frac{1}{1+k^2}$ converges, the alternating series $\sum_{k=0}^{\infty}(-1)^k\frac{1}{1+k^2}$ must be convergent too, — in fact, it converges *absolutely.*

For what it is worth, *Mathematica* readily provides the values

$$\sum_{k=0}^{\infty}\frac{1}{1+k^2} = \tfrac{1}{2}\Big(1+\frac{\pi\cosh(\pi)}{\sinh(\pi)}\Big) \approx 2.07667\ldots, \qquad \sum_{k=0}^{\infty}\frac{(-1)^k}{1+k^2} = \tfrac{1}{2}\Big(1+\frac{\pi}{\sinh(\pi)}\Big) \approx 0.636015\ldots \tag{5}$$

in agreement with the above computations.

[20 pt] ***c.*** $\sum_{k=0}^{\infty}(-1)^k \frac{k^3}{3^k}(x+1)^k$.

Solution

Applying the ratio test, we get:

$$1 \overset{?}{>} \lim_{k\to\infty}\left|\frac{(-1)^{k+1}\,\frac{(k+1)^3}{3^{k+1}}\,(x+1)^{k+1}}{(-1)^k\,\frac{k^3}{3^k}\,(x+1)^k}\right| = \lim_{k\to\infty}\left|\frac{\frac{(k\cancel{+1})^3}{\cancel{3^k}\cdot 3^1}\,(x+1)\cdot\cancel{(x+1)^k}}{\frac{k^3}{\cancel{3^k}}\,\cancel{(x+1)^k}}\right| = \lim_{k\to\infty}\left|\frac{\cancel{k^3}}{3\cancel{k^3}}\right||x+1| = \tfrac{1}{3}|x+1|. \tag{6}$$

Solving for x, this implies that the series converges absolutely (point-by-point) for

$$|x+1| < 3, \qquad \text{i.e.,} \qquad -4 < x < 2. \tag{7}$$

As for *uniform convergence,* we may apply Abel's test, we should factor $(-1)^k \frac{k^3}{3^k}(x+1)^k = a_m\, f_k(x)$, in such a way that: (**1**) $\sum_{k=0}^{\infty} a_k = A < \infty$, and (**2**) $0 \leqslant f_k(x) \leqslant f_{k+1}(x) \leqslant M < \infty$ for all k over the region of convergence, $x \in [a, b]$.

The simplest such factorization has $a_k = (-1)^k \frac{k^3}{3^k}$ and $f_k(x) = (x+1)^k$, for which indeed:

1. $\sum_{k=0}^{\infty}(-1)^k \frac{k^3}{3^k}$ converges, in fact absolutely, since $\lim_{k\to\infty}\left|\frac{(k\cancel{+1})^3/3^{k+1}}{k^3/3^k}\right| = \frac{1}{3} < 1$;
2. $0 \leqslant (x+1)^k \leqslant (x+1)^{k+1} \leqslant 1$ for $x \in (-1, 0)$.

This factorization choice thus guarantees uniform convergence only within the subset $(-1, 0) \subset [-4, 2]$ of the absolute convergence range.

This can be improved: For the convergence of $\sum_{k=0}^{\infty}(-1)^k \frac{k^3}{\alpha^k}$, it suffices that $\alpha > 1$. So, we may factorize $3 = (\alpha)(3/\alpha)$, where $\alpha = 1{+}\epsilon$, and $0 < \epsilon \ll 1$ is as small as our calculators/computers can manage. Then,

1. $\sum_{k=0}^{\infty}(-1)^k \frac{k^3}{\alpha^k}$ converges, in fact absolutely, since $\lim_{k\to\infty}\left|\frac{(k\cancel{+1})^3/\alpha^{k+1}}{k^3/\alpha^k}\right| = \frac{1}{\alpha} < 1$;
2. $0 \leqslant \big(\alpha(x+1)/3\big)^k \leqslant \big(\alpha(x+1)/3\big)^{k+1} \leqslant 1$ for $x \in (-1, 2-\delta_\alpha)$, where $\lim_{\alpha\to 1}\delta_\alpha = 0$.

(If this is not clear, plot the first dozen or so functions $\big(\alpha(x+1)/3\big)^k$ for α such as $1.00\ldots01$, and verify their monotony and boundedness by inspection.)

In fact, this latter property can be verified separately also by applying Weierstrass' majorant test, but I'll leave that to the diligent student.

2. Given a vector field specified in (circular) cylindrical coordinates as $\vec{A} = \varphi\,\hat{\mathrm{e}}_\rho$ for $n = 1, 2, 3\ldots$

[5 pt] ***a.*** Compute $\vec{\nabla}\cdot\vec{A}$.

Solution

Using [1, (3.148)], we have:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\,\varphi) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}(0) + \frac{\partial}{\partial z}(0) = \frac{\varphi}{\rho}. \tag{8}$$

[5 pt] ***b.*** Compute $\vec{\nabla}\times\vec{A}$.

Solution

Using [1, (3.150)], we have:

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\varphi} & \hat{z}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ \varphi & \rho{\cdot}0 & 0\end{vmatrix} = \frac{1}{\rho}(+\hat{z})\begin{vmatrix}\frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi}\\ \varphi & \rho{\cdot}0\end{vmatrix} = \frac{1}{\rho}(+\hat{z})(-1) = -\frac{1}{\rho}\hat{z}. \tag{9}$$

[5 pt] ***c.*** Compute $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$.

Solution

Using again [1, (3.150)] as well as the result from ***b***, we have:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\varphi} & \hat{z}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ 0 & \rho{\cdot}0 & -\frac{1}{\rho}\end{vmatrix} = \frac{1}{\rho}(-\rho\hat{\varphi})\begin{vmatrix}\frac{\partial}{\partial\rho} & \frac{\partial}{\partial z}\\ 0 & -\frac{1}{\rho}\end{vmatrix} = -\hat{\varphi}\big(\tfrac{1}{\rho^2}\big) = -\frac{1}{\rho^2}\hat{\varphi}. \tag{10}$$

[3×5 pt] ***d.*** Compute the three components of $\vec{\nabla}^2\vec{A}$.

Solution

The three components may, in turn, be computed using [1, (3.151)], but it is much faster to use the above results and compute:

$$\vec{\nabla}^2\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\Big(\frac{\varphi}{\rho}\Big) - \Big(-\frac{1}{\rho^2}\hat{\varphi}\Big) = \Big[\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{z}\frac{\partial}{\partial z}\Big]\Big(\frac{\varphi}{\rho}\Big) + \frac{1}{\rho^2}\hat{\varphi}, \tag{11}$$

$$= \Big[\hat{\rho}\Big(-\frac{\varphi}{\rho^2}\Big) + \frac{1}{\rho}\hat{\varphi}\Big(\frac{1}{\rho}\Big)\Big] + \frac{1}{\rho^2}\hat{\varphi} = -\frac{\varphi}{\rho^2}\hat{\rho} + \frac{2}{\rho^2}\hat{\varphi}, \tag{12}$$

Thus: $\vec{\nabla}^2\vec{A}|_\rho = -\frac{\varphi}{\rho^2}$, $\vec{\nabla}^2\vec{A}|_\varphi = \frac{2}{\rho^2}$, and $\vec{\nabla}^2\vec{A}|_z = 0$.

Indeed, let's verify this by using [1, (3.151)]:

$$\vec{\nabla}^2\vec{A}|_\rho = \Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho^2}\Big](\varphi) - \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}(0) = \Big[-\frac{\varphi}{\rho^2}\Big], \quad\checkmark \tag{13}$$

$$\vec{\nabla}^2\vec{A}|_\varphi = \Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho^2}\Big](0) + \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}(\varphi) = +\frac{2}{\rho^2}, \quad\checkmark \tag{14}$$

$$\vec{\nabla}^2\vec{A}|_z = \Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho^2}\Big](0) = 0. \quad\checkmark \tag{15}$$

[10 pt] ***e.*** Compute $\oint_S \mathrm{d}^2\vec{r}\times\vec{A}$ where S is the (complete, closed) surface of the height-h right cone with the radius-R circular base in the (x,y)-plane.

Solution

I will calculate this is two different ways — both of which are tricky.

e.1: To evaluate the integral directly, we need that the outward normal to the surface of this right cone is given in to parts:

1. The base, in the (x,y)-plane, has a downward normal: $\hat{n}_B = -\hat{z}$.

2. The conical sheath has a tilted normal: $\hat{n}_S = \cos(\alpha)\hat{\rho} + \sin(\alpha)\hat{z}$, where $\alpha = \arctan(R/h)$ is the (half-)opening angle of the cone.

The integrations over the base and the conical sheath may both be expressed as:

1. Circular base: $\int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi\ (-\hat{z}) \times \big[\vec{A}\big]_{z=0}$.
2. Conical sheath: $\int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi \big(\cos(\alpha)\hat{\rho} + \sin(\alpha)\hat{z}\big) \times \big[\vec{A}\big]_{z=h(1-\rho/R)}$

Since $\vec{A} = \varphi\,\hat{\rho}$ is $z-$independent (both the component, $A_\rho = \varphi$, and the unit-vector, $\hat{\rho}$), the two contributions may be summed straightforwardly, and we have:

$$\oint_S \mathrm{d}^2\vec{r} \times \vec{A} \ = \int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi\ \Big\{\Big[\underbrace{(-\hat{z})}_{z=0} \ + \ \underbrace{\big(\cos(\alpha)\hat{\rho} + \sin(\alpha)\hat{z}\big)}_{z=h(1-\rho/R)}\Big] \times \big(\varphi\hat{\rho}\big)\Big\}. \tag{16}$$

We now use that $\hat{\rho} \times \hat{\rho} = 0$, and $\hat{z} \times \hat{\rho} = \hat{\varphi}$, so that:

$$\oint_S \mathrm{d}^2\vec{r} \times \vec{A} \ = \int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi\ \Big\{\big(\sin(\alpha){-}1\big)\hat{z} \times \hat{\rho}\,\varphi\Big\} = \big(\sin(\alpha){-}1\big) \int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi\ \varphi\,\hat{\varphi}. \tag{17}$$

Recall that the unit-vector $\hat{\varphi}$ *changes its direction,* and so is not a constant to be "moved" outside the integral. Instead, if we rewrite it as $\hat{\varphi} = \hat{y}\cos\varphi - \hat{x}\sin\varphi$ [1, Exc. 3.10.6, p. 197], the *constant* unit-vectors can "move" out of the integrals:

$$\begin{aligned}
\oint_S \mathrm{d}^2\vec{r} \times \vec{A} \ &= \big(\sin(\alpha){-}1\big) \int_0^R \rho\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi\ \varphi(\hat{y}\cos\varphi - \hat{x}\sin\varphi), && (18)\\
&= \big(\sin(\alpha){-}1\big)\Big(\frac{R^2}{2}\Big)\Big(\hat{y}\int_0^{2\pi} \mathrm{d}\varphi\ \varphi\,\cos\varphi - \hat{x}\int_0^{2\pi} \mathrm{d}\varphi\ \varphi\sin\varphi\Big), && (19)\\
&= \big(\sin(\alpha){-}1\big)\Big(\frac{R^2}{2}\Big)\Big(\hat{y}\Big[\cos\varphi + \varphi\,\sin\varphi\Big]_0^{2\pi} - \hat{x}\Big[\sin\varphi - \varphi\cos\varphi\Big]_0^{2\pi}\Big), && (20)\\
&= \big(\sin(\alpha){-}1\big)\Big(\frac{R^2}{2}\Big)\Big(\hat{y}(0) - \hat{x}(-2\pi)\Big) = \big(\sin(\alpha){-}1\big)\pi R^2\,\hat{x}, && (21)\\
&= \big(\sin(\alpha){-}1\big)\pi R^2\big(\cos\varphi\,\hat{\rho} - \sin\varphi\,\hat{\varphi}\big), && (22)
\end{aligned}$$

where in the last line I reverted to the cylindrical coordinate unit-vectors, although the previous line, with $\hat{x}$ may well look simpler.

***e*.2:** It is tempting to use Gauss-like substitution, $\oint_{S=\partial V} \mathrm{d}^2\vec{r} \times \vec{A} = \int_V \mathrm{d}^3\vec{r}\ \vec{\nabla} \times \vec{A}$, since we have computed the $\vec{\nabla} \times \vec{A}$ integrand above:

$$\begin{aligned}
\oint_S \mathrm{d}^2\vec{r} \times \vec{A} \ &= \int_V \mathrm{d}^3\vec{r}\,\vec{\nabla} \times \vec{A} = \int_V \mathrm{d}^3\vec{r}\,\big(-\hat{z}/\rho\big) = -\hat{z}\int_0^R \rho\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \int_0^{h(1-\rho/R)} \mathrm{d}z\ (1/\rho), && (23)\\
&= -\hat{z}(2\pi)\int_0^R \mathrm{d}\rho\ h\Big(1 - \frac{\rho}{R}\Big) = -\hat{z}(2\pi)(h)\Big[\rho - \frac{\rho^2}{2R}\Big]_0^R = -(\pi R\,h)\hat{z}, && (24)
\end{aligned}$$

which looks *nothing* like (22)!

Was Gauss wrong?! No. It's we who would be wrong to use Gauss' theorem.
Why? Because the integral $\oint_S \mathrm{d}^2\vec{r}\times\vec{A}$ has an *ambiguous* integrand. The angle φ is periodic, with the evidently discontinuous — and more importantly, *double-valued* — specification $(\varphi=0)\equiv(\varphi=2\pi)$. The integrand in the integral (16) is therefore also ambiguous.

Discontinuity itself is not an obstruction to the $\oint_S \mathrm{d}^2\vec{r}\times\vec{A}$ integral: discontinuous functions with a finite number of separated finite discontinuities are all integrable in the usual sense. The derivatives (8), (9), (10) and (12) *should* however be regarded as suspect, and therefore also the evaluation of the right-hand side integral $\int_V \mathrm{d}^3\vec{r}\,\vec{\nabla}\times\vec{A}$ to be employed in Gauss' theorem.

In fact, it *is* possible to modify the computations of the derivatives $(\vec{\nabla}\cdot\vec{A})$ (8), $(\vec{\nabla}\times\vec{A})$ (9), $\big(\vec{\nabla}\times(\vec{\nabla}\cdot\vec{A})\big)$ (10) and $(\vec{\nabla}^2\vec{A})$ (12) so as to include the discontinuity of $\vec{A}$ at $\varphi=0\equiv 2\pi$, using a suitable multiple of a $\delta(\varphi)$-function — just as done in [1, pp. 175–179]; "we must be more devious." I'll leave these "suitable modifications" to the diligent reader.

For the test itself, no such "deviousness" was required or expected.

3. For the following differential equations, compute the general solution and then the particular one satisfying the indicated boundary condition(s):

[10 pt] ***a.*** Solve $\frac{\mathrm{d}y}{\mathrm{d}x}=-e^y\,x$, so that $y(0)=1$.

Solution

This differential equation separates:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-e^y\,x \quad\to\quad e^{-y}\,\mathrm{d}y=-x\,\mathrm{d}x \quad\to\quad \int e^{-y}\,\mathrm{d}y=-\int x\,\mathrm{d}x \quad\to\quad -e^{-y}=-\tfrac{1}{2}x^2-C, \tag{25}$$

$$\to\quad e^{-y}=C+\tfrac{1}{2}x^2 \quad\to\quad y(x)=-\ln\big(C+\tfrac{1}{2}x^2\big). \tag{26}$$

To satisfy the "boundary" condition, we solve for "C":

$$y(0)=-\ln\big(C+\tfrac{1}{2}0^2\big)=-\ln\big(C\big)\overset{!}{=}1, \quad\to\quad \ln(C)=-1, \quad\to\quad C=e^{-1}=\tfrac{1}{e}. \tag{27}$$

Thus, the solution satisfying the "boundary" condition is $\boxed{y(x)=-\ln\big(\frac{1}{e}+\frac{1}{2}x^2\big)}$.

[10 pt] ***b.*** Solve $x^2\,y''-3x\,y'+3y=0$, so that $y(1)=1$ and $y'(1)=1$.

Solution

This differential equation conforms precisely to the "recipe" #7 in "Know Thy Math" notes (listed as #6 in class):

$$x^2\,y''+px\,y'+qy=0 \quad\to\quad y(x)=x^{\frac{1-p}{2}}\Big(C_1\,x^{\sqrt{(\frac{1-p}{2})^2-q}}+C_2\,x^{-\sqrt{(\frac{1-p}{2})^2-q}}\Big), \tag{28}$$

so that

$$x^2\,y''-3x\,y'+3y=0 \quad\to\quad y(x)=x^{\frac{1+3}{2}}\Big(C_1\,x^{\sqrt{(\frac{1+3}{2})^2-3}}+C_2\,x^{-\sqrt{(\frac{1+3}{2})^2-3}}\Big), \tag{29}$$

$$=x^2\big(C_1\,x^1+C_2\,x^{-1}\big)=C_1\,x^3+C_2\,x. \tag{30}$$

To satisfy the "boundary" conditions, $y(1)=1$ and $y'(1)=1$, we substitute and solve for C_1, C_2:

$$\left.\begin{aligned} y(1) &= \big[C_1\,(x^3)+C_2\,(x)\big]_{x=1} &\overset{!}{=} 1\\ y'(1) &= \big[C_1\,(3{\cdot}x^2)+C_2\,(1)\big]_{x=1} &\overset{!}{=} 1\end{aligned}\right\} \quad\Rightarrow\quad \begin{bmatrix}1&1\\3&1\end{bmatrix}\begin{bmatrix}C_1\\C_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \quad\Rightarrow\quad \begin{cases}C_1 &= 0,\\ C_2 &= 1,\end{cases} \tag{31}$$

so that the solution that satisfies the "boundary" conditions is $y(x)=x$.

4. Solve the 3-parameter family of differential equations $xy''(x)-(A{+}Bx)y'(x)-C\,y(x)=0$ by assuming the solution to be of the form $y(x)=\sum_{k=0}^{\infty}c_k\,x^{k+s}$ and that A,B,C are real parameters:

[8 pt] ***a.*** Substitute the series in the differential equation, then combine the terms into a single infinite series by shifting the summation index in some of the terms, and isolating any extra terms.

Solution

Fo substitution into the 2nd-order ODE, we compute:

$$y(x)=\sum_{k=0}^{\infty}c_k\,x^{k+s},\quad y'(x)=\sum_{k=0}^{\infty}c_k(k{+}s)\,x^{k+s-1},\quad y''(x)=\sum_{k=0}^{\infty}c_k(k{+}s)(k{+}s{-}1)\,x^{k+s-2}, \tag{32}$$

and substitute into the ODE, $xy''(x)-(A{+}Bx)y'(x)-C\,y(x)=0$:

$$x\sum_{k=0}^{\infty}c_k(k{+}s)(k{+}s{-}1)\,x^{k+s-2}-(A{+}Bx)\sum_{k=0}^{\infty}c_k(k{+}s)\,x^{k+s-1}-C\sum_{k=0}^{\infty}c_k\,x^{k+s}=0,$$

where we multiply through with the leading "x" in the first term, and distribute the multiplication by the series across the binomial coefficient:

$$\underbrace{\sum_{k=0}^{\infty}c_k(k{+}s)(k{+}s{-}1)\,x^{k+s-1}}_{k\mapsto k+1}-A\underbrace{\sum_{k=0}^{\infty}c_k(k{+}s)\,x^{k+s-1}}_{k\mapsto k+1}-B\sum_{k=0}^{\infty}c_k(k{+}s)\,x^{k+s}-C\sum_{k=0}^{\infty}c_k\,x^{k+s}=0,$$

where we shift the summation-index in the first two series so as to bring the exponents of x to the same form. While at it, we also combine the first two and the last two series:

$$\sum_{k=-1}^{\infty}c_{k+1}(k{+}1{+}s)(k{+}s{-}A)\,x^{k+s}-\sum_{k=0}^{\infty}c_k[B(k{+}s)+C]\,x^{k+s}=0, \tag{33}$$

whence it becomes obvious that the first of these has an extra term — the one with x^{s-1}, which does not appear in the second series. Extracting this leaves a $0\leqslant k<\infty$ series that is readily combined with the second one:

$$c_0(s)(s{-}A{-}1)\,x^{s-1}+\sum_{k=0}^{\infty}\big\{c_{k+1}(k{+}1{+}s)(k{+}s{-}A)-c_k[B(k{+}s)+C]\big\}\,x^{k+s}=0. \tag{34}$$

[5 pt] ***b.*** Determine the possible value(s) of the index s.

Solution

All the different powers of x being linearly independent, the above infinite sum can vanish only if every (sum-total) coefficient multiplying distinct powers of x vanishes separately. To being with, the coefficient of x^{s-1} must vanish:

$$c_0(s)(s{-}A{-}1) = 0 \quad \overset{c_0\neq 0}{\Longrightarrow} \quad (s)(s{-}A{-}1) = 0, \tag{35}$$

which has two solutions: $s = 0$ and $s = A{+}1$. It is stipulated here that $c_0 \neq 0$ — since the series must have an initial term. Were we to set $c_0 = 0$, that would merely promote $c_1 x^{1+s}$ to become the initial term, and the result would be the same, except for shifting $k \mapsto k+1$ throughout.

[5 pt] ***c.*** Determine the recursion relation that specifies all the coefficients c_k in terms of one or more of the initial coefficient(s).

Solution

The remaining infinitely many terms in the series (34) must vanish separately for each distinct power of x, producing:

$$c_{k+1}(k{+}1{+}s)(k{+}s{-}A) = c_k[C + B(k{+}s)] \tag{36}$$

which we easily solve:

$$c_{k+1} = \frac{C + B(k{+}s)}{(k{+}1{+}s)(k{+}s{-}A)}\, c_k, \tag{37}$$

which is the required recursion relation — which is, owing to two solutions for s, actually two different recursion relations:

$$c^{(0)}_{k+1} = \frac{C + B(k)}{(k{+}1)(k{-}A)}\, c^{(0)}_k, \qquad c^{(A+1)}_{k+1} = \frac{C + B(k{+}A{+}1)}{(k{+}2{+}A)(k{+}1)}\, c^{(A+1)}_k. \tag{38}$$

[5 pt] ***d.*** Determine the range/radius of convergence of $y(x) = \sum_{k=0}^{\infty} c_k\, x^{k+s}$, and specify any conditions on the parameters A, B, C possibly required for this convergence.

Solution

Radius of Convergence: To test the convergence of the power-series solution, we require

$$1 > \lim_{k\to\infty}\left|\frac{c_{k+1}x^{k+1+s}}{c_k x^{k+s}}\right| = \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right||x| = \lim_{k\to\infty}\left|\frac{C + B(k{+}s)}{(k{+}1{+}s)(k{+}s{-}A)}\right||x|, \tag{39}$$

from which we obtain

$$|x| < R := \lim_{k\to\infty}\left|\frac{(k{+}1{+}s)(k{+}s{-}A)}{C + B(k{+}s)}\right| = \lim_{k\to\infty}\left|\frac{(k{+}1)(k{-}A)}{C + Bk}\right| = \infty, \qquad s = 0; \tag{40}$$

$$= \lim_{k\to\infty}\left|\frac{(k{+}2{+}A)(k{+}1)}{C + B(k{+}A{+}1)}\right| = \infty, \quad s = 0 \tag{41}$$

For generic (non-special, see below) choices of A, B, C, both choices of s imply infinite radius of (absolute) convergence.

Conditions: From (38), we see that:

1. If A is a positive integer, the computation of $c^{(0)}_{A+1}$ in (38) fails as the r.h.s. diverges, $\sim \frac{1}{0}$:

$$c^{(0)}_{A+1} = \frac{C + B(A)}{(A+1)(A-A)}\, c^{(0)}_{A} \sim \frac{1}{0}\, c^{(0)}_{A}. \tag{42}$$

   ...unless also $C = -AB$, so that the r.h.s. acquires the $\sim \frac{0}{0}$ indeterminate form as $k \to A$. In that doubly-special case, the original recursion relation (38) simplifies:

$$c^{(0)}_{k+1} = \frac{-AB + B(k)}{(k+1)(k-A)}\, c^{(0)}_{k} = \frac{B\cancel{(k-A)}}{(k+1)\cancel{(k-A)}}\, c^{(0)}_{k} = \frac{B}{A+1}\, c^{(0)}_{k}, \tag{43}$$

   so that its iteration proceeds unhindered.

2. If $A = -2, -3, -4, \ldots$, the computation of $c^{(A+1)}_{-A-1}$ in (38) fails as the r.h.s. diverges, $\sim \frac{1}{0}$:

$$c^{(A+1)}_{-A-1} = \frac{C + B[(-A-2)+A+1]}{[(-A-2)+2+A][(-A-2)+1]}\, c^{(A+1)}_{-A-2} \sim \frac{1}{0}\, c^{(A+1)}_{-A-2}. \tag{44}$$

   ...unless also $C = B$, so that the r.h.s. acquires the $\sim \frac{0}{0}$ indeterminate form as $k \to -A-2$. In that doubly-special case, the original recursion relation (38) simplifies:

$$c^{(0)}_{k+1} = \frac{B + B(k+A+1)}{(k+2+A)(k+1)}\, c^{(A+1)}_{k} = \frac{B\cancel{(k+A+2)}}{\cancel{(k+2+A)}(k+1)}\, c^{(A+1)}_{k} = \frac{B}{A+1}\, c^{(A+1)}_{k}, \tag{45}$$

   so that its iteration proceeds unhindered.

[2 pt] ***e.*** Determine for which values of A, B, C does this produce two linearly independent solutions.

Solution

It is the differences between the two recursion relations (38) that insures the distinctness of the two so-defined series. For these two recursion relations to become the same, it would have to be the case that

$$\frac{C + B(k)}{(k+1)(k-A)} = \frac{C + B(k+A+1)}{(k+2+A)(k+1)}, \quad \text{i.e.,} \quad (C + Bk)(k+2+A) = [C + B(k+A+1)](k-A) \tag{46}$$

for all values of k. Expanding and reordering, this produces the condition

$$[-(A+1)B]\, k - [(A+1)(AB+2C)] = 0. \tag{47}$$

For this to be true for all k (*different powers of which are linearly independent!*), it must be the case that the two square-bracketed coefficients vanish separately:

$$(A+1)B = 0 = (A+1)(AB+2C) \tag{48}$$

which happens if either $A = -1$, B, C arbitrary, or $A \neq -1$ but $B = 0 = C$.

Notice that the condition (47) was supposed to be quadratic in k, producing three independent conditions on A, B, C, but the quadratic term cancelled out.

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*. Academic Press, 7 ed., 2012.