Vec-Tac-Toe Calculus

(Vector Tic-Tac-Toe of Calculus)

The following assumes very little prerequisite knowledge except partial derivatives and elementary integration, but *does assume* that you learn fast.

1 Vectors

We need to distinguish *vectors* from *scalars*: the latter are quantities that only have a sense of magnitude (temperature, mass, pressure...), whereas the former have both a sense of magnitude and of *direction* (velocity, force, acceleration...). Notationally, we'll distinguish vectors from scalars by an arrow atop the symbol.

The following is then *assumed* to hold:

1. Scalars form a *(ground) field*, $a, b \in \mathbb{Q}, \mathbb{R}, \mathbb{C}$: they can be added and multiplied as usual, and multiplication distributed over addition; the only scalar that does not have a multiplicative inverse is 0—the additive "unit" ($a + 0 = 0 + a = a$).
2. Any *linear combination*, *i.e.*, algebraic sum, $a\vec{A}+b\vec{B}$, of any two vectors, $\vec{A}, \vec{B}$, is again a vector, for any $a, b \in \mathbb{R}$, and the multiplication of vectors by scalars has to be distributive with respect to addition of both scalars and of vectors:

$$(a+b)\vec{A}+(a+b)\vec{B} = (a+b)(\vec{A}+\vec{B}) = a(\vec{A}+\vec{B}) + b(\vec{A}+\vec{B}) \ . \tag{1.1}$$

In general, it behooves us to distinguish types of vectors (specified in a *coordinate system* where $i, j = 1, 2, 3$ label/count the independent coordinates[1]):

1. **Covariant:** such as $(\vec{\nabla}')_i = \dfrac{\partial}{\partial \xi^i} = \dfrac{\partial x^j}{\partial \xi^i}\dfrac{\partial}{\partial x^j} = \dfrac{\partial x^j}{\partial \xi^i}(\vec{\nabla})_j$, and
2. **Contravariant:** such as $(\mathrm{d}\vec{r}\,')^i = \mathrm{d}\xi^i = \dfrac{\partial \xi^i}{\partial x^j}\mathrm{d}x^j = \dfrac{\partial \xi^i}{\partial x^j}(\mathrm{d}\vec{r})^j$,

with respect to *any* change of variables/coordinates:

$$x^i \to \xi^i = \xi^i(x^1, x^2, \ldots), \quad \text{which must be invertible } x^i = x^i(\xi^1, \xi^2, \ldots). \tag{1.2}$$

We adopt the *Einstein summation convention*: whenever an index appears exactly twice, precisely once as a subscript (as in $\mathrm{d}x^j$) and precisely once as a superscript (as in $\frac{\partial}{\partial x^j}$), the summation symbol ($\sum_{j=1}^{3}$) will be omitted, but implied. Note that the index j is used here to indicate the summation given explicitly within the parentheses) and is thus fully "used up": it's values are not free to be chosen at will; they are akin to a variable that has been integrated over and replaced with the limits of integration. Such indices are called *dummy*, and they may be renamed at will—as long as that will incur no confusion.

[1] And, our manifesto: no concrete choice of coordinates should ever affect "the physics" of a problem—so we must be able to freely transform from any one to any other coordinate system!

Note that co- and contra-variant vectors transform under a change of coordinates by being multiplied by mutually *inverse* transformation matrices, $\frac{\partial x^j}{\partial \xi^i}$ and $\frac{\partial \xi^i}{\partial x^j}$, respectively:

$$\frac{\partial x^j}{\partial \xi^i}\frac{\partial \xi^i}{\partial x^k} = \underbrace{\frac{\partial x^j}{\partial \xi^i}\frac{\partial}{\partial x^k}\xi^i(x) = \frac{\partial}{\partial \xi^i}\xi^i(x)}_{\text{chain-rule, in reverse}} = \frac{\partial x^j}{\partial x^k} = \delta_i^j := \begin{cases} 1 & \text{if } i=j, \quad \text{e.g. } \frac{\partial x^1}{\partial x^1} \equiv 1; \\ 0 & \text{if } i \neq j, \quad \text{e.g. } \frac{\partial x^2}{\partial x^1} \equiv 0. \end{cases} \tag{1.3}$$

Exceptionally however, when ξ^i are obtained from x^j by *rotation*, $\frac{\partial x^j}{\partial \xi^i} = \frac{\partial \xi^i}{\partial x^j}$ — that is, the *matrix* of $\frac{\partial \xi}{\partial x}$-derivatives is the *transpose* of the *matrix* of $\frac{\partial x}{\partial \xi}$-derivatives (verify this).

OK, consider a 2-dimensional rotation:

$$\begin{aligned} \xi^1 &= \cos(\alpha)\,x^1 + \sin(\alpha)\,x^2 \\ \xi^2 &= -\sin(\alpha)\,x^1 + \cos(\alpha)\,x^2 \end{aligned} \qquad \textit{so that} \qquad \begin{aligned} x^1 &= \cos(\alpha)\,\xi^1 - \sin(\alpha)\,\xi^2 \\ x^2 &= \sin(\alpha)\,\xi^1 + \cos(\alpha)\,\xi^2 \end{aligned} \tag{1.4}$$

Then,

$$\frac{\partial \xi^1}{\partial x^1} = \cos\alpha, \quad \frac{\partial \xi^1}{\partial x^2} = \sin\alpha, \quad \frac{\partial \xi^2}{\partial x^1} = -\sin\alpha, \quad \frac{\partial \xi^2}{\partial x^2} = \cos\alpha; \tag{1.5}$$

$$\frac{\partial x^1}{\partial \xi^1} = \cos\alpha, \quad \frac{\partial x^1}{\partial \xi^2} = -\sin\alpha, \quad \frac{\partial x^2}{\partial \xi^1} = \sin\alpha, \quad \frac{\partial x^2}{\partial \xi^2} = \cos\alpha \tag{1.6}$$

So, indeed:

$$\begin{aligned} \frac{\partial \xi^1}{\partial x^1} &= \cos(\alpha) = \frac{\partial x^1}{\partial \xi^1}, & \frac{\partial \xi^1}{\partial x^2} &= \sin(\alpha) = \frac{\partial x^2}{\partial \xi^1}, \\ \frac{\partial \xi^2}{\partial x^1} &= -\sin(\alpha) = \frac{\partial x^1}{\partial \xi^2}, & \frac{\partial \xi^2}{\partial x^2} &= \cos(\alpha) = \frac{\partial x^2}{\partial \xi^2}, \end{aligned} \qquad \frac{\partial x^j}{\partial \xi^i} = \frac{\partial \xi^i}{\partial x^j}. \tag{1.7}$$

Most importantly: $\frac{\partial x^j}{\partial \xi^i} \neq \left(\frac{\partial \xi^i}{\partial x^j}\right)^{-1}$, *as you might be tempted to guess, guided by the 1-variable calculus fact that* $\frac{\mathrm{d}\xi}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-1}$. *Instead:*

$$\begin{bmatrix} \frac{\partial \xi^1}{\partial x^1} & \frac{\partial \xi^2}{\partial x^1} \\ \frac{\partial \xi^1}{\partial x^2} & \frac{\partial \xi^2}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^1} \\ \frac{\partial x^1}{\partial \xi^2} & \frac{\partial x^2}{\partial \xi^2} \end{bmatrix}^{-1} : \qquad \frac{\partial \xi^i}{\partial x^j}\frac{\partial x^k}{\partial \xi^i} = \underbrace{\frac{\partial \xi^i}{\partial x^j}\frac{\partial}{\partial \xi^i}x^k(\xi) = \frac{\partial}{\partial x^j}x^k}_{\text{chain-rule, in reverse}} = \frac{\partial x^k}{\partial x^j} = \delta_j^k = [\mathbb{1}]_j^k. \tag{1.8}$$

Since the *most general* coordinate transformation of a Cartesian coordinate system into another must be a combination of a constant translation and a uniform and constant rotation

$$\begin{bmatrix} \xi^1(x) \\ \xi^2(x) \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\alpha)\cdots & \sin(\alpha)\cdots & \cdots \\ -\sin(\alpha)\cdots & \cos(\alpha)\cdots & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{\left[\frac{\partial(\xi^1,\xi^2,\cdots)}{\partial(x^1,x^2,\cdots)}\right] \ \leftarrow \text{ a rotation matrix}} \begin{bmatrix} x^1 \\ x^2 \\ \vdots \end{bmatrix} + \begin{bmatrix} c^1 \\ c^2 \\ \vdots \end{bmatrix}, \tag{1.9}$$

and *all rotation matrices* are "*orthogonal*," $\mathbb{O}^T = \mathbb{O}^{-1}$, covariant and contravariant vectors are indistinguishable—*as long as we restrict to Cartesian systems only* (which Arfken, Webber & Harris do, in this chapter), and we need not distinguish between sub- and super-scripts.

To specify vectors, we may *choose* a particular coordinate system, *i.e.*, a system of reference with a (complete) set of possible independent unit-vectors[2], $\{\hat{\mathrm{e}}_i, \forall i\}$, so:

$$\vec{A} \;:=\; A^i\,\hat{\mathrm{e}}_i\;(:= A^1\,\hat{\mathrm{e}}_1 + A^2\,\hat{\mathrm{e}}_2 + A^3\,\hat{\mathrm{e}}_3)\;, \qquad A^i \in \mathbb{Q}, \mathbb{R}, \mathbb{C}\;. \tag{1.10}$$

Cartesian coordinates will be used throughout for which $\hat{\mathrm{e}}_x, \hat{\mathrm{e}}_y, \hat{\mathrm{e}}_z$ are *constant* unit-vectors in the "usual" x, y, z (forward, to the left, upward) directions of the (real, 3-dimensional vector) space, respectively. We will also write $x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$ and $\hat{\mathrm{e}}_1 \equiv \hat{\mathrm{e}}_x, \hat{\mathrm{e}}_2 \equiv \hat{\mathrm{e}}_y, \hat{\mathrm{e}}_3 \equiv \hat{\mathrm{e}}_z$.

2 Products (Algebra)

We define how to multiply unit vectors (in two different ways), and then use that for general vectors = linear combinations of those unit vectors.

Given the $\hat{\mathrm{e}}_i$'s, we can form a *scalar product*[3]:

$$\hat{\mathrm{e}}_i\cdot\hat{\mathrm{e}}_j \;=\; \delta_{ij} \;:=\; \begin{cases} 1 & \text{if } i = j\;, \\ 0 & \text{if } i \neq j\;, \end{cases} \tag{2.1}$$

since every $\hat{\mathrm{e}}_i$ is orthogonal to every *other* $\hat{\mathrm{e}}_j$, and the length, $\sqrt{(\hat{\mathrm{e}}_i\cdot\hat{\mathrm{e}}_i)} = 1$ for each $\hat{\mathrm{e}}_i$, as they are *unit* vectors. Notice that the value of $\hat{\mathrm{e}}_i\cdot\hat{\mathrm{e}}_i$ is a *scalar*, so the assigned values (1 and 0) make sense. We will also need

$$\delta^{ij} \;:=\; \begin{cases} 1 & \text{if } i = j\;, \\ 0 & \text{if } i \neq j\;, \end{cases} \tag{2.2}$$

The σ-symbols appearing in (1.3), (2.1) and (2.2) are all the same "*Kronecker symbol*," defined in (1.3) by the simple fact that the different coordinates within any coordinate system are independent of each other.

We can also form a *vector product*:

$$\hat{\mathrm{e}}_i \times \hat{\mathrm{e}}_j = \sum_{k=1}^{3} \epsilon_{ijk}\hat{\mathrm{e}}_k\;, \quad \text{where} \quad \epsilon_{ijk} \;:=\; \begin{cases} +1 & \text{if } i, j, k \text{ an even permutation of 1,2,3,} \\ -1 & \text{if } i, j, k \text{ an odd permutation of 1,2,3,} \\ \;\;0 & \text{otherwise,} \end{cases} \tag{2.3}$$

since its result is a vector. The "*Levi-Civita symbol*" ϵ_{ijk} is simply the pattern of $\pm 1, 0$ that defines the expansion of a determinant of a 3×3 matrix: This expansion contains no product of two

[2] See below for the definition of unit vectors.

[3] Recall: using Cartesian coordinates only means that we need not worry about subscripts (indicating covariant components) vs. superscripts (indicating contravariant components) of vectors.

elements from the same row or same column, so $\epsilon_{112} \equiv 0$ for example; the non-zero terms acquire alternating signs as we pass from one column to the next, and one row to the next. Thus:

$$\det \begin{bmatrix} a_{11} & a_{12} & 13 \\ a_{21} & a_{22} & 23 \\ a_{31} & a_{32} & 33 \end{bmatrix} = \underbrace{\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\, a_{i1}\, a_{j2}\, a_{k3}}_{\text{exp. by rows}} = \underbrace{\sum_{i,j,k=1}^{3} \epsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}}_{\text{exp. by columns}}$$
$$= \sum_{i,j,k=1}^{3}\sum_{l,m,n=1}^{3} \frac{1}{3!}\,\epsilon_{ijk}\,\epsilon_{lmn}\, a_{il}\, a_{jm}\, a_{kn}, \tag{2.4}$$

the last of which looks like a *very redundant* way of writing out the same thing: there are $(3{\cdot}3{\cdot}3){\cdot}(3{\cdot}3{\cdot}3) = 729$ terms in that expansion, most of which are zero owing to the zeros in the ϵ's (2.3). This last formula however makes evident all the (anti)symmetries in the expansion, which is the quality that can make it useful at times. In particular:

$$\epsilon_{ijk} = 1 = \epsilon_{jki} = \epsilon_{kij}, \quad = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}. \tag{2.5}$$

Notice that, according to the *strict* Einstein summation convention, the k-summation *must* be explicitly written in (2.3) and (2.4), since the *strict* Einstein summation convention does not apply: both occurrences of the index k have it as a subscript! Alternatively, we could have introduced the unit vectors:

$$\hat{\mathrm{e}}^i := \delta^{ij}\hat{\mathrm{e}}_j\ , \tag{2.6}$$

and then write

$$\hat{\mathrm{e}}_i \times \hat{\mathrm{e}}_j = \epsilon_{ijk}\,\hat{\mathrm{e}}^k \ = \ \epsilon_{ijk}\,\delta^{kl}\,\hat{\mathrm{e}}_l \ = \ \epsilon_{ij}{}^{l}\,\hat{\mathrm{e}}_l\ . \tag{2.7}$$

Using (2.2), we can now also define:

$$\epsilon_{ij}{}^{k} := \epsilon_{ijn}\,\delta^{nk}\ , \quad \epsilon_{i}{}^{jk} := \epsilon_{imn}\,\delta^{mj}\,\delta^{nk}\ , \quad \epsilon^{ijk} := \epsilon_{lmn}\,\delta^{li}\,\delta^{mj}\,\delta^{nk}\ . \tag{2.8}$$

That is, the Kronecker symbols, δ_{ij} and δ^{ij}, may be used (in Cartesian coordinate systems only!) to raise and lower indices.

Furthermore, notice that in both equations, the indices i, j are *free*: their values are free to be chosen at will; they are not being summed over, but they are akin to a variable of a function. If renamed, free indices must be consistently renamed throughout the expression and equation.

The Kronecker symbol, δ_{ij}, and the Levi-Civita symbol, ϵ_{ijk}, are related through the identity:

$$\epsilon_{ijk}\epsilon_{lmn} \equiv \delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}\ . \tag{2.9}$$

From this, it follows that

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{lmk} \equiv \sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{klm} \equiv \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\ , \qquad \text{and} \qquad \sum_{j,k=1}^{3} \epsilon_{ijk}\epsilon_{ljk} \equiv \delta_{il}\ . \tag{2.10}$$

It then also follows (work this out yourself!) that:

$$\hat{\mathrm{e}}_x \times \hat{\mathrm{e}}_y = \hat{\mathrm{e}}_z \qquad \hat{\mathrm{e}}_y \times \hat{\mathrm{e}}_z = \hat{\mathrm{e}}_x \qquad \hat{\mathrm{e}}_z \times \hat{\mathrm{e}}_x = \hat{\mathrm{e}}_y\ , \tag{2.11}$$

(notice the cyclicity of the second batch of relations).

For general vectors then:

$$\vec{A}\cdot\vec{B} = (A^i\,\hat{\mathrm{e}}_i)\cdot(B^j\,\hat{\mathrm{e}}_j) = A^i\,B^j(\hat{\mathrm{e}}_i\cdot\hat{\mathrm{e}}_j) = A^i\,B^j\,\delta_{ij} = \sum_{i=1} A^i\,B^i \;=\; A^i\,B_i \;=\; A_i\,B^i\;, \tag{2.12}$$

and

$$\begin{aligned}\vec{A}\times\vec{B} &= (A^i\,\hat{\mathrm{e}}_i)\times(B^j\,\hat{\mathrm{e}}_j) = A^i\,B^j(\hat{\mathrm{e}}_i\times\hat{\mathrm{e}}_j) = A^i\,B^j\sum_{k=1}^{3}\epsilon_{ijk}\,\hat{\mathrm{e}}_k = \sum_{k=1}(A^i\,B^j\,\epsilon_{ijk})\,\hat{\mathrm{e}}_k\;,\\ &= A^i\,B^j\,\epsilon_{ijk}\,\delta^{kl}\,\hat{\mathrm{e}}_l \;=\; A^i\,B^j\,\epsilon_{ijk}\,\hat{\mathrm{e}}^k \;=\; A^i\,B^j\,\epsilon_{ij}{}^k\,\hat{\mathrm{e}}_k\;.\end{aligned} \tag{2.13}$$

We may also write $(\vec{A}\times\vec{B})_k = A^iB^j\epsilon_{ijk}$ for the k^{th} component of the vector $\vec{A}\times\vec{B}$.

Let's examine these two calculations:

1. It should be clear that

$$\vec{A}\cdot\vec{A} = \sum_{i=1}^{3}(A^i)^2 \geq 0\;, \qquad \text{since} \qquad A^i\in\mathbb{R},\; i=1,2,3\;. \tag{2.14}$$

Thus, we define the length of $\vec{A}$ to be

$$|\vec{A}| := \sqrt{\vec{A}\cdot\vec{A}}\;, \tag{2.15}$$

and it is then clear that $|\hat{\mathrm{e}}_i| = 1$, $i = 1,2,3$, so that $\hat{\mathrm{e}}_i$ are indeed unit vectors—vetors of unit length.

2. It should also be clear that

$$\vec{A}\cdot\vec{B} = +\vec{B}\cdot\vec{A}\;, \qquad \text{but} \qquad \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}\;. \tag{2.16}$$

3. Also (since ϵ_{ijk} encodes the expansion of a 3×3 determinant):

$$\vec{A}\cdot\big(\vec{B}\times\vec{C}\big) \;=\; \vec{A}\cdot\begin{vmatrix}\hat{\mathrm{e}}_x & \hat{\mathrm{e}}_y & \hat{\mathrm{e}}_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix} \;=\; \begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}\;, \tag{2.17}$$

and so, as determinants stay the same if rows are permuted cyclicly,

$$\vec{A}\cdot\big(\vec{B}\times\vec{C}\big) \;=\; \vec{B}\cdot\big(\vec{C}\times\vec{A}\big) \;=\; \vec{C}\cdot\big(\vec{A}\times\vec{B}\big)\;. \tag{2.18}$$

Now, careful with those parentheses! For example, $(\vec{A}\cdot\vec{B})\times\vec{C}$ simply makes no sense whatsoever! (The scalar product $(\vec{A}\cdot\vec{B})$ is a *scalar*, so one cannot put it in a vector product with the vector $\vec{C}$.) A little less dramatic, but still important:

$$\vec{A}\times\big(\vec{B}\times\vec{C}\big) \;\neq\; \big(\vec{A}\times\vec{B}\big)\times\vec{C}\;. \tag{2.19}$$

For one thing, the left-hand-side must be perpendicular to $\vec{A}$ (why?), while the right-hand-side must be perpendicular to $\vec{C}$; so, if the two sides were equal, the result would have to be along the unique normal to the plane spanned by $\vec{A}, \vec{C}$—regardless of which way $\vec{B}$ points! Another quick way to see that these cannot be the same (except in very special cases), is to evaluate the magnitudes of the cross products as:

$$|\vec{A}||\vec{B}||\vec{C}| \ \sin\theta_{B,C} \ \sin\theta_{A,BC} \ \neq \ |\vec{A}||\vec{B}||\vec{C}| \ \sin\theta_{A,B} \ \sin\theta_{AB,C} \ , \tag{2.20}$$

where $\theta_{A,B}$ is the angle between $\vec{A}$ and $\vec{B}$, and where $\theta_{AB,C}$ is the angle between $(\vec{A}\times\vec{B})$ and $\vec{C}$. So, it should now be (perhaps a little more) clear that, *in general*

$$\sin\theta_{B,C} \ \sin\theta_{A,BC} \ \neq \ \sin\theta_{A,B} \ \sin\theta_{AB,C} \ , \tag{2.21}$$

although there will be *special cases* where this is true.

3 Derivatives (Calculus 1)

Given that $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are *partial* derivatives with respect to Cartesian coordinates x, y, z, we define the vector-derivative *operator* $\vec{\nabla} := \hat{\mathrm{e}}_x\frac{\partial}{\partial x} + \hat{\mathrm{e}}_y\frac{\partial}{\partial y} + \hat{\mathrm{e}}_z\frac{\partial}{\partial z}$.

Given the scalar function $f = f(x,y,z)$ and the vector function

$$\vec{A} = \vec{A}(x,y,z) = A^i\hat{\mathrm{e}}_i = A_x\hat{\mathrm{e}}_x + A_y\hat{\mathrm{e}}_y + A_z\hat{\mathrm{e}}_z \ , \tag{3.1}$$

the possible derivatives are:

1. $\vec{\nabla}f$; there is only one way to multiply the vector $\vec{\nabla}$ with a scalar function, so:

$$\mathbf{grad}(f) \equiv \vec{\nabla}f \ := \ \sum_{i=1}^{3}\hat{\mathrm{e}}_i\frac{\partial f}{\partial x^i} \ = \ \hat{\mathrm{e}}_x\frac{\partial f}{\partial x} + \hat{\mathrm{e}}_y\frac{\partial f}{\partial y} + \hat{\mathrm{e}}_y\frac{\partial f}{\partial y} \ , \tag{3.2}$$

the result of which is a vector function called the **gradient** of f.

There are two ways to multiply the vector $\vec{\nabla}$ with a vector function, so...

2. $\vec{\nabla}\cdot\vec{A}$ uses the *scalar product*:

$$\mathbf{div}(\vec{A}) \equiv \vec{\nabla}\cdot\vec{A} \ := \ \frac{\partial A^i}{\partial x^i} \ = \ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \ , \tag{3.3}$$

the result of which is a scalar function called the **divergence** of $\vec{A}$.

3. $\vec{\nabla}\times\vec{A}$ uses the *vector product*:

$$\begin{aligned} \mathbf{curl}(\vec{A}) \equiv \mathbf{rot}(\vec{A}) = (\vec{\nabla}\times\vec{A}) \ &:= \ \sum_{j,k=1}^{3}\epsilon_{ijk}\frac{\partial A^i}{\partial x^j}\hat{\mathrm{e}}_k = \begin{vmatrix} \hat{\mathrm{e}}_x & \hat{\mathrm{e}}_y & \hat{\mathrm{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \ , \\ &= \ \hat{\mathrm{e}}_x\Big(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\Big) + \hat{\mathrm{e}}_y\Big(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\Big) + \hat{\mathrm{e}}_z\Big(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\Big) \ . \end{aligned} \tag{3.4}$$

the result of which is a vector function called the **curl** (a.k.a. "**rot**or") of $\vec{A}$.

Taking the results of the preceding *first* order derivatives, we can form second order derivatives by iteration:

4. $(\vec{\nabla}\cdot(\vec{\nabla}f)) = (\vec{\nabla}^2 f)$:

$$\vec{\nabla}^2 f \;=\; \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x^i \partial x^i} \;=\; \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\;, \tag{3.5}$$

the result of which is a scalar function called the **Laplacian** of f. Note that by the same token,

$$\vec{\nabla}^2 \vec{A} \;=\; \sum_{i=1}^{3} \frac{\partial^2 \vec{A}}{\partial x^i \partial x^i} \;=\; \sum_{i=1}^{3} \frac{\partial^2 (A^j\,\hat{\mathrm{e}}_j)}{\partial x^i \partial x^i} \;=\; \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2} \tag{3.6}$$

is a vector function called the **Laplacian** of $\vec{A}$—notice that the derivatives do act on unit vectors $\hat{\mathrm{e}}_j$. In Cartesian coordinates, the $\hat{\mathrm{e}}_j$ are constant, but not so in any other coordinate system! That is, *only in Cartesian coordinates* is the Laplacian of a vector equal to the vector of Laplacians of its vector components.

5. $\vec{\nabla}\times(\vec{\nabla}f)$:

$$\vec{\nabla}\times(\vec{\nabla}f) \;=\; \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial^2 f}{\partial x^i \partial x^j}\,\hat{\mathrm{e}}_k \;=\; \epsilon^{ijk} \frac{\partial^2 f}{\partial x^i \partial x^j}\,\hat{\mathrm{e}}_k \;=\; \begin{vmatrix} \hat{\mathrm{e}}_x & \hat{\mathrm{e}}_y & \hat{\mathrm{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}\;, \tag{3.7}$$

vanishes for any f (easily seen on straightforward expansion, *and* noting that partial derivatives commute by definition); in the index notation, this should be clear since

$$\begin{aligned} \sum_{i,j=1}^{3} \epsilon_{ijk} \frac{\partial^2}{\partial x^i \partial x^j} &= \sum_{i,j=1}^{3} (-\epsilon_{jik}) \frac{\partial^2}{\partial x^i \partial x^j} = -\sum_{j',i'=1}^{3} \epsilon_{i'j'k} \frac{\partial^2}{\partial x^{j'} \partial x^{i'}} \\ &= -\sum_{j',i'=1}^{3} \epsilon_{i'j'k} \frac{\partial^2}{\partial x^{i'} \partial x^{j'}} = -\sum_{j,i=1}^{3} \epsilon_{ijk} \frac{\partial^2}{\partial x^i \partial x^j}\;, \end{aligned} \tag{3.8}$$

where in the first equality we used that $\epsilon_{ijk} = -\epsilon_{jik}$, obtained the third expression by renaming $i,j \to j',i'$, then using that the order of taking partial derivatives does not matter, $\frac{\partial^2}{\partial x^{j'} \partial x^{i'}} = \frac{\partial^2}{\partial x^{i'} \partial x^{j'}}$, and finally we dropped the primes from the names of the dummy indices. This proves that the 2nd order derivative operator in Eq. (3.7) equals its own negative—and so must be zero.

6. $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$:

$$\begin{aligned} \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) &= \sum_{i=1}^{3} \hat{\mathrm{e}}_i \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial x^i} = \hat{\mathrm{e}}^i \frac{\partial}{\partial x^i}\Big(\frac{\partial A^j}{\partial x^j}\Big) \\ &= \hat{\mathrm{e}}_x \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial x} + \hat{\mathrm{e}}_y \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial y} + \hat{\mathrm{e}}_y \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial y}\;, \\ &= \hat{\mathrm{e}}_x \frac{\partial(\frac{\partial A^1}{\partial x} + \frac{\partial A^2}{\partial y} + \frac{\partial A^3}{\partial z})}{\partial x} + \hat{\mathrm{e}}_y \frac{\partial(\frac{\partial A^1}{\partial x} + \frac{\partial A^2}{\partial y} + \frac{\partial A^3}{\partial z})}{\partial y} + \hat{\mathrm{e}}_y \frac{\partial(\frac{\partial A^1}{\partial x} + \frac{\partial A^2}{\partial y} + \frac{\partial A^3}{\partial z})}{\partial y}\;. \end{aligned} \tag{3.9}$$

7. $(\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}))$:

$$(\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})) \;=\; \epsilon^{ijk}\frac{\partial^2 A_k}{\partial x^i \partial x^j} \;=\; \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}\,, \tag{3.10}$$

vanishes for any $\vec{A}$ (easily seen on straightforward expansion, *and* noting that partial derivatives commute by definition); in the index notation, it is clear that this vanishes for the same reason that (3.8) does.

8. $(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))$:

$$(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) \;=\; (\vec{\nabla}(\vec{\nabla}\cdot\vec{A})) - (\vec{\nabla}^2\vec{A})\,. \tag{3.11}$$

This a straightforward application of the "BAC−CAB" rule. However, *one must* keep the object of differentiation, $\vec{A}$ always on the far right so the derivatives would keep on acting on it as they do in the original expression. In the index notation, this merits a little more detail:

$$(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) \;=\; \sum_{i,j,k=1}^{3}\epsilon^{ijk}\frac{\partial}{\partial x^i}\big(\vec{\nabla}\times\vec{A}\big)_j\,\hat{\mathrm{e}}_k = \sum_{i,j,k=1}^{3}\epsilon^{jki}\frac{\partial}{\partial x^i}\Big(\sum_{l=1}^{3}\epsilon^{lmj}\frac{\partial}{\partial x^l}A_m\Big)\,\hat{\mathrm{e}}_k \tag{3.12a}$$

$$= \sum_{i,k,l=1}^{3}\Big(\sum_{j=1}^{3}\epsilon^{jki}\epsilon^{lmj}\Big)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^l}A_m\,\hat{\mathrm{e}}_k = \sum_{i,k,l=1}^{3}\Big(\delta^{kl}\delta^{im} - \delta^{km}\delta^{il}\Big)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^l}A_m\,\hat{\mathrm{e}}_k \tag{3.12b}$$

$$= \sum_{i,k,l=1}^{3}\delta^{kl}\delta^{im}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^l}A_m\,\hat{\mathrm{e}}_k - \sum_{i,k,l=1}^{3}\delta^{km}\delta^{il}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^l}A_m\,\hat{\mathrm{e}}_k \tag{3.12c}$$

$$= \sum_{k=1}^{3}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^l}A^i\,\hat{\mathrm{e}}^l - \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}A^k\,\hat{\mathrm{e}}_k = \sum_{k=1}^{3}\hat{\mathrm{e}}^l\,\frac{\partial}{\partial x^l}\Big(\frac{\partial A^i}{\partial x^i}\Big) - \sum_{i=1}^{3}\frac{\partial^2(A^k\,\hat{\mathrm{e}}_k)}{\partial x^i \partial x^i} \tag{3.12d}$$

$$= \vec{\nabla}\big(\vec{\nabla}\cdot\vec{A}\big) - \vec{\nabla}^2\,\vec{A}\,. \tag{3.12e}$$

Here, in the first row, we iteratively wrote out the curls in the index notation (and substituted $\epsilon_{ijk} = \epsilon_{jki}$); in the second, we regrouped the factors in the summands and used the first of the identities (2.10); in the third, we simply wrote the two summands separately; in the fourth, we used that, *e.g.*, $\delta_{kl} = 0$ unless in fact $l = k$, whereby dropping the zero terms in which $l \neq k$ and so also the independent k-summation (and the same for the m-summation), and then we regrouped the remaining terms; in the fifth row we just rewrote the result in the arrow-notation.

Yes, the expansion (3.12) is *looong and teeedious*, but it is also true that it is herein expanded *extremely meticulously*—to the last detail; practice makes skipping (*correctly*, without missing a sign, a factor, an index,…) over such details routine.

4 Integration (Calculus 2)

As for integration, we'll have to deal with line- surface- and volume-integration. Any recipe for integration calls for three ingredients:

1. an *integration domain*, the region over which the relevant variables are varied;

2. an *integration element*, the infinitesimal portion of the integration domain;
3. an *integrand*, the function to be integrated.

Thereafter, integration means “sample the integrand point-by-point over the domain, *multiply* each sample with the integration element, add up the results”.

4.1 1D

Line- or contour-integration is 1-dimensional, the domain is a line (curve, contour), often denoted C. The integration- or line-element is an *infinitesimal vector* along the line, denoted $\mathrm{d}\vec{r}$; in Cartesian coordinates, $\mathrm{d}\vec{r} = \hat{\mathrm{e}}_x \mathrm{d}x + \hat{\mathrm{e}}_y \mathrm{d}y + \hat{\mathrm{e}}_z \mathrm{d}z$. Given a scalar or a vector integrand, there are three possible line-integrals:

$$\int_C \mathrm{d}\vec{r}\, f\,, \qquad \int_C \mathrm{d}\vec{r}\cdot\vec{A}\,, \quad \text{and} \quad \int_C \mathrm{d}\vec{r}\times\vec{A}\,, \tag{4.1}$$

the results of which are: a vector, a scalar and a vector, respectively.

4.2 2D

Area- or surface-integration is 2-dimensional, the domain is a surface, often denoted S. The integration,- area- or surface-element is an *infinitesimal vector* perpendicular to the surface (the direction is conventional), denoted “$\mathrm{d}^2\vec{r}$”—the “d^2” indicates that this is a square of differentials. Recalling the geometrical interpretation of the vector product, one example of a surface element (for a surface in the x,y-plane) is, using (2.3),

$$\mathrm{d}^2\vec{r} \;=\; (\hat{\mathrm{e}}_x \mathrm{d}x)\times(\hat{\mathrm{e}}_y \mathrm{d}y) \;=\; \hat{\mathrm{e}}_z\, \mathrm{d}x\, \mathrm{d}y\;; \tag{4.2}$$

more generally,

$$\mathrm{d}^2\vec{r} := \sum_{k=1}^{3}\Big(\sum_{i,j=1}^{3}\epsilon_{ijk}\,\mathrm{d}x_i\,\mathrm{d}x_j\Big)\,\hat{\mathrm{e}}_k = \hat{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z + \hat{\mathrm{e}}_y\,\mathrm{d}z\,\mathrm{d}x + \hat{\mathrm{e}}_z\,\mathrm{d}x\,\mathrm{d}y. \tag{4.3}$$

Notice that, true to form, $\mathrm{d}^2\vec{a}$ does *not* include the literal square of any single differential: for example, $\mathrm{d}^2\vec{a}$ does not include $(\mathrm{d}x)^2$, which is (as standard in calculus) still considered vanishing as compared to $(\mathrm{d}x)\cdot$(anything else)—including as compared to $(\mathrm{d}x)\cdot(\mathrm{d}y)$.

BTW, this *heuristic rule* of calculus follows from the fact that $\epsilon_{iik} \equiv 0$. It also shows that the differentials are *not* multiplied as ordinary numbers, commutatively:

$$\mathrm{d}^2\vec{r} = \hat{\mathrm{e}}_x\,\mathrm{d}y\,\mathrm{d}z + \cdots = -\hat{\mathrm{e}}_x\,\mathrm{d}z\,\mathrm{d}y + \ldots \tag{4.4}$$

This is easy to verify by performing a change of variables $(x,y,z)\to(x,z,y)$, for which the Jacobian of the transformation (the determinant of the transformation matrix $[\frac{\partial\xi^i}{\partial x^j}]$ in $\mathrm{d}\xi^i = \frac{\partial\xi^i}{\partial x^j}\mathrm{d}x^j$) in the transformation $(x,y,z)\to(\xi,\eta,\zeta)=(x,z,y)$ equals to -1.

Why haven’t you noticed this before?!
Usually, we choose and fix the orientation of $\mathrm{d}^2\vec{r}$, and then really compute with $|\mathrm{d}^2\vec{r}|$.

Given a scalar or a vector integrand, there are three possible surface-integrals:

$$\int_S \mathrm{d}^2\vec{r}\; f\;, \qquad \int_S \mathrm{d}^2\vec{r}\cdot\vec{A}\;, \quad \text{and} \quad \int_S \mathrm{d}^2\vec{r}\times\vec{A}\;, \tag{4.5}$$

the results of which are: a vector, a scalar and a vector, respectively.

4.3 3D

Volume-integration is 3-dimensional, the domain is a volume, often denoted V. The integration- or volume-element is an *infinitesimal scalar*, denoted "$\mathrm{d}\tau$," or (my preference) "$\mathrm{d}^3\vec{r}$"[4]. Recalling the geometrical interpretation of the mixed product, the volume element in Cartesian coordinates can be written as, using (2.3) and (2.1),

$$\mathrm{d}^3\vec{r} \;=\; (\hat{\mathrm{e}}_x \mathrm{d}x)\cdot\big[(\hat{\mathrm{e}}_y \mathrm{d}y)\times(\hat{\mathrm{e}}_z \mathrm{d}z)\big] \;=\; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;. \tag{4.6}$$

Given a scalar or a vector integral, there are two possible volume-integrals:

$$\int_V \mathrm{d}^3\vec{r}\; f\;, \quad \text{and} \quad \int_V \mathrm{d}^3\vec{r}\;\vec{A}\;, \tag{4.7}$$

the results of which are: a scalar and a vector, respectively. *Usually,* we choose and fix the orientation of $\mathrm{d}^3\vec{r}$, and then really compute with $|\mathrm{d}^3\vec{r}|$.

5 Reduction of Multiple Integrations

The so-called *fundamental theorem of calculus* should be well known for the case of single-variable functions:

$$\int_a^b \mathrm{d}x\,\frac{\mathrm{d}f}{\mathrm{d}x} \;=\; f(b) - f(a)\;. \tag{5.1}$$

While this may look trivial at the moment, let us examine this a little bit further. The left-hand side is an integral of a derivative, while the right-hand-side is simply an evaluation of the function—the integration and the derivative cancel each other in a sense. Note however, that the function is evaluated at the endpoints of the integration interval, and that the endpoints are the boundary points of the interval. Note that we may consider the evaluation of the function at the points a, b as an *integration* over the 0-dimensional domain consisting of the two points and where the point-integration-element is "1"; silly as it may look, this does satisfy the basic integration recipe.

We now wish to find generalizations of this "integration-order-reduction" formula for (at least some cases of) the above integrals.

Generally speaking, a line-integral of a derivative should become some simple evaluation; a surface-integral of a derivative should become some line-integral; a volume integral of a derivative should become some surface-integral. In addition, the lower-order integral should range

[4] Yes, this is very agreeably confusing, as $\mathrm{d}^3\vec{r}$ *looks* like a vector; but it isn't. However, $\mathrm{d}\tau$ looks 1-dimensional and $\mathrm{d}V$ (for volume) may be confusing since V is also used for the electrostatic potential (voltage), the potential energy, and sometimes to denote a vector…

over the boundary of the higher-order integral. We will use the symbol ∂ in place of writing "the boundary of". Next, note that the boundary is closed, *i.e.*, has no boundary of its own: this is trivial for the two endpoints of an interval, since the boundary of these (0-dimensional) endpoints would have to be -1-dimensional. In higher dimensions, this is no longer so trivial, but is nevertheless straightforward: the boundary ∂X of some space X by definition contains all limit points of (Cauchy) sequences within X, which are not in X. If ∂X had a boundary, it would consist of points which are limit points of sequences within ∂X which are not in ∂X. But such limit points would also be limit points of sequences in X, and would therefore have to have been in ∂X in the first place, hence contradiction. Integrals over closed surfaces will be written $\oint$ instead of $\int$.

5.1 One-to-zero

Start with a simple generalization of (5.1):

$$\int_a^b \mathrm{d}\vec{r}\cdot\vec{\nabla} f \;=\; f(b) - f(a) \;=\; f\Big|_a^b , \tag{5.2}$$

where $f\,\big|_a^b$ is read "f evaluated in the limits b, a". This can also be written as

$$\int_C \mathrm{d}\vec{r}\cdot\vec{\nabla} f \;=\; f\Big|_{\partial C} , \tag{5.3}$$

where $\partial C = \{a, b\}$ is the boundary of the curve C connecting the points a, b. Besides the hinted analogy with (5.1), this can be proven rather easily by cutting up the curve C into sufficiently many pieces such that each looks fairly straight, and then applying (5.1) piece by piece (changing integration variables for each piece to be aligned with the particular piece of C). The valuations at the in-between pieces cancel upon summation and only the end-point contributions remain.

Now, given a vector $\vec{A}$, the derivative $\vec{A}\cdot\vec{\nabla}$ is easily seen to be a derivative in the direction of $\vec{A}$, sometimes called the "directional derivative." In the above integral, $\mathrm{d}\vec{r}\cdot\vec{\nabla}$ is the directional derivative in the direction of the line-element, *i.e.*, along the integration contour C.

By the same token, replacing the scalar function above with a vector:

$$\int_C \mathrm{d}\vec{r}\cdot\vec{\nabla}\;\vec{A} \;=\; \int_C \Big(\mathrm{d}x\frac{\partial\vec{A}}{\partial x} + \mathrm{d}y\frac{\partial\vec{A}}{\partial y} + \mathrm{d}z\frac{\partial\vec{A}}{\partial z}\Big) \;=\; \vec{A}\Big|_{\partial C} , \tag{5.4}$$

and a proof can be written pretty much along the same line as above, except it would have to be written out separately for the x, the y and the z component of $\vec{A}$.

Rather more formally, we note that the left-hand-side of (5.3) is a scalar. Dropping the order of integration by one, and the derivative operator, we remain with the scalar function itself. Its values on the two points of ∂C are of course also scalars, so the formula *makes sense*. Furthermore, the formula holds because quite simply, there is multiplying the symbols $\mathrm{d}\vec{r}$, $\vec{\nabla}$ and $\vec{A}$ so as to have the integration cancel the derivative (within the integral) and yet have both sides of such a putative identiy be vectors. Indeed, the alternate candidates for the left-hand-side:

$$\int_C \mathrm{d}\vec{r}\,(\vec{\nabla}\cdot\vec{A}) , \qquad \int_C \mathrm{d}\vec{r}\times(\vec{\nabla}\times\vec{A}) \quad \text{and} \quad \int_C (\mathrm{d}\vec{r}\times\vec{\nabla})\times\vec{A} \tag{5.5}$$

could not have possibly contributed to the left-hand-side of (5.4): in all three, the direction of the resulting vector depends on the direction of $\mathrm{d}\vec{r}$ along contour, whereas the intended right-hand-side, $\vec{A}|_{\partial C}$ is manifestly independent of the path/contour C and only depends on its boundary, i.e., the ordered pair of its end-points. In fact, we may well write this as an equality of operators:

$$\int_C \mathrm{d}\vec{r}\cdot\vec{\nabla}\,[\,\cdots] \;=\; [\,\cdots]\Big|_{\partial C}\,. \tag{5.6}$$

5.2 Two-to-one

Now consider possible formulas relating surface- to line-integrals. For the left-hand-side, the surface integration, we have two vectors, $\mathrm{d}^2\vec{r}$ and $\vec{\nabla}$. Given then a scalar function, we can either form a scalar, $\mathrm{d}^2\vec{r}\cdot(\vec{\nabla}f)$, or a vector, $\mathrm{d}^2\vec{r}\times(\vec{\nabla}f)$. Consider a possible integration-order-reduction formula:

$$\int_S \mathrm{d}^2\vec{r}\cdot\vec{\nabla}f \;\stackrel{?}{=}\; \oint_{\partial S} \mathrm{d}\vec{r}\,f \tag{5.7}$$

cannot possibly be correct, since the left-hand-side is a scalar, and one cannot make a scalar of the right-hand-side. So then try

$$\int_S \mathrm{d}^2\vec{r}\times\vec{\nabla}f \;=\; \oint_{\partial S} \mathrm{d}\vec{r}\,f\,, \tag{5.8}$$

which in fact must be correct, since there is no other way to make a vector from a surface integral of the gradient of the scalar function f on the left-hand-side, and from a line integral of the scalar f on the right hand side. So, we have obtained our first "trivially proven" integration-order-reduction formula.

In (5.8), the domains of integration, S and ∂S, are crucially related: ∂S is the ***oriented*** *boundary of the oriented surface S. For a disc (or rectangle, or any other, however lumpy—but* ***simply covered***—*region, i.e., any region with no self-crossing) in the (x,y)-plane, we typically (often implicitly, without saying so)* ***choose*** *the right-hand coordinate system, and so* ***choose*** *the surface domain of integration, S, to be oriented in the positive z-direction. The boundary of this region then* ***acquires*** *the* ***induced*** *orientation, again typically (implicitly, without saying so) by the right-hand rule: viewed from the positive z-direction, ∂S is oriented counter-clockwise.*

Given a vector function, there is again a unique way to make a surface integral of a derivative of a vector such that the result would be a scalar:

$$\int_S \mathrm{d}^2\vec{r}\cdot(\vec{\nabla}\times\vec{A}) \equiv \int_S (\mathrm{d}^2\vec{r}\times\vec{\nabla})\cdot\vec{A} \;=\; \oint_{\partial S} \mathrm{d}\vec{r}\cdot\vec{A}\,, \tag{5.9}$$

which must again be true, since both sides are unique scalar expressions with the correct number of derivatives and integrations. This, in fact is the well-known "Stokes' formula", or the "fundamental theorem of curls".

The third formula

$$\int_S (\mathrm{d}^2\vec{r}\times\vec{\nabla})\times\vec{A} \;=\; \oint_{\partial S} \mathrm{d}\vec{r}\times\vec{A}\,, \tag{5.10}$$

is harder to pinpoint: while the right-hand-side is the unique vector made from the line element $\mathrm{d}\vec{r}$ and $\vec{A}$, the left-hand-side is not unique. By far the easiest way to prove by substituting $\vec{A} \to \vec{A} \times \vec{P}$ in (5.9), where $\vec{P}$ is a constant vector. Upon using the cyclic property of the mixed product (2.18) (twice in the left-hand-side and once on the right hand side):

$$\vec{P} \cdot \int_S (\mathrm{d}^2\vec{r} \times \vec{\nabla}) \times \vec{A} \;=\; \vec{P} \cdot \oint_{\partial S} \mathrm{d}\vec{r} \times \vec{A}\,. \tag{5.11}$$

So, since the constant vector $\vec{P}$ is arbitrary, this must be true even if $\vec{P}\cdot$ is "stripped off", whence (5.10) follows.

A neat way to remember that the left-hand-side of (5.10) contains $(\mathrm{d}^2\vec{r} \times \vec{\nabla}) \times \vec{A}$ rather than $\mathrm{d}^2\vec{r} \times (\vec{\nabla} \times \vec{A})$ is that the derivative will cancel one of the two integrations in $\int \mathrm{d}^2\vec{r}$, so that the cross-product between $\mathrm{d}^2\vec{r}$ and $\vec{\nabla}$ is the first to be evaluated, and the resulting integration operator is the applied to $\vec{A}$. Indeed, comparing formulae (5.8), (5.9) and (5.10), the general pattern is again "operatorial"—quite akin to (5.6):

$$\int_S \mathrm{d}^2\vec{r} \times \vec{\nabla}\,[\,\cdots] \;=\; \int_{\partial S} \mathrm{d}\vec{r}\,[\,\cdots], \tag{5.12}$$

where now "$[\,\cdots]$" includes not only the integrand function (scalar or vector) but also how it multiplies with the integration operator to its left: ordinary multiplication if $[\,\cdots]$ includes only a scalar function, but either "$\cdot$" or "$\times$" if $[\,\cdots]$ includes a vector function.

5.3 Three-to-two

Finally, we come to volume integration. This time, we have the scalar volume element $\mathrm{d}^3\vec{r}$, the derivative $\vec{\nabla}$ and either a scalar or a vector. Given a scalar f, both left and right-hand-sides of

$$\int_V \mathrm{d}^3\vec{r}\,\big(\vec{\nabla} f\big) \;=\; \oint_{\partial V} \mathrm{d}^2\vec{r}\; f \tag{5.13}$$

are unique, whence the formula must hold. Similarly, given a vector $\vec{A}$, there is a unique scalar on both left and right-hand-sides of

$$\int_V \mathrm{d}^3\vec{r}\,\big(\vec{\nabla} \cdot \vec{A}\big) \;=\; \oint_{\partial V} \mathrm{d}^2\vec{r} \cdot \vec{A}\,, \tag{5.14}$$

and also there is a unique vector on both left and right-hand-sides of

$$\int_V \mathrm{d}^3\vec{r}\,\big(\vec{\nabla} \times \vec{A}\big) \;=\; \oint_{\partial V} \mathrm{d}^2\vec{r} \times \vec{A}\,. \tag{5.15}$$

Therefore, Eqs.(5.13)–(5.15) are all "trivially true". The second of these is known as "Gauss's law" or the "fundamental theorem of divergences"; all such formulas however fall under the general category of "Stokes' formulae".

The domains of integration, V and ∂V, are again related by compatible orientation: $\mathrm{d}^2\vec{r}$ within the volume domain V is typically defined using a right-hand coordinate system and the surface ∂V which bounds the volume domain is then compatibly oriented *outwardly*.

6 All together now...

To summarize, we find that the formulae (5.3), (5.4), (5.8), (5.9), (5.10), (5.13), (5.14) and (5.15) must hold, simply because it is not possible to write anything else on either side of the equation. The "fundamental theorems of gradients, curls and divergences", Eqs.(5.3), (5.9) and (5.14), *proven herein "trivially"* (by virtue solely of not being able to write down anything else sensible), tend to occur more often in physics applications than the other ones, so one tends to remember them better after some practice. In all cases, however, it is possible to extract an operatorial identity in the manner of Eq.(5.6), and we do this at the bottom of the following table:

One-to-Zero	**Two-to-One**	**Three-to-Two**
$\int_C \mathrm{d}\vec{r}\cdot\vec{\nabla}f = f\Big\vert_{\partial C}$	$\int_S(\mathrm{d}^2\vec{r}\times\vec{\nabla})f \;=\; \oint_{\partial S}\mathrm{d}\vec{r}\;f$	$\int_V \mathrm{d}^3\vec{r}\,\vec{\nabla}f \;=\; \oint_{\partial V}\mathrm{d}^2\vec{r}\;f$
$\int_C \mathrm{d}\vec{r}\cdot\vec{\nabla}\;\vec{A} = \vec{A}\Big\vert_{\partial C}$	$\int_S(\mathrm{d}^2\vec{r}\times\vec{\nabla})\cdot\vec{A} \;=\; \oint_{\partial S}\mathrm{d}\vec{r}\cdot\vec{A}$	$\int_V \mathrm{d}^3\vec{r}\,\vec{\nabla}\cdot\vec{A} \;=\; \oint_{\partial V}\mathrm{d}^2\vec{r}\cdot\vec{A}$
	$\int_S(\mathrm{d}^2\vec{r}\times\vec{\nabla})\times\vec{A} \;=\; \oint_{\partial S}\mathrm{d}\vec{r}\times\vec{A}$	$\int_V \mathrm{d}^3\vec{r}\,\vec{\nabla}\times\vec{A} \;=\; \oint_{\partial V}\mathrm{d}^2\vec{r}\times\vec{A}$
$\int_C \mathrm{d}\vec{r}\cdot\vec{\nabla}\,[\,\cdots] = [\,\cdots]\Big\vert_{\partial C}$	$\int_S(\mathrm{d}^2\vec{r}\times\vec{\nabla})[\,\cdots] \;=\; \oint_{\partial S}\mathrm{d}\vec{r}\,[\,\cdots]$	$\int_V \mathrm{d}^3\vec{r}\,\vec{\nabla}\,[\,\cdots] \;=\; \oint_{\partial V}\mathrm{d}^2\vec{r}\,[\,\cdots]$

Note that the arguments used here do not fix the relative sign; that is, any one of the above formulae argued herein could have, in principle, a "$-$" in front of the right-hand-side. However, the sign is straightforward to determine by substituting particularly simple functions, f or $\vec{A}$ as the case may be, and calculating directly. In any case, recall that these signs are *conventional but compatible* (e.g., right-hand-rules), regarding the relative orientations of the integration elements in the interior of the domain and on the boundary.

Bonus: Well-defined integrals over *void* (empty) domains automatically vanish.

Therefore, if the higher-dimensional integral in the left-hand side integration in the above (Stokes-type) formulae is a *closed domain*, it has no boundary: a closed contour C has no endpoints: $\partial C = \varnothing$; a closed surface S has no "rim": $\partial S = \varnothing$. If so, you are lucky: the right-hand side integrals automatically vanish, regardless of how complicated the integrand function may be or how lumpy/convoluted/awkwardly-shaped the left-hand side integration domain is!

The key-phrase, however, was "well-defined," and this is also assumed of all integrals in the above table. The hallmark exception to these theorems ($\rightarrow$ Dirac δ-function) will be discussed later.