Tensor Calculus WoEs:

(Worked-Out Examples*)

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The subsequent text is intended to ease the Students' adoption of the *doggedly correct* index-notation; it is designed to *complement* rather than *supplant* the exposition in standard texts on Mathematical Methods for Physicists (and Engineers).

Preface

Although we frequently use Cartesian coordinates, other so-called *generalized* coordinates are often better suited to describe a particular coordinate system—not infrequently, because of the geometry of the boundaries or (twisted) periodicity conditions.

For example, the description of the vertical vibrations of a taut string, such as on a horizontally held guitar, is readily accomplished using Cartesian coordinates. However, the use of circular polar coordinates is much better suited for describing the (vertical) vibrations of a horizontally held circular drumhead since the boundary conditions (the drumhead does not move at the circular rim) is straightforward therein, but unnecessarily arduous in Cartesian coordinates. In turn, the configuration space of a double planar pendulum (idealized so the first pendulum may rotate freely in a plane, and the second one freely about the bead of the first, permitting their otherwise rigid support cranks to pass through each other if need be) is a torus. Thus, while the configuration of the double planar pendulum *can* be described in terms of the Cartesian coordinates of the plane, the use of a two-angular coordinate system simplifies the problem tremendously. In addition, the geometry of this doubly periodic space of the two angles, (ϕ_1, ϕ_2) permits the introduction of a complex coordinate $z = \phi_1 + i\phi_2$, which allows us to use the powerful techniques of complex analysis and geometry. A little further thought then reveals that the configuration space of an n-fold similarly idealized *spherical* pendulum is

$$(S^2)^{\times n} = \underbrace{S^2 \times \cdots \times S^2}_{n \text{ times}} = (\mathbb{CP}^1)^{\times n}, \qquad S^2 := \big\{(x,y,z) \in \mathbb{R}^3 : \quad x^2 + y^2 + z^2 = R^2\big\}, \tag{0.1}$$

and benefits similarly from the use of complex geometry. In general, $\mathbb{CP}^n$ is recursively defined for all $n \in \mathbb{N}$ as: "$\mathbb{C}^n$ with $\mathbb{CP}^{n-1}$ glued in at infinity", and where $\mathbb{CP}^0$ is a point. So, $\mathbb{CP}^1$ is $\mathbb{C}^1$ (the complex plane[1]) with a point glued at infinity, *i.e.*, where "all points at infinity" are identified; $\mathbb{CP}^1 \simeq S^2$. Generally, a *generalized* coordinate may well turn out to be an arbitrarily complicated function of relative positions of the constituents of the physical system being described, their velocities, and possibly also their higher derivatives. Also, generalized coordinates may have arbitrary physical units.

Our approach then is "from ground up" (inductive)—in the pragmatic manner of a practicing physicist/engineer, rather than "from top down" (axiomatic-deductive)—as often done in the mathematical literature. In this spirit, the material is uncovered by examining worked-out examples and generalizing from there. Nevertheless, one hopes, no real loss of rigor has been committed. Finally, the Reader should be cautioned that the material presented herein is conceptually not as difficult as the variety of notational standards in the literature makes it appear; only practice, however, makes it familiar.

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* This document is meant to complement/supplement Refs. [1,2,3,4,5,6,7,8], not to supplant them.

[1] This is isomorphic to the 2-dimensional real plane, $\mathbb{R}^2$, equipped with $\underline{a}$ "complex structure", meaning $\underline{a}$ choice of a linear map $i : \mathbb{R}^2 \to \mathbb{R}^2$ that squares to multiplication by -1.

Contents

1 Basics

We will need to recall two most basic rules in calculus:

$$\textbf{product rule:}\quad \frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) = \Big(\frac{\mathrm{d}\,f(x)}{\mathrm{d}x}\Big)g(x) + f(x)\Big(\frac{\mathrm{d}\,g(x)}{\mathrm{d}x}\Big), \tag{1.1}$$

$$\textbf{chain rule:}\quad \frac{\mathrm{d}}{\mathrm{d}x}\Big(f\big(g(x)\big)\Big) = \Big(\frac{\mathrm{d}\,f(g)}{\mathrm{d}g}\Big)\Big(\frac{\mathrm{d}\,g(x)}{\mathrm{d}x}\Big), \tag{1.2}$$

and their multi-variable calculus generalizations:

$$\boxed{\textbf{product rule:}\quad \frac{\partial}{\partial\xi^i}\big(f(\xi)g(\xi)\big) = \Big(\frac{\partial f(\xi)}{\partial\xi^i}\Big)g(\xi) + f(\xi)\Big(\frac{\partial g(\xi)}{\partial\xi^i}\Big),} \tag{1.3}$$

$$\boxed{\textbf{chain rule:}\quad \frac{\partial}{\partial\xi^i}\Big(\eta^j\big(\zeta(\xi)\big)\Big) = \Big(\frac{\partial\eta^j}{\partial\zeta^k}\Big)\Big(\frac{\partial\zeta^k}{\partial\xi^i}\Big).} \tag{1.4}$$

These formulae hold within a "coordinate chart"—an n-dimensional region isomorphic to $\mathbb{R}^n$ (perhaps with an additional complex or other *structure*) within (possibly all of) the space of interest[2]. Not infrequently, the space of interest *cannot* be covered completely by a single chart: a 2-sphere, S^2, requires at least two such charts. Then, the space of interest, X, is to be covered by an *atlas* of such charts, $A := \{U_i, i = 1, 2, \dots\}$, such that: **(1)** $\cup_i U_i = X$, and **(2)** we know how to transform the coordinates specified for U_i into those specified for U_j wherever $U_i \cap U_j \neq \varnothing$. Without loss of generality then, we restrict to such a coordinate chart, and furthermore use the fact that $\mathbb{R}^n$ is a vector space over the ground field of real numbers.

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For a given n-dimensional vector space, we will choose a reference frame within which the n-tuple of coordinates $(\xi^1, \cdots, \xi^n)$ can represent any point $\vec{r}$ in the given space. In typical physics/engineering applications, we will need to compute with an assortment of quantities, many of which are *vectors* $(\vec{A}, \vec{B}, \dots)$ in the physics/engineering sense; see definition 1.2 on page 5. Since we may equally well choose a different reference frame, with an n-tuple $(\eta^1, \cdots, \eta^n)$ representing the same point (vector), it is important to be able to change $\xi \to \eta$ and back $\eta \to \xi$. Indeed, most of the subsequent material is introduced with this notion in mind. This application-motivated approach is pragmatic and realistic: vector spaces are collections of concrete objects (such as the red, green and blue quark; the modes of oscillation of a guitar string; stationary states in the Hydrogen atom; ...) the specification of which is subject to routine redefinitions (coordinate changes, *e.g.*, the easier to compute), rather than an abstract axiomatic structure. The latter, of course, does exist; it merely is not as compelling a framework from which to start—with an ultimate physics application, and a typical physics prerequisite education in mind.

Note that $\mathbb{R}^n$ being a vector space implies that each coordinate chart *admits* a choice of coordinates $(\xi^1, \cdots, \xi^n)$ which satisfy the vector space axioms, so that if all n-tuples $(\xi^1, \cdots, \xi^n)$

[2] Physical applications include the *configuration space*, *momentum space* and *phase space* in classical physics, *space of extrinsic variables* in statistical physics, *parameter spaces* in various branches of physics, the *Hilbert space* of states in quantum physics, the space of *normal modes* of dynamics of spatially extended bodies (the latter two of which, admittedly, are most often infinite-dimensional, so that some of the results presented herein simply do not apply therein), *etc.*...

represent points in the coordinate chart, so do all real linear combinations $\alpha_i\xi^i + \beta_i\xi^i$, $\alpha_i, \beta_i \in \mathbb{R}$. However, we are by no means *mandated* to use such coordinates, and nonlinear transformations $\xi^i \to \eta^i = \eta^i(\xi)$ are perfectly welcome—often even indispensable. For example, $\vec{r}_1 + \vec{r}_2$ in the usual 3-dimensional space is indeed a linear combination in the Cartesian coordinate representation of $\vec{r}_1, \vec{r}_2$, but is well known *not to be the case* in the spherical coordinates. (It makes even no physical sense to form linear combinations of radii and the two types of angles measured in two different ways.) This may then be seen as a consequence of the nonlinearity of the coordinate transformation $(x, y, z) \leftrightarrow (r, \theta, \phi)$.

1.1 Vector Variations

In many typical physics/engineering considerations, we compute with an assortment of quantities, some of which are *vectors*. Intuitively understood as quantities that in addition to "magnitude" also have a "sense of direction"—which is woefully imprecise for an actual definition, these *vectors* turn out to be a specific special case of what mathematicians call so.

The math-formal notion of *vectors* turns out to be very useful—in fact, ubiquitous throughout physics/engineering—and is as follows:

Definition 1.1 *Given a ground field*[3] *of **scalars**, $\Bbbk$ (such as $\mathbb{R}, \mathbb{C}$), a (linear) **vector space** V (over the field $\Bbbk$) is a collection of objects, $\vec{v}$, such that*

$$\sum_i c_i\vec{v}_i \in V, \qquad \text{for all } c_i \in \Bbbk, \quad \vec{v}_i \in V. \tag{1.5}$$

That is, vectors form a closed set under $\Bbbk$-linear superposition.

Remark 1.1: The use of the word *superposition* is not at all accidental: one might say that these "math-formal vectors are quantities that are *superposable*," and in turn, any set of superposable quantities are math-formal vectors. In particular, the physics/engineering notion of *vectors* satisfies this definition, thus justifying the name. However, this math-formal notion of vectors is *vastly* more general: all matrices of a given $p \times q$ size are superposable and so *vectors*, so are all quaternions, all spin-$\frac{1}{2}$ wave-functions satisfying a pre-selected Schrödinger equation, so are all binary numbers (with the ground field reduced to $\{0, 1\}$), *etc.*

It will behoove us therefore to provide a more precise definition, one that singles out the physics/engineering vectors (forces, velocities, accelerations, electric/magnetic fields, *etc.*). Many of these physics/engineering quantities that are vectors may well be depicted within our "real" 3-dimensional space, and graphed/plotted in it. One should be careful, however, not to identify the two: Clearly, any n-tuple of arbitrary *generalized* coordinates (as the notion is carefully defined within classical mechanics) will in general depend nonlinearly on, say, Cartesian coordinates of the 3-dimensional space. Therefore, an n-tuple of generalized coordinates $(\xi^1, \cdots, \xi^n)$ will hardly ever be a vector in the sense of the definition 1.1. Nevertheless, certain derived quantities such

[3] "Field" here is meant in the *mathematical* sense of the word: A **field** is a collection of elements equipped with two binary operations, an 'addition' and a 'multiplication' with respect to both of which the elements of a field form a group, except that "0" (the 'additive' unit) does not have a multiplicative inverse; furthermore, these two operations must satisfy the usual distributive laws.

as the n-tuple of appropriately scaled differentials $(h_1\mathrm{d}\xi^1, \cdots, h_1\mathrm{d}\xi^n)$ and of appropriately scaled derivatives $\big(h_1^{-1}\frac{\partial}{\partial\xi^1}, \cdots, h_n^{-1}\frac{\partial}{\partial\xi^n}\big)$ *always are* vectors with a suitable choice of scaling functions $(h_1, \cdots, h_n)$, and we now turn to some of these; we return to this issue in section 1.1.3.

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Given the rules for changing coordinates, standard rules of calculus imply that:

$$\mathrm{d}\xi^i = \Big(\frac{\partial\xi^i}{\partial\eta^j}\Big)\mathrm{d}\eta^j, \tag{1.6a}$$

$$\frac{\partial}{\partial\xi^i} = \Big(\frac{\partial\eta^k}{\partial\xi^i}\Big)\frac{\partial}{\partial\eta^k}. \tag{1.6b}$$

In particular, note that with respect to the change of variables $\xi^i \mapsto \xi^i(\eta)$, the quantity $\mathrm{d}\xi^i$, the i^{th} component of the differential $\mathrm{d}\vec{r}$ in the ξ-coordinates, transforms ***inversely*** from $\frac{\partial}{\partial\xi^i}$, the i^{th} component of the derivative operator $\vec{\nabla}$ in the ξ-coordinates. Whereas it is irrelevant whether we superscript coordinates (as done herein, consistently), or subscript them, it is important that we do distinguish the relatively inverse nature of the two transformation rules (1.6a)–(1.6b), and that we do distinguish them one from another:

Definition 1.2 *Any n-tuple of quantities that transform akin to the components of $\mathrm{d}\vec{r}$, as specified in (1.6a):*

$$A^i(\xi) = \Big(\frac{\partial\xi^i}{\partial\eta^j}\Big)A^j(\eta), \tag{1.7}$$

will be called ***contravariant*** *vectors and will be superscripted, just as coordinates are. Any n-tuple of quantities that transform akin to the components of $\vec{\nabla}$, as specified in (1.6b):*

$$B_i(\xi) = \Big(\frac{\partial\eta^k}{\partial\xi^i}\Big)B_j(\eta), \tag{1.8}$$

will be called ***covariant*** *vectors and will be subscripted, opposite from how coordinates are indexed.*

Remark 1.2: The nonlinearity (curvilinearity) of the coordinates $(\xi^1, \cdots, \xi^n)$ in no way prevents forming superpositions of the differentials $\mathrm{d}\xi^i$ on one hand, and/or superpositions of the partial derivatives $\frac{\partial}{\partial\xi^i}$ on the other. In concrete applications, this most often violates a *separate* fact—that the generalized coordinates, and so also their differentials and partial derivatives with respect to them, have definite and distinct physics/engineering units/dimensions. At the moment, we are concerned with the quantities such as (1.6a)–(1.6b) regardless of additional properties that might stem from their application in physics/engineering; those additional properties can always be included subsequently.

In turn, one can also *define* the superposition to involve coefficients that themselves have appropriate physics/engineering units/dimensions, thus permitting us to form a linear combination of physically disparate objects, such as:

$$\mathrm{d}z + \ell\sin\theta\,\mathrm{d}\phi, \qquad \mathrm{d}E + mc\,\mathrm{d}v + mf\,\mathrm{d}a, \qquad \textit{etc.}, \tag{1.9}$$

where z is the vertical coordinate, θ, ϕ the spherical angles, ℓ a constant of the units/dimensions of length, E, m, v, f, a the energy, mass, speed, frequency and acceleration, respectively, and c denotes the speed of light in vacuum.

By linearity of the defining differential conditions, the twin definition 1.2 specifies two vector spaces over any given field, $\Bbbk$, since any $\Bbbk$-linear superposition of covariant vectors is still covariant, and similarly, any $\Bbbk$-linear combination of contravariant vectors is still contravariant.

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Note that there will definitely exist quantities that do not transform at all with respect to a change of coordinates $\xi^i \mapsto \eta^i(\xi)$: those are *invariants* and are also called *scalars*. In turn, there exist quantities that transform, but unlike either of the two rules (1.6a)–(1.6b): we'll see later what some of those might be.

WoE 1.1 *(the exterior derivative)*: The quantity $\mathrm{d} := \mathrm{d}\xi^i \frac{\partial}{\partial \xi^i}$ does not transform at all:

$$\mathrm{d}\xi^i \frac{\partial}{\partial \xi^i} = \mathrm{d}\eta^j \frac{\partial \xi^i}{\partial \eta^j} \frac{\partial \eta^k}{\partial \xi^i} \frac{\partial}{\partial \eta^k} = \mathrm{d}\eta^j \, \delta_j^k \frac{\partial}{\partial \eta^k} = \mathrm{d}\eta^j \frac{\partial}{\partial \eta^j}, \tag{1.10}$$

since

$$\frac{\partial \xi^i}{\partial \eta^j} \frac{\partial \eta^k}{\partial \xi^i} = \frac{\partial \xi^i}{\partial \eta^j} \frac{\partial}{\partial \xi^i} \eta^k \overset{*}{=} \frac{\partial}{\partial \eta^j} \eta^k = \frac{\partial \eta^k}{\partial \eta^j} = \delta_j^k, \tag{1.11}$$

where $\overset{*}{=}$ follows on using the "chain rule" backwards.

Remark 1.3: The invariant pairing (1.10) of these co- and contra-variant vector components is closely related to the duality between any given vector space V and its formal dual, V^*. Here, if $\mathrm{d}\xi^i \in V$, then $\frac{\partial}{\partial \xi^i} \in V^*$.

1.1.1 The Kronecker δ-Symbol

Let ξ^i denote the i^{th} coordinate in a system $(\xi^1, \cdots, \xi^n)$ of *independent* coordinates. Then:

$$\boxed{\frac{\partial \xi^i}{\partial \xi^j} =: \delta_j^i := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}} \tag{1.12}$$

We pause here to observe that the indices i, j are ***free***: one can freely substitute any of their possible values in their place. Also, we will adhere to the convention that the index (counter) of the coordinates is written as a superscript. This definition of the Kronecker δ-symbol (1.12) is simply a reflection of what we mean by *independent variables* and *partial derivatives*.

This statement of linear independence defines an immensely important quantity (1.12):

WoE 1.2 *(The Kronecker δ-Symbol)*: The quantity δ_j^i as defined in (1.12) is invariant with respect to all invertible coordinate changes. The proof of this in fact requires no actual computation, merely the observation that Eq. (1.12) asserts the mutual independence of the coordinates $(\xi^1, \cdots, \xi^n)$ and the maximality of the collection that forms a basis. All invertible changes of coordinates must then result in a system of coordinates which must be mutually independent just the same.

Nevertheless, consider the transformation of δ^i_j with respect to the coordinate change[4], $\xi^i \mapsto \xi^i(\eta)$:

$$\delta^i_j = \frac{\partial \xi^i}{\partial \xi^j} \;\mapsto\; \Big(\frac{\partial \eta^l}{\partial \xi^j}\,\frac{\partial (\xi^i)}{\partial \eta^l}\Big) = \Big(\frac{\partial \eta^l}{\partial \xi^j}\,\frac{\partial \eta^k}{\partial \eta^l}\,\frac{\partial \xi^i}{\partial \eta^k}\Big) \stackrel{*}{=} \Big(\frac{\partial \eta^l}{\partial \xi^j}\,\delta^k_l\,\frac{\partial \xi^i}{\partial \eta^k}\Big) = \Big(\frac{\partial \eta^k}{\partial \xi^j}\,\frac{\partial \xi^i}{\partial \eta^k}\Big) = \frac{\partial \xi^i}{\partial \xi^j} = \delta^i_j, \tag{1.13}$$

where $\stackrel{*}{=}$ follows using that $\frac{\partial \eta^k}{\partial \eta^l} = \delta^k_l$ since the coordinates η^i are independent from each other; colors were used to help tracking the use of the 'chain rule' in changing variables: red and blue initially, and green when using the 'chain rule' backwards.

Conversely, we can also write:

$$\frac{\partial \xi^i}{\partial \xi^j} = [\delta_{(\xi)}]^i_j \;\mapsto\; \frac{\partial \eta^i}{\partial \xi^k}\,[\delta_{(\xi)}]^k_l\,\frac{\partial \xi^l}{\partial \eta^j} = \frac{\partial \eta^i}{\partial \xi^k}\,\frac{\partial \xi^k}{\partial \eta^j} = \frac{\partial \eta^i}{\partial \eta^j} = [\delta_{(\eta)}]^i_j, \tag{1.14}$$

which shows that the δ^i_j stating the independence of the ξ-coordinates simply transforms into the δ^i_j stating the independence of the η-coordinates.

WoE 1.3 *(The Kronecker δ-Symbol, Again)*: Consider changing coordinates $\xi \to \eta$:

$$\delta^i_j = \frac{\partial \xi^i}{\partial \xi^j} = \frac{\partial \eta^k}{\partial \xi^j}\,\frac{\partial \xi^i}{\partial \eta^k} \tag{1.15}$$

and multiply this by $\frac{\partial \xi^j}{\partial \eta^m}$ from the left, and by $\frac{\partial \eta^n}{\partial \xi^i}$ from the right to obtain:

$$\begin{aligned} \frac{\partial \xi^j}{\partial \eta^m}\,\delta^i_j\,\frac{\partial \eta^n}{\partial \xi^i} &= \frac{\partial \xi^j}{\partial \eta^m}\,\frac{\partial \eta^k}{\partial \xi^j}\,\frac{\partial \xi^i}{\partial \eta^k}\,\frac{\partial \eta^n}{\partial \xi^i} \\ \frac{\partial \xi^j}{\partial \eta^m}\,\frac{\partial \eta^n}{\partial \xi^j} = \frac{\partial \eta^n}{\partial \eta^m} &= \frac{\partial \eta^k}{\partial \eta^m}\,\frac{\partial \eta^n}{\partial \eta^k}. \end{aligned} \tag{1.16}$$

In the first row, we used that $\xi \to \eta$ is an invertible change of variables, so that the matrix $[\frac{\partial \xi^i}{\partial \eta^j}]$ is invertible. The last equality states that the matrix $[\frac{\partial \eta^n}{\partial \eta^m}]$ is idempotent (it squares to itself) and so is a projection matrix. We must assume that this matrix is also of maximum rank (equivalently, it has a nonzero determinant)—were it not, the collection $\eta^1, \cdots, \eta^n$ would turn out linearly dependent. Since the only projection matrix the rank of which equals its size is the identity matrix, it follows that

$$\frac{\partial \eta^n}{\partial \eta^m} = \delta^n_m. \tag{1.17}$$

That implies:

$$\delta^i_j(\xi) = \delta^i_j(\eta) \tag{1.18}$$

is invariant with respect to the general coordinate transformation $\xi \to \eta$.

The definition (1.12) and the result (1.16) imply (separately):

Corollary 1.1 *The Kronecker symbol satisifes:*

$$\delta^i_j\,\delta^j_k = \delta^i_k, \qquad \delta^i_i = n. \tag{1.19}$$

[4] Here and hereafter, we use the Einstein convention, according to which—unless otherwise stated—we sum over each pair of repeated indices; thus: $X^i_j Y^j_k := \sum_{j=1}^n X^i_j Y^j_k$, *etc.*, for any X^i_j and any Y^j_k.

Using δ^i_j, we can construct other invariant objects, by means of *linear algebra*, *i.e.*, by Cartesian products and (anti)symmetrization:

$$
\begin{aligned}
\delta^{ik}_{[jl]} = \delta^{[ik]}_{jl} &:= \tfrac{1}{2!}\big(\delta^i_j\,\delta^k_l - \delta^i_l\,\delta^k_j\big), \qquad \text{(antisymmetrized)} && \text{(1.20a)}\\
\delta^{ik}_{(jl)} = \delta^{(ik)}_{jl} &:= \tfrac{1}{2!}\big(\delta^i_j\,\delta^k_l + \delta^i_l\,\delta^k_j\big), \qquad \text{(symmetrized)} && \text{(1.20b)}\\
\delta^{ikm}_{[jln]} = \delta^{[ikm]}_{jln} &:= \tfrac{1}{3!}\big(\delta^i_j\,\delta^k_l\,\delta^m_n - \delta^i_l\,\delta^k_j\,\delta^m_n + \delta^i_l\,\delta^k_n\,\delta^m_j - \delta^i_n\,\delta^k_l\,\delta^m_j + \delta^i_n\,\delta^k_j\,\delta^m_l - \delta^i_j\,\delta^k_n\,\delta^m_l\big), && \text{(1.20c)}\\
\delta^{ikm}_{(jln)} = \delta^{(ikm)}_{jln} &:= \tfrac{1}{3!}\big(\delta^i_j\,\delta^k_l\,\delta^m_n + \delta^i_l\,\delta^k_j\,\delta^m_n + \delta^i_l\,\delta^k_n\,\delta^m_j + \delta^i_n\,\delta^k_l\,\delta^m_j + \delta^i_n\,\delta^k_j\,\delta^m_l + \delta^i_j\,\delta^k_n\,\delta^m_l\big), && \text{(1.20d)}\\
\delta^{ik\,m}_{[jl]n} &:= \delta^{ik}_{[jl]}\,\delta^m_n - \delta^{ikm}_{[jln]},\\
&= \tfrac{1}{3!}\big(2\delta^i_j\,\delta^k_l\,\delta^m_n - 2\delta^i_l\,\delta^k_j\,\delta^m_n - \delta^i_l\,\delta^k_n\,\delta^m_j + \delta^i_n\,\delta^k_l\,\delta^m_j - \delta^i_n\,\delta^k_j\,\delta^m_l + \delta^i_j\,\delta^k_n\,\delta^m_l\big), && \text{(1.20e)}\\
\delta^{i\,km}_{j[ln]} &:= \delta^i_j\,\delta^{km}_{[ln]} - \delta^{ikm}_{[jln]},\\
&= \tfrac{1}{3!}\big(2\delta^i_j\,\delta^k_l\,\delta^m_n + \delta^i_l\,\delta^k_j\,\delta^m_n - \delta^i_l\,\delta^k_n\,\delta^m_j + \delta^i_n\,\delta^k_l\,\delta^m_j - \delta^i_n\,\delta^k_j\,\delta^m_l - 2\delta^i_j\,\delta^k_n\,\delta^m_l\big), && \text{(1.20f)}\\
\delta^{ik\,m}_{(jl)n} &:= \delta^{ik}_{(jl)}\,\delta^m_n - \delta^{ikm}_{(jln)},\\
&= \tfrac{1}{3!}\big(2\delta^i_j\,\delta^k_l\,\delta^m_n + 2\delta^i_l\,\delta^k_j\,\delta^m_n - \delta^i_l\,\delta^k_n\,\delta^m_j - \delta^i_n\,\delta^k_l\,\delta^m_j - \delta^i_n\,\delta^k_j\,\delta^m_l - \delta^i_j\,\delta^k_n\,\delta^m_l\big), && \text{(1.20g)}
\end{aligned}
$$

and so on, indefinitely; the normalizations are included for computational convenience.

WoE 1.4 *(Kronecker Identities)*: Note that

$$
\begin{aligned}
\delta^{ik}_{[jl]} + \delta^{ik}_{(jl)} &= \delta^i_j\,\delta^k_l; \qquad \delta^j_i\,\delta^{ik}_{[jl]} = 0; \qquad \delta^j_i\,\delta^{ik}_{(jl)} = \delta^k_l; && \text{(1.21)}\\
\delta^{ik}_{[jl]}\delta^{\,j\,l}_{(mn)} &= 0; && \text{(1.22)}\\
\delta^{ik}_{[jl]}\delta^m_n &= \tfrac{1}{3!}(3\delta^i_j\,\delta^k_l - 3\delta^i_l\,\delta^k_j)\delta^m_n \pm \tfrac{1}{3!}(\delta^i_l\,\delta^k_n\,\delta^m_j - \delta^i_n\,\delta^k_l\,\delta^\mu_j + \delta^i_n\,\delta^k_j\,\delta^m_l - \delta^i_j\,\delta^k_n\,\delta^\mu_l),\\
&= \tfrac{1}{3!}\big(\delta^i_j\,\delta^k_l\,\delta^m_n - \delta^i_l\,\delta^k_j\,\delta^m_n + \delta^i_l\,\delta^k_n\,\delta^m_j - \delta^i_n\,\delta^k_l\,\delta^m_j + \delta^i_n\,\delta^k_j\,\delta^m_l - \delta^i_j\,\delta^k_n\,\delta^m_l\big)\\
&\quad + \tfrac{1}{3!}\big(2\delta^i_j\,\delta^k_l\,\delta^m_n - 2\delta^i_l\,\delta^k_j\,\delta^m_n - \delta^i_l\,\delta^k_n\,\delta^m_j + \delta^i_n\,\delta^k_l\,\delta^m_j - \delta^i_n\,\delta^k_j\,\delta^m_l + \delta^i_j\,\delta^k_n\,\delta^m_l\big)\\
&= \delta^{ikm}_{[jln]} \;+\; \delta^{ikm}_{[jl]n}; && \text{(1.23)}\\
\delta^{ik}_{[jl]}\delta^{\,jm}_{(np)} &= \tfrac{1}{2!}\big(\delta^i_j\,\delta^k_l - \delta^i_l\,\delta^k_j\big)\tfrac{1}{2!}\big(\delta^j_n\,\delta^m_p + \delta^j_p\,\delta^m_n\big),\\
&= \tfrac{1}{4}\big(\delta^i_j\,\delta^k_l\,\delta^j_n\,\delta^m_p - \delta^i_l\,\delta^k_j\,\delta^j_n\,\delta^m_p + \delta^i_j\,\delta^k_l\,\delta^j_p\,\delta^m_n - \delta^i_l\,\delta^k_j\,\delta^j_p\,\delta^m_n\big),\\
&= \tfrac{1}{4}\big(\delta^i_n\,\delta^k_l\,\delta^m_p - \delta^i_l\,\delta^k_n\,\delta^m_p + \delta^i_p\,\delta^k_l\,\delta^m_n - \delta^i_l\,\delta^k_p\,\delta^m_n\big),\\
&= \tfrac{1}{2}\big(\delta^{ik}_{[nl]}\,\delta^m_p + \delta^{ik}_{[pl]}\,\delta^m_n\big),\\
&= \tfrac{1}{2}\big(\delta^{ik\,m}_{[nl]p} + \delta^{ikm}_{[nlp]} + \delta^{ik\,m}_{[pl]n} + \delta^{ikm}_{[pln]}\big),\\
&= \tfrac{1}{2}\big(\delta^{ik\,m}_{[nl]p} + \delta^{ik\,m}_{[pl]n}\big); && \text{(1.24)}
\end{aligned}
$$

$$
\begin{aligned}
\delta^{ikm}_{[jl]n} + \delta^{ikm}_{[ln]j} + \delta^{ikm}_{[nj]l} &= 0, && \text{(1.25)}\\
\text{since:}\qquad &= \tfrac{1}{3!}\big(2\delta^i_j\,\delta^k_l\,\delta^m_n - 2\delta^i_l\,\delta^k_j\,\delta^m_n - \delta^i_l\,\delta^k_n\,\delta^m_j + \delta^i_n\,\delta^k_l\,\delta^m_j - \delta^i_n\,\delta^k_j\,\delta^m_l + \delta^i_j\,\delta^k_n\,\delta^m_l\big)\\
&\quad + \tfrac{1}{3!}\big(-\delta^i_j\,\delta^k_l\,\delta^m_n + \delta^i_l\,\delta^k_j\,\delta^m_n + 2\delta^i_l\,\delta^k_n\,\delta^m_j - 2\delta^i_n\,\delta^k_l\,\delta^m_j - \delta^i_n\,\delta^k_j\,\delta^m_l + \delta^i_j\,\delta^k_n\,\delta^m_l\big)\\
&\quad + \tfrac{1}{3!}\big(-\delta^i_j\,\delta^k_l\,\delta^m_n + \delta^i_l\,\delta^k_j\,\delta^m_n - \delta^i_l\,\delta^k_n\,\delta^m_j + \delta^i_n\,\delta^k_l\,\delta^m_j + 2\delta^i_n\,\delta^k_j\,\delta^m_l - 2\delta^i_j\,\delta^k_n\,\delta^m_l\big);
\end{aligned}
$$

but, then also:

$$
\begin{aligned}
\delta^{ikm}_{[jl]n} + \delta^{ikm}_{[ln]j} + \delta^{ikm}_{[nj]l} &= \delta^{ik}_{[jl]}\,\delta^m_n + \delta^{ik}_{[ln]}\,\delta^m_j + \delta^{ik}_{[nj]}\,\delta^m_l - 3\delta^{ikm}_{[jln]}, \quad \text{implying that}\\
\delta^{ikm}_{[jln]} &= \tfrac{1}{3}\big(\delta^{ik}_{[jl]}\,\delta^m_n + \delta^{ik}_{[ln]}\,\delta^m_j + \delta^{ik}_{[nj]}\,\delta^m_l\big). && \text{(1.26)}
\end{aligned}
$$

This clearly just barely starts the combinatorial avalanche of such identities.

1.1.2 Levi-Civita

Let $(\xi^1, \cdots, \xi^n)$ and $(\eta^1, \cdots, \eta^n)$ be two coordinate systems. Assuming that both systems equally well describe all of the space they are intended to describe, a change of variables $\eta^i \mapsto \xi^i = \xi^i(\eta)$ must be invertible, which implies that:

$$\left.\begin{aligned} \xi^i &= \xi^i(\eta) \\ \eta^j &= \eta^j(\xi) \end{aligned}\right\} \qquad \Rightarrow \qquad \left|\frac{\partial \xi}{\partial \eta}\right| := \det\left[\frac{\partial(\xi^1, \cdots, \xi^n)}{\partial(\eta^1, \cdots, \eta^n)}\right] := \det\begin{bmatrix} \frac{\partial \xi^1}{\partial \eta^1} & \cdots & \frac{\partial \xi^1}{\partial \eta^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi^n}{\partial \eta^1} & \cdots & \frac{\partial \xi^n}{\partial \eta^n} \end{bmatrix} \neq 0. \tag{1.27}$$

Note that the elements of this matrix—and so also its determinant, called the Jacobian (matrix) of the transformation, can be arbitrarily varying functions. The invertibility condition is then supposed to hold at every point in the space described by $(\xi^1, \cdots, \xi^n)$, *i.e.*, $(\eta^1, \cdots, \eta^n)$—that is, for all permissible values of the ξ- and η-variables, *i.e.*, for all points (vectors) in this vector space.

Writing out the determinant (1.27) in some detail reveals another immensely useful quantity:

$$\begin{aligned} \left|\frac{\partial \xi}{\partial \eta}\right| &= \frac{\partial \xi^1}{\partial \eta^1}\left(\frac{\partial \xi^2}{\partial \eta^2}\left(\cdots\left(\frac{\partial \xi^{n-1}}{\partial \eta^{n-1}}\frac{\partial \xi^n}{\partial \eta^n} - \frac{\partial \xi^{n-1}}{\partial \eta^n}\frac{\partial \xi^n}{\partial \eta^{n-1}}\right)\cdots\right)\right) \\ &\quad - \frac{\partial \xi^1}{\partial \eta^2}\left(\frac{\partial \xi^2}{\partial \eta^1}\left(\cdots\left(\frac{\partial \xi^{n-1}}{\partial \eta^{n-1}}\frac{\partial \xi^n}{\partial \eta^n} - \frac{\partial \xi^{n-1}}{\partial \eta^n}\frac{\partial \xi^n}{\partial \eta^{n-1}}\right)\cdots\right)\right) \\ &\quad \vdots \\ &\quad + (-1)^n\frac{\partial \xi^1}{\partial \eta^n}\left(\frac{\partial \xi^2}{\partial \eta^1}\left(\cdots\left(\frac{\partial \xi^{n-1}}{\partial \eta^{n-2}}\frac{\partial \xi^n}{\partial \eta^{n-1}} - \frac{\partial \xi^{n-1}}{\partial \eta^{n-1}}\frac{\partial \xi^n}{\partial \eta^{n-2}}\right)\cdots\right)\right) \\ &= \boxed{\varepsilon^{ij\cdots kl}}\,\frac{\partial \xi^1}{\partial \eta^i}\frac{\partial \xi^2}{\partial \eta^j}\cdots\frac{\partial \xi^{n-1}}{\partial \eta^k}\frac{\partial \xi^n}{\partial \eta^l} = \boxed{\varepsilon_{ij\cdots kl}}\,\frac{\partial \xi^i}{\partial \eta^1}\frac{\partial \xi^j}{\partial \eta^2}\cdots\frac{\partial \xi^k}{\partial \eta^{n-1}}\frac{\partial \xi^l}{\partial \eta^n}. \end{aligned} \tag{1.28}$$

A little experimentation[5] should convince the diligent Reader that the Levi-Civita permutation symbols $\varepsilon^{ij\cdots kl}$ and $\varepsilon_{ij\cdots kl}$—defined implicitly by their alternative occurrence in the expansion of the determinant (1.28)—may also be specified as follows:

$$\boxed{\varepsilon^{i_1\cdots i_n} = \begin{cases} +1 & \text{for} \quad i_1, \cdots, i_n = \text{even permutation of } 1, \cdots, n; \\ -1 & \text{for} \quad i_1, \cdots, i_n = \text{odd permutation of } 1, \cdots, n; \\ \ \ 0 & \text{for} \quad \text{all other cases;} \end{cases}} \tag{1.29}$$

and identically for $\varepsilon_{i_1\cdots i_n}$.

WoE 1.5 *(Levi-Civita)*: For example, when $n = 2$:

$$\varepsilon^{12} = 1 = -\varepsilon^{21}, \qquad \varepsilon^{11} = 0 = \varepsilon^{22}; \tag{1.30}$$

and when $n = 3$:

$$\begin{aligned} &\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = 1, && \varepsilon^{112} = \varepsilon^{121} = \varepsilon^{211} = \varepsilon^{122} = \varepsilon^{212} = \varepsilon^{221} = 0, \\ &\varepsilon^{132} = \varepsilon^{321} = \varepsilon^{213} = -1, && \varepsilon^{113} = \varepsilon^{131} = \varepsilon^{311} = \varepsilon^{133} = \varepsilon^{313} = \varepsilon^{331} = 0, \\ &\varepsilon^{111} = \varepsilon^{222} = \varepsilon^{333} = 0, && \varepsilon^{223} = \varepsilon^{232} = \varepsilon^{322} = \varepsilon^{233} = \varepsilon^{323} = \varepsilon^{332} = 0; \end{aligned} \tag{1.31}$$

[5] Start with $n = 2, 3$, not $n = 213\ldots$ *please*.

and so on, for $n = 4, 5 \ldots$

The definition (1.28)–(1.29) has an immediate consequence:

Corollary 1.2 *The Levi-Civita symbol is totally antisymmetric. That is, it changes sign upon swapping any two if its indices:*

$$\varepsilon^{\cdots i \cdots j \cdots} = -\varepsilon^{\cdots j \cdots i \cdots}. \tag{1.32}$$

Upon defining the covariant Levi-Civita symbol, $\varepsilon_{ij\cdots kl}$, to have the same values as its contravariant cousin (1.29), a little low-n experimentation and mathematical induction proves:

Proposition 1.1 *The Levi-Civita symbol satisfies:*

$$\begin{aligned}\varepsilon^{i_1 i_2 \cdots i_{n-1} i_n} \varepsilon_{j_1 j_2 \cdots j_{n-1} j_n} = n!\, \delta^{\,i_1 \cdots i_n}_{[j_1 \cdots j_n]} &= \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_{n-1}}_{j_{n-1}} \delta^{i_n}_{j_n} - \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_{n-1}}_{j_n} \delta^{i_n}_{j_{n-1}} + \cdots \\ &\quad - \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \cdots \delta^{i_{n-1}}_{j_{n-1}} \delta^{i_n}_{j_n} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \cdots \delta^{i_{n-1}}_{j_n} \delta^{i_n}_{j_{n-1}} - \cdots \\ &\qquad \ddots \qquad \textit{(a total of } n! \textit{ terms, in all permutations).}\end{aligned} \tag{1.33}$$

WoE 1.6: For $n = 2$, we have:

$$\varepsilon^{ij} \varepsilon_{kl} = 2!\, \delta^{\,ij}_{[kl]} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k, \quad \Rightarrow \quad \varepsilon^{ij} \varepsilon_{kj} = \delta^i_k, \quad \Rightarrow \quad \varepsilon^{ij} \varepsilon_{ij} = 2; \tag{1.34}$$

where the latter two equations are obtained from the first one, like so:

$$\begin{aligned}\delta^l_j \big(\varepsilon^{ij} \varepsilon_{kl}\big) &= \delta^l_j \Big(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\Big), \\ \varepsilon^{ij} \big(\delta^l_j \varepsilon_{kl}\big) &= \delta^i_k \big(\delta^l_j \delta^j_l\big) - \big(\delta^l_j \delta^i_l\big) \delta^j_k, \\ \varepsilon^{ij} \varepsilon_{kj} &= \delta^i_k \big(\delta^j_j\big) - \big(\delta^i_j\big) \delta^j_k = \delta^i_k (2) - \delta^i_k = (2-1) \delta^i_k = \delta^i_k,\end{aligned} \tag{1.35}$$

and

$$\delta^k_i \big(\varepsilon^{ij} \varepsilon_{kj}\big) = \delta^k_i \big(\delta^i_k\big) \qquad \Rightarrow \qquad \varepsilon^{ij} \varepsilon_{ij} = \delta^i_i = 2. \tag{1.36}$$

For $n = 3$, we have:

$$\varepsilon^{ijk} \varepsilon_{lmn} = 3!\, \delta^{\,i\,j\,k}_{[lmn]} = \delta^i_l \delta^j_m \delta^k_n - \delta^i_l \delta^j_n \delta^k_m + \delta^i_n \delta^j_l \delta^k_m - \delta^i_n \delta^j_m \delta^k_l + \delta^i_m \delta^j_n \delta^k_l - \delta^i_m \delta^j_l \delta^k_n, \tag{1.37a}$$

$$\Rightarrow \quad \varepsilon^{ijk} \varepsilon_{lmk} = 2!\, \delta^{\,i\,j}_{[lm]} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l, \tag{1.37b}$$

$$\Rightarrow \quad \varepsilon^{ijk} \varepsilon_{ljk} = 2\, \delta^i_l, \tag{1.37c}$$

$$\Rightarrow \quad \varepsilon^{ijk} \varepsilon_{ijk} = 3! = 6. \tag{1.37d}$$

and so on, for $n = 4, 5 \ldots$

1.1.3 Linearization Owing to Infinitesimality

A cautionary note is in order: we have insisted that we are in fact considering general invertible changes of variables $\xi^i \to \eta^i = \eta^i(\xi)$. On the other hand, the erudite Reader will realize that the transformation rules (1.6a)–(1.6b) are examples of *linear transformations*. Surely, we have not blundered using linear transformations but calling them "general invertible".

To clear up a confusing bit of terminology, recall that what is called "a linear transformation" of an n-tuple $(x^1, \cdots, x^n)$ may be written:

$$x^i \to y^i = a^i{}_j\, x^j + b^i, \qquad \textit{i.e.}, \qquad \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \to \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} = \begin{bmatrix} a^1{}_1 & \cdots & a^1{}_n \\ \vdots & \ddots & \vdots \\ a^n{}_1 & \cdots & a^n{}_n \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} + \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}. \tag{1.38}$$

The simplest example of this is the very well known $n = 1$ case: $x \to y = ax + b$, and indeed, it is *standard* to call $y(x) = ax + b$ a "linear function."

On the other hand, the term "linear operator/operation" is subject to the more formal

Definition 1.3 *An operator (operation) $\mathscr{L}$ over (on elements of) a vector space (as per definition 1.1) is called* ***linear*** *if*

$$\mathscr{L}\Big(\sum_i c_i \vec{v}_i\Big) = \sum_i c_i\, \mathscr{L}(\vec{v}_i). \tag{1.39}$$

In turn, if an operator (operation) $\mathscr{A}$ over (on elements of) a vector space (as per definition 1.1) satisfies

$$\begin{aligned} \mathscr{A}\Big(\textstyle\sum_i c_i \vec{v}_i\Big) &= \textstyle\sum_i c_i^*\, \mathscr{A}(\vec{v}_i) \quad \textit{for} \quad c_i \in \mathbb{C}, \\ \textit{or, less often,} \quad \mathscr{A}\Big(\textstyle\sum_i c_i \vec{v}_i\Big) &= -\textstyle\sum_i c_i\, \mathscr{A}(\vec{v}_i), \end{aligned} \tag{1.40}$$

it is called ***anti-linear****. (The latter choice is standard within quantum theory.)*

Somewhat confusingly, the *linear transformation* (1.38) is not a *linear operation* in the sense of definition 1.3:

$$\mathscr{L}(x) = ax + b, \qquad \text{then} \qquad \mathscr{L}(cx + c'x') = a(cx + c'x') + b = acx + ac'x' + b, \tag{1.41}$$

which is equal to

$$c\mathscr{L}(x) + c'\mathscr{L}(x') = c(ax + b) + c'(ax' + b) = acx + ac'x' + (c + c')b \tag{1.42}$$

only if $b = 0$. The analogous observation applies to the full, n-tuple linear transformation (1.38).

The additive terms b^i in (1.38) are called *inhomogeneities* or *translations*, and may be thought of as an obstruction for Eq. (1.38) to be linear in the sense of the definition 1.3. In turn, the *homogeneous* linear transformation

$$x^i \to y^i = a^i{}_j\, x^j, \qquad \textit{i.e.}, \qquad \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \to \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} = \begin{bmatrix} a^1{}_1 & \cdots & a^1{}_n \\ \vdots & \ddots & \vdots \\ a^n{}_1 & \cdots & a^n{}_n \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \tag{1.43}$$

is also linear in the sense of definition 1.3.

Finally, examining the transformation rules (1.6a)–(1.6b), which we may write akin to (1.43):

$$\begin{bmatrix} \mathrm{d}\eta^1 \\ \vdots \\ \mathrm{d}\eta^n \end{bmatrix} = \begin{bmatrix} \frac{\partial\eta^1}{\partial\xi^1} & \cdots & \frac{\partial\eta^1}{\partial\xi^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\eta^n}{\partial\xi^1} & \cdots & \frac{\partial\eta^n}{\partial\xi^n} \end{bmatrix} \begin{bmatrix} \mathrm{d}\xi^1 \\ \vdots \\ \mathrm{d}\xi^n \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \frac{\partial}{\partial\eta^1} \\ \vdots \\ \frac{\partial}{\partial\eta^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial\xi^1}{\partial\eta^1} & \cdots & \frac{\partial\xi^n}{\partial\eta^1} \\ \vdots & \ddots & \vdots \\ \frac{\partial\xi^1}{\partial\eta^n} & \cdots & \frac{\partial\xi^n}{\partial\eta^n} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial\xi^1} \\ \vdots \\ \frac{\partial}{\partial\xi^n} \end{bmatrix}, \tag{1.44}$$

we see that they are indeed *homogeneous* linear transformations as defined in (1.43).

The matrices $\left[\frac{\partial\eta^i}{\partial\xi^j}\right]$ and $\left[\frac{\partial\xi^i}{\partial\eta^j}\right]$ are far from constant, in general: for a general invertible transformation $\xi^i \to \eta^i = \eta^i(\xi)$, they will be appropriate non-trivial functions. However, the equations (1.44) are linear in the n-tuples of differentials and partial derivatives, respectively. We refer to the homogeneous linear transformations (1.44) as the linearization of the general invertible transformation $\xi^i \to \eta^i = \eta^i(\xi)$, which itself is nonlinear in general.

WoE 1.7: Consider the nonlinear change of variables

$$\eta^1 = \xi^1\xi^2, \quad \eta^2 = \frac{\xi^1}{\xi^2}; \qquad \xi^1 = \sqrt{\eta^1\eta^2}, \quad \xi^2 = \sqrt{\frac{\eta^1}{\eta^2}}. \tag{1.45}$$

Then, having in mind WoE 1.1, we have

$$\begin{bmatrix}\mathrm{d}\eta^1 & \mathrm{d}\eta^2\end{bmatrix} = \begin{bmatrix}\mathrm{d}\xi^1 & \mathrm{d}\xi^2\end{bmatrix}\begin{bmatrix}\xi^2 & \frac{1}{\xi^2}\\ \xi^1 & -\frac{\xi^1}{(\xi^2)^2}\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}\frac{\partial}{\partial\eta^1}\\ \frac{\partial}{\partial\eta^2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2\xi^2} & \frac{1}{2\xi^1}\\ \frac{1}{2}\xi^2 & -\frac{(\xi^2)^2}{2\xi^1}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial\xi^1}\\ \frac{\partial}{\partial\xi^2}\end{bmatrix}, \tag{1.46}$$

are both non-constant, homogeneous linear transformations $(\mathrm{d}\xi^1, \mathrm{d}\xi^2) \to (\mathrm{d}\eta^1, \mathrm{d}\eta^2)$ and $\left(\frac{\partial}{\partial\xi^1}, \frac{\partial}{\partial\xi^2}\right) \to \left(\frac{\partial}{\partial\eta^1}, \frac{\partial}{\partial\eta^2}\right)$, respectively. Note that

$$\begin{bmatrix}\xi^2 & \frac{1}{\xi^2}\\ \xi^1 & -\frac{\xi^1}{(\xi^2)^2}\end{bmatrix}\begin{bmatrix}\frac{1}{2\xi^2} & \frac{1}{2\xi^1}\\ \frac{1}{2}\xi^2 & -\frac{(\xi^2)^2}{2\xi^1}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2\xi^2} & \frac{1}{2\xi^1}\\ \frac{1}{2}\xi^2 & -\frac{(\xi^2)^2}{2\xi^2}\end{bmatrix}\begin{bmatrix}\xi^2 & \frac{1}{\xi^2}\\ \xi^1 & -\frac{\xi^1}{(\xi^2)^2}\end{bmatrix} \tag{1.47}$$

verifies that

$$\begin{bmatrix}\frac{\partial\eta^1}{\partial\xi^1} & \frac{\partial\eta^1}{\partial\xi^2}\\ \frac{\partial\eta^2}{\partial\xi^1} & \frac{\partial\eta^2}{\partial\xi^1}\end{bmatrix} = \begin{bmatrix}\frac{\partial\xi^1}{\partial\eta^1} & \frac{\partial\xi^1}{\partial\eta^2}\\ \frac{\partial\xi^2}{\partial\eta^1} & \frac{\partial\xi^2}{\partial\eta^2}\end{bmatrix}^{-1}, \qquad \text{so} \qquad \begin{bmatrix}\mathrm{d}\eta^1 & \mathrm{d}\eta^2\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial\eta^1}\\ \frac{\partial}{\partial\eta^2}\end{bmatrix} = \begin{bmatrix}\mathrm{d}\xi^1 & \mathrm{d}\xi^2\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial\xi^1}\\ \frac{\partial}{\partial\xi^2}\end{bmatrix}. \tag{1.48}$$

In fact, this is also easily verified by explicitly writing out the product of the two matrices of partials. ⌟

Thus, the n-tuples $(\mathrm{d}\xi^1, \cdots, \mathrm{d}\xi^n)$ and $(\mathrm{d}\eta^1, \cdots, \mathrm{d}\eta^n)$ may serve as two (contravariant) bases, whereas $\left(\frac{\partial}{\partial\xi^1}, \cdots, \frac{\partial}{\partial\xi^n}\right)$ and $\left(\frac{\partial}{\partial\eta^1}, \cdots, \frac{\partial}{\partial\eta^n}\right)$ may serve as two (covariant) bases. In fact, in any general coordinate system, $\mathrm{d}\vec{r}$ defines a contravariant basis for an n-dimensional vector space, and $\vec{\nabla}$ a covariant basis for a dual n-dimensional vector space[6].

One last remark is in order: In typical applications in physics, the generalized coordinates may well not have the same physical dimensions, and neither will then have their differentials or partial derivatives. For example, in spherical coordinates, r and $\mathrm{d}r$ have dimensions of *length* and $\frac{\partial}{\partial r}$ of *length*$^{-1}$, while $\theta, \mathrm{d}\theta, \frac{\partial}{\partial\theta}$ are all dimension-less. Thus, for $(\mathrm{d}r, \mathrm{d}\theta, \mathrm{d}\phi)$ to form a basis for a vector space, this *physics* consideration forces us to re-scale the differentials by scaling factors. To this end alone, the basis $(\mathrm{d}r, r\mathrm{d}\theta, r\mathrm{d}\phi)$ on one hand and $(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r}\frac{\partial}{\partial\phi})$ on the other would perfectly suffice, since all basis elements indeed have the same dimensions. However, examining the detailed decomposition of $\mathrm{d}\vec{r}$ in spherical coordinates, we see that the triple $(\mathrm{d}r, r\mathrm{d}\theta, r\sin(\theta)\mathrm{d}\phi)$ is appropriate, to which $(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi})$ is the dual basis. Here, we have used elementary geometry to determine the scaling factors $(1, r, r\sin(\theta))$ in spherical coordinate; see Eq. (1.112) for a definition of the proper scaling functions in the general case.

[6] The vector space obtained using the basis of $\mathrm{d}\vec{r}$ is called the cotangent space, and the vector space obtained using the basis of $\vec{\nabla}$ is called the tangent space.

1.1.4 Products of Vectors

Motivated by the result of WoE 1.1, we have

> ***Definition 1.4*** *Given two vectors, $\vec{A}$ specified in terms of a covariant basis (with contravariant coefficients A^i) and $\vec{B}$ specified in terms of a contravariant basis (with covariant coefficients B_j),*
>
> $$\boxed{\textbf{\textit{scalar product:}}\quad \vec{A}\cdot\vec{B} := A^i\,B_i.} \tag{1.49}$$
>
> *Two vectors are said to be* ***orthogonal*** *if $\vec{A}\cdot\vec{B} = 0$.*

WoE 1.8 *(General invariance of the scalar product)*: The scalar product of a covariant and a contravariant vector is invariant with respect to general invertible transformations:

$$A^i B_i \to A'^{\,i}\,B'_i = \Big(\frac{\partial\eta^i}{\partial\xi^j}A^j\Big)\Big(\frac{\partial\xi^k}{\partial\eta^i}B_k\Big) = A^j\,\frac{\partial\eta^i}{\partial\xi^j}\frac{\partial\xi^k}{\partial\eta^i}\,B_k = A^j\,\frac{\partial\xi^k}{\partial\xi^j}\,B_k = A^j\,\delta^k_j\,B_k = A^j B_j. \tag{1.50}$$

We recall that the Jacobian (determinant) of a transformation $\xi^i \to \eta^i$ occurs in the transformation of the volume differential:

$$\mathrm{d}\eta^1\cdots\mathrm{d}\eta^n = \Big|\frac{\partial\eta}{\partial\xi}\Big|\,\mathrm{d}\xi^1\cdots\mathrm{d}\xi^n, \tag{1.51a}$$

$$= \varepsilon_{ij\cdots kl}\,\frac{\partial\eta^i}{\partial\xi^1}\frac{\partial\eta^j}{\partial\xi^2}\cdots\frac{\partial\eta^k}{\partial\xi^{n-1}}\frac{\partial\eta^l}{\partial\xi^n}\,\mathrm{d}\xi^1\cdots\mathrm{d}\xi^n, \tag{1.51b}$$

$$= \varepsilon_{ij\cdots kl}\,\frac{\partial\eta^i}{\partial\xi^m}\frac{\partial\eta^j}{\partial\xi^n}\cdots\frac{\partial\eta^k}{\partial\xi^p}\frac{\partial\eta^l}{\partial\xi^q}\,\tfrac{1}{n!}\delta^{mn\cdots pq}_{[r\,s\cdots tu]}\,\mathrm{d}\xi^r\mathrm{d}\xi^s\cdots\mathrm{d}\xi^t\mathrm{d}\xi^u, \tag{1.51c}$$

$$= \underbrace{\varepsilon_{ij\cdots kl}\,\frac{\partial\eta^i}{\partial\xi^m}\frac{\partial\eta^j}{\partial\xi^n}\cdots\frac{\partial\eta^k}{\partial\xi^p}\frac{\partial\eta^l}{\partial\xi^q}\,\tfrac{1}{n!}\varepsilon^{mn\cdots pq}}_{}\;\underbrace{\tfrac{1}{n!}\varepsilon_{rs\cdots tu}\,\mathrm{d}\xi^r\mathrm{d}\xi^s\cdots\mathrm{d}\xi^t\mathrm{d}\xi^u}_{}. \tag{1.51d}$$

WoE 1.9 *(Antisymmetry of $\mathrm{d}\xi^i$)*: The expressions (1.51b)–(1.51d) imply that the product of the coordinate differentials is antisymmetric. At first, this may seem odd: "Shouldn't it be irrelevant whether we first integrate over $\mathrm{d}x$ and then over $\mathrm{d}y$ or the other way around?" However, the antisymmetry of this product has little to do with integration methods and all to do with the chosen orientation of the coordinate system.

As a simple illustration, consider the (x,y)-plane. One writes a surface integral as $\int \mathrm{d}x\,\mathrm{d}y\,\big[\cdots\big]$. With suitable integrands and limits (domains in the (x,y)-plane), this may well be computed integrating either first "over $\mathrm{d}y$" and then "over $\mathrm{d}x$," or the other way around—and the results of course must be independent of this order. Note, however, that the coordinate system of the (x,y)-plane is chosen typically so that the positive x-direction *precedes* the positive y-direction by $90°$, viewing the (x,y)-plane "from above" and regarding the counter-clockwise direction of rotations as positive. This is the familiar "right-handed" coordinate system.

The left-handed coordinate system would order the positive directions of the x- and y-axes in the opposite order, and the change from the left-handed to the right-handed coordinate system is easily obtained by performing the formal coordinate substitution $(x,y)\to(y,x)$.

As with *any* coordinate substitution, the integration element must be computed:

$$\begin{bmatrix}x\\ y\end{bmatrix} \to \begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, \tag{1.52}$$

$$\mathrm{d}x\,\mathrm{d}y \to \mathrm{d}\xi\,\mathrm{d}\eta = \det\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathrm{d}x\,\mathrm{d}y = -\mathrm{d}x\,\mathrm{d}y, \tag{1.53}$$

which is precisely the indication that the antisymmetric product of the (lexicographically ordered) coordinate differentials encodes the standard choice of the right-handed coordinate system.

Identifying the Jacobian (determinant) of the transformation:

$$\left|\frac{\partial\eta}{\partial\xi}\right| = \det\left[\frac{\partial\eta}{\partial\xi}\right] = \tfrac{1}{n!}\,\varepsilon_{ij\cdots kl}\,\frac{\partial\eta^i}{\partial\xi^m}\frac{\partial\eta^j}{\partial\xi^n}\cdots\frac{\partial\eta^k}{\partial\xi^p}\frac{\partial\eta^l}{\partial\xi^q}\varepsilon^{mn\cdots pq}, \tag{1.54}$$

and motivated by the form of (1.51), we define

Definition 1.5 *Given n vectors $\vec{A}_1,\cdots,\vec{A}_n$, we define their*

$$\boxed{\textbf{\textit{determinant product:}}\quad \det\left[\vec{A}_1,\ldots,\vec{A}_n\right] := \begin{cases}\varepsilon_{i_1\cdots i_n}\,A_1^{i_1}\cdots A_n^{i_n},\\ \varepsilon^{i_1\cdots i_n}\,A_{1,i_1}\cdots A_{n,i_n}\end{cases},} \tag{1.55}$$

depending on whether the contravariant or the covariant components have been supplied.

Remark 1.4: By comparison with (1.51b), the determinant product may be identified with the determinant of the matrix formed by stacking the n-tuples $\vec{A}_i$ in the i^{th} row. This makes it clear that $\det\left[\vec{A}_1,\ldots,\vec{A}_n\right]$ vanishes unless the n vectors $\vec{A}_1,\cdots,\vec{A}_n$ are all linearly independent.

WoE 1.10 *(Volume product transformation)*: The determinant product of n contravariantly specified vectors transforms:

$$\det\left[\vec{A}'_1,\ldots,\vec{A}'_n\right]_{\text{contr.}} = \varepsilon_{i_1\cdots i_n}\,A'^{\,i_1}_1\cdots A'^{\,i_n}_n = \varepsilon_{i_1\cdots i_n}\left(\frac{\partial\eta^{i_1}}{\partial\xi^{j_1}}A_1^{j_1}\right)\cdots\left(\frac{\partial\eta^{i_n}}{\partial\xi^{j_n}}A_n^{j_n}\right), \tag{1.56a}$$

$$= \varepsilon_{i_1\cdots i_n}\frac{\partial\eta^{i_1}}{\partial\xi^{j_1}}\cdots\frac{\partial\eta^{i_n}}{\partial\xi^{j_n}}\big(A_1^{j_1}\cdots A_n^{j_n}\big), \tag{1.56b}$$

$$= \varepsilon_{i_1\cdots i_n}\frac{\partial\eta^{i_1}}{\partial\xi^{j_1}}\cdots\frac{\partial\eta^{i_n}}{\partial\xi^{j_n}}\delta^{[j_1\cdots j_n]}_{k_1\cdots k_n}\big(A_1^{k_1}\cdots A_n^{k_n}\big), \tag{1.56c}$$

$$= \underbrace{\varepsilon_{i_1\cdots i_n}\frac{\partial\eta^{i_1}}{\partial\xi^{j_1}}\cdots\frac{\partial\eta^{i_n}}{\partial\xi^{j_n}}\varepsilon^{j_1\cdots j_n}\,\tfrac{1}{n!}}\;\underbrace{\varepsilon_{k_1\cdots k_n}\big(A_1^{k_1}\cdots A_n^{k_n}\big)}, \tag{1.56d}$$

$$\det\left[\vec{A}'_1,\ldots,\vec{A}'_n\right]_{\text{contr.}} = \left|\frac{\partial\eta}{\partial\xi}\right|\,\det\left[\vec{A}_1,\ldots,\vec{A}_n\right]_{\text{contr.}} \tag{1.56e}$$

with an overall factor of the Jacobian (determinant) of the general invertible coordinate change. The Reader should have no difficulty proving that

$$\det\left[\vec{A}'_1,\ldots,\vec{A}'_n\right]_{\text{cov.}} = \left|\frac{\partial\xi}{\partial\eta}\right|\,\det\left[\vec{A}_1,\ldots,\vec{A}_n\right]_{\text{cov.}} \tag{1.56f}$$

The result (1.56) proves:

Proposition 1.2 *The determinant product $\det\left[\vec{A}_1,\ldots,\vec{A}_n\right]$ is invariant with respect to coordinate transformations of unit Jacobian determinant, called "volume-preserving transformations" owing to the well-known result (1.51a).*

1.1.5 Vector Squares and the Metric

How do we square a vector? $\vec{A}^2 = A_i A^i$ is invariant under general transformations—as per Eq. (1.4)—but it requires *two* n-plets of components—$(A^1, \cdots, A^n)$ and $(A_1, \cdots, A_n)$—representing one and the same vector. Surely, there must exist a more economical way; after all, with the A^i's and the A_i's both representing the same vector, there must exist a (homogeneous) linear relationship between the two, presumably of the form $A_i = (\textit{something})_{ij} A^j$.

We are now after determining that "$(\textit{something})_{ij}$."

A second glance at the definitions 1.2 and 1.4 and WoE 1.8 confirms that the scalar product (1.49) is defined only as a product of a covariant and a contravariant vector. In particular, it applies neither to a product of two n-ples of covariant vector components nor to a product of two n-ples of contravariant ones. Nevertheless, we clearly do have such a thing, to wit:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \cdots = \mathrm{d}\vec{r} \cdot \mathrm{d}\vec{r} \tag{1.57}$$

is well-known to be the square of the infinitesimal line element in Euclidean geometry (where the Pythagorean theorem holds as usual). Writing this, however, in the more formal way:

$$\textbf{Euclidean/Pythagorean line element:} \quad \mathrm{d}s^2 = \sum_{i=1}^{n} (\mathrm{d}x^i)^2 = \mathrm{d}x^i\, \delta_{ij}\, \mathrm{d}x^j \tag{1.58}$$

reveals the implicit use of

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.59}$$

This object δ_{ij} is deceptively similar to δ^i_j, defined in Eq. (1.12), *but they are not the same*!

WoE 1.11: By contrast to this generally invariant δ^i_j and motivated by (1.58), consider the quantity δ_{ij}, needed in (1.58) and with values as in Eq. (1.59) but specified while referring to some general ξ-coordinates. Then change to some other, also general η-coordinates:

$$[\delta_{(\xi)}]_{ij} \;\mapsto\; [\delta_{(\eta)}]_{ij} = \frac{\partial \xi^k}{\partial \eta^i}\, [\delta_{(\xi)}]_{kl}\, \frac{\partial \xi^l}{\partial \eta^j} = \sum_k \frac{\partial \xi^k}{\partial \eta^i}\, \frac{\partial \xi^k}{\partial \eta^j}, \tag{1.60}$$

which does not simplify any further—certainly not for general initial ξ-coordinates and a general, invertible change into some other, general η-coordinates. This implies that the δ_{ij} quantity certainly does not stay invariant when changing coordinates in general—even if the initial ξ-coordinates had in fact been Cartesian coordinates!

We wish, however, to use the very useful Euclidean/Pythagorean square (1.58), and so we track how it varies when changing from Cartesian x-coordinates into some other, general ξ-coordinates:

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}x^i\, \delta_{ij}\, \mathrm{d}x^j &= \Big(\frac{\partial x^i}{\partial \xi^k}\, \mathrm{d}\xi^k\Big)\, \delta_{ij}\, \Big(\frac{\partial x^j}{\partial \xi^l}\, \mathrm{d}\xi^l\Big), \\ &= \mathrm{d}\xi^k \left[\frac{\partial x^i}{\partial \xi^k}\, \delta_{ij}\, \frac{\partial x^j}{\partial \xi^l}\right] \mathrm{d}\xi^l =: \mathrm{d}\xi^k\, g_{kl}(\xi)\, \mathrm{d}\xi^l \end{aligned} \tag{1.61}$$

where we recognize the δ_{ij}-like quantity:

$$\boxed{\textbf{the metric:}\quad g_{kl}(\xi) := \left[\frac{\partial x^i}{\partial \xi^k}\,\delta_{ij}\,\frac{\partial x^j}{\partial \xi^l}\right],} \tag{1.62}$$

which, for use in general ξ-coordinates, replaces the δ_{ij} used in the Cartesian coordinates to define the square of the line element (1.58). The metric (1.62) thereby permits us to measure distances in an Euclidean/Pythagorean fashion also in ξ-coordinates, and so is the key quantity in doing geometry in general coordinates.

Definition 1.6 *A coordinate system $(\xi^1, \cdots, \xi^n)$ is said to be **orthogonal** if its metric is diagonal: $g_{ij}(\xi) = 0$ for $i \neq j$.*

WoE 1.12: Consider the coordinates (ξ^1, ξ^2) specified in terms of the Cartesian ones as

$$\xi^1 = x^1 x^2, \quad \xi^2 = \frac{x^1}{x^2}; \qquad x^1 = \sqrt{\xi^1\xi^2}, \quad x^2 = \sqrt{\frac{\xi^1}{\xi^2}}. \tag{1.63}$$

Then

$$\left|\frac{\partial x}{\partial \xi}\right| = \det\begin{bmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} \end{bmatrix} = \det\begin{bmatrix} \frac{1}{2}\sqrt{\frac{\xi^2}{\xi^1}} & \frac{1}{2}\sqrt{\frac{\xi^1}{\xi^2}} \\ \frac{1}{2\sqrt{\xi^1\xi^2}} & -\frac{1}{2}\sqrt{\frac{\xi^1}{(\xi^2)^3}} \end{bmatrix} = -\frac{1}{2\xi^2} \tag{1.64}$$

is the Jacobian (determinant) of the coordinate transformation $(x^1, x^2) \to (\xi^1, \xi^2)$, so that

$$\mathrm{d}x^1\mathrm{d}x^2 = -\frac{1}{2\xi^2}\mathrm{d}\xi^1\mathrm{d}\xi^2, \tag{1.65}$$

and

$$g_{11}(\xi) = \frac{\partial x^1}{\partial \xi^1}\frac{\partial x^1}{\partial \xi^1} + \frac{\partial x^2}{\partial \xi^1}\frac{\partial x^2}{\partial \xi^1} = \frac{1+(\xi^2)^2}{4\xi^1\xi^2}, \tag{1.66}$$

$$g_{12}(\xi) = \frac{\partial x^1}{\partial \xi^1}\frac{\partial x^1}{\partial \xi^2} + \frac{\partial x^2}{\partial \xi^1}\frac{\partial x^2}{\partial \xi^2} = \frac{1}{4} - \frac{1}{4(\xi^2)^2} = g_{21}(\xi), \tag{1.67}$$

$$g_{22}(\xi) = \frac{\partial x^1}{\partial \xi^2}\frac{\partial x^1}{\partial \xi^2} + \frac{\partial x^2}{\partial \xi^2}\frac{\partial x^2}{\partial \xi^2} = \frac{\xi^1(1+(\xi^2)^2)}{4(\xi^2)^3}, \tag{1.68}$$

so that

$$[g_{ij}(\xi)] = \begin{bmatrix} \frac{1+(\xi^2)^2}{4\xi^1\xi^2} & \frac{1}{4} - \frac{1}{4(\xi^2)^2} \\ \frac{1}{4} - \frac{1}{4(\xi^2)^2} & \frac{\xi^1(1+(\xi^2)^2)}{4(\xi^2)^3} \end{bmatrix} \tag{1.69}$$

is the metric in the (ξ^1, ξ^2) coordinate system. Then

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2, \tag{1.70a}$$

$$= \frac{1+(\xi^2)^2}{4\xi^1\xi^2}(\mathrm{d}\xi^1)^2 + \Big(\frac{1}{2} - \frac{1}{2(\xi^2)^2}\Big)\mathrm{d}\xi^1\mathrm{d}\xi^2 + \Big(\frac{\xi^1(1+(\xi^2)^2)}{4(\xi^2)^3}\Big)(\mathrm{d}\xi^2)^2. \tag{1.70b}$$

Note that the multiplication of differentials in the line element—unlike in the volume element (1.65), and more generally (1.51)—is *symmetric*.

Upon comparing Eq. (1.58) and the end result in the derivation (1.61), we conclude in hindsight that

$$\boxed{\delta_{ij} \textit{ is the metric in Cartesian coordinates.}} \tag{1.71}$$

This reveals the tremendous simplicity—and so speciality—of geometry in Cartesian coordinates: just consider (1.70).

One might complain that the definition (1.62) still relies on Cartesian coordinates. In some sense, this is correct: given that we know the (extremely simple!) metric in Cartesian coordinates, we use (1.62) to compute metrics in other coordinates designed to describe the same space. However, if we happen to be given (by any means) the metric in *any* general η-coordinate system, $g_{ij}(\eta)$, from this we can compute the metric in any other, general ξ-coordinate system describing the same space:

$$g_{kl}(\xi) = \left[\frac{\partial\eta^i}{\partial\xi^k}\, g_{ij}(\eta)\, \frac{\partial\eta^j}{\partial\xi^l}\right]. \tag{1.72}$$

Remark 1.5: The formula (1.72), generalizing (1.62), in fact tells how g_{ij} transforms from one coordinate system to another. By comparing with (1.6b), we see that g_{ij} is covariant—in fact, *twice* so, since it picks up *two* $\frac{\partial\eta}{\partial\xi}$-factors when changing coordinates $\eta \to \xi$. We will return to this below.

Remark 1.6: Eqs. (1.62) and (1.72) imply that $g_{ij}(\xi) = g_{ji}(\xi)$, so that the components of a metric, $g_{ij}(\xi)$, form a symmetric matrix. We will write $g^{kl}(\xi)$ for the matrix-inverse of the metric:

$$g^{ij}(\xi) \;:\qquad g^{ij}(\xi)\, g_{jk}(\xi) = \delta^i_k = g_{kj}(\xi)\, g^{ji}(\xi). \tag{1.73}$$

Clearly, $g^{ij}(\xi) = g^{ji}(\xi)$ also.

Remark 1.7: Finally, comparing the results (1.27), (1.28) and (1.72), we see that

$$g(\xi) := \det\big[\, g_{ij}(\xi)\,\big] = \left|\frac{\partial\eta}{\partial\xi}\right|^2 \det\big[\, g_{ij}(\eta)\,\big] = \left|\frac{\partial\eta}{\partial\xi}\right|^2 g(\eta) \tag{1.74}$$

This allows us to re-define the determinant product so it is invariant:

Definition 1.7 *Given n vectors $\vec{A}_1, \cdots, \vec{A}_n$, we define their*

$$\boxed{\textbf{\textit{volume product:}}\quad \langle\vec{A}_1, \ldots, \vec{A}_n\rangle := \begin{cases} \sqrt{g}\,\varepsilon_{i_1\cdots i_n}\, A_1^{i_1}\cdots A_n^{i_n}, \\ \frac{1}{\sqrt{g}}\,\varepsilon^{i_1\cdots i_n}\, A_{1,i_1}\cdots A_{n,i_n}, \end{cases}} \tag{1.75}$$

depending on whether their contravariant or covariant components are provided; clearly, Eqs. (1.82) and (1.79), below, can always be employed to adapt a ragtag specification.

Remark 1.8: The transformation of the $\sqrt{g}$ pre-factor precisely balances the transformation of the "bare" volume-product (1.55). We therefore also have that

Corollary 1.3 *For a general coordinate system $(\xi^1, \cdots, \xi^n)$, we have the*

$$\boxed{\textbf{\textit{invariant volume element:}}\quad \mathrm{d}^n\vec{r} \;:=\; \tfrac{1}{n!}\sqrt{g(\xi)}\,\varepsilon_{i_1\cdots i_n}\,\mathrm{d}\xi^{i_1}\cdots\mathrm{d}\xi^{i_n} \;=\; \sqrt{g(\xi)}\,\mathrm{d}\xi^1\cdots\mathrm{d}\xi^n.} \tag{1.76}$$

The parenthetical $\frac{1}{n!}$ pre-factor is conventional, so that the reduction to the Cartesian case would turn out to coincide precisely with the usual $\mathrm{d}x^1\cdots\mathrm{d}x^n$.

1.1.6 Raising, Lowering and Contracting

The fundamental purpose—and definition—of the metric is so that one would be able to define the infinitesimal line element for a curve specified in general ξ-coordinates:

$$\boxed{\mathrm{d}s := \sqrt{g_{ij}(\xi)\,\mathrm{d}\xi^i\,\mathrm{d}\xi^j},} \tag{1.77}$$

but that is not all this quantity can do.

For, suppose we specify a vector in terms of its contravariant components, $(A^1(\xi), \cdots, A^n(\xi))$. Then we have that

$$A^i(\xi) \;\mapsto\; A^i(\eta) = \frac{\partial \eta^i}{\partial \xi^j}\,A^j(\xi), \tag{1.78}$$

but

$$A_i(\xi) := g_{ij}(\xi)\,A^j(\xi), \tag{1.79}$$

$$\begin{aligned} g_{ij}(\xi)\,A^j(\xi) \;\mapsto\; g_{ij}(\eta)\,A^j(\eta) &= \Big(\frac{\partial \xi^k}{\partial \eta^i}\,\frac{\partial \xi^l}{\partial \eta^j}\,g_{\kappa l}(\xi)\Big)\Big(\frac{\partial \eta^j}{\partial \xi^m}\,A^m(\xi)\Big),\\ &= \frac{\partial \xi^k}{\partial \eta^i}\,\frac{\partial \xi^l}{\partial \eta^j}\,\frac{\partial \eta^j}{\partial \xi^m}\,g_{\kappa l}(\xi)\,A^m(\xi) = \frac{\partial \xi^k}{\partial \eta^i}\,\frac{\partial \xi^l}{\partial \xi^m}\,g_{\kappa l}(\xi)\,A^m(\xi),\\ &= \frac{\partial \xi^k}{\partial \eta^i}\,\delta^l_m\,g_{\kappa l}(\xi)\,A^m(\xi) = \frac{\partial \xi^k}{\partial \eta^i}\,\big(g_{kl}(\xi)\,A^l(\xi)\big). \end{aligned} \tag{1.80}$$

That is, whereas $\{A^i, i = 1, \cdots, n\}$ are contravariant vector components, $\{(g_{ij}A^j), i = 1, \cdots, n\}$ transform as covariant components: *contraction* with the metric has effectively lowered[7] the index. More to the point, this demonstrates that for every vector of which the contravariant components are specified, we can construct the corresponding covariant(ized) components, and *vice versa*.

Conversely, given the covariant vector components, $(B_i(\xi), \cdots, B_n(\xi))$, we have that

$$B_i(\xi) \;\mapsto\; B_i(\eta) = \frac{\partial \xi^j}{\partial \eta^i}\,B_j(\xi), \tag{1.81}$$

but

$$B^i(\xi) := g^{ij}(\xi)\,B_j(\xi), \tag{1.82}$$

$$\begin{aligned} g^{ij}(\xi)\,B_j(\xi) \;\mapsto\; g^{ij}(\eta)\,B_j(\eta) &= \Big(\frac{\partial \eta^i}{\partial \xi^k}\,\frac{\partial \eta^j}{\partial \xi^l}\,g^{\kappa l}(\xi)\Big)\Big(\frac{\partial \xi^m}{\partial \eta^j}\,B_m(\xi)\Big),\\ &= \frac{\partial \eta^i}{\partial \xi^k}\,\frac{\partial \eta^j}{\partial \xi^l}\,\frac{\partial \xi^m}{\partial \eta^j}\,g^{kl}(\xi)\,B_m(\xi) = \frac{\partial \eta^i}{\partial \xi^k}\,\frac{\partial \xi^m}{\partial \xi^l}\,g^{kl}(\xi)\,B_m(\xi),\\ &= \frac{\partial \eta^i}{\partial \xi^k}\,\delta^m_l\,g^{kl}(\xi)\,B_m(\xi) = \frac{\partial \eta^i}{\partial \xi^k}\,\big(g^{kl}(\xi)\,B_l(\xi)\big). \end{aligned} \tag{1.83}$$

[7] This then is referred to as the *musical isomorphism* ***flat***, and denoted: $(\vec{A}^\flat)|_i = (g_{ij}A^j)$.

That is, whereas $\{B_i, i=1,\cdots,n\}$ are covariant vector components, $\{(g^{ij}B_j), i=1,\cdots,n\}$ transform as components of a contravariant one: *contraction* with the *inverse* metric has effectively raised[8] the index.

Thus, declaring a vector itself as covariant or contravariant depends on the choice of how the components have been specified—and so is not an invariant property. However, the relative difference between the transformation properties of $\mathrm{d}\vec{r}$ and $\vec{\nabla}$—within the same framework and system of conventions—is an invariant property: Within the framework and conventions where the components of $\mathrm{d}\vec{r}$ are *naturally* contravariant, those of $\vec{\nabla}$ are *naturally* covariant. It is of course possible to define

$$\mathrm{d}\vec{r}: \quad \mathrm{d}\xi^i \mapsto \mathrm{d}\xi_i := g_{ij}(\xi)\,\mathrm{d}\xi^j \qquad \text{covariant(ized) } \mathrm{d}\vec{r} \text{ components,} \tag{1.84}$$

$$\vec{\nabla}: \quad \frac{\partial}{\partial \xi^i} \mapsto \frac{\partial}{\partial \xi_i} := g^{ij}(\xi)\,\frac{\partial}{\partial \xi^j} \qquad \text{contravariant(ized) } \vec{\nabla} \text{ components,} \tag{1.85}$$

but it is plain that this redefinition requires the use of the metric. Another indication that $\mathrm{d}\vec{r}$ and $\vec{\nabla}$ are *naturally dual* to each other is the intrinsic invariance of the exterior derivative (1.10) in WoE 1.1 and its formal analogue (1.49), defined without the use of the metric and so requiring no knowledge thereof.

To summarize: had we decided to index coordinates by a subscript instead, all index positions would be reversed throughout—but would not change the relative *inverseness* of the two definitions 1.2. It is standard to pick one convention ($\mathrm{d}\xi^i$ to be called contravariant), and understand all definitions to be relative to this choice. Given this preferred choice, we will say that a vector itself is contra- or co-variant, depending on how its components transform—as defined, without using the metric to raise or lower indices, and with respect to such a preferred choice.

— ⋆ —

Using the metric, it is possible to generalize the definition 1.4:

Definition 1.8 *Given two vectors, $\vec{A}$ and $\vec{B}$, and the metric g_{ij} we define:*

$$\boxed{\textbf{\textit{scalar product:}} \quad \vec{A}\cdot\vec{B} := A^i\,B_i := A^i\,g_{ij}\,B^j := A_i\,g^{ij}\,B_j,} \tag{1.86}$$

by defining

$$A_i := g_{ij}\,A^j = A^j\,g_{ij} = A^j\,g_{ji}, \qquad \textit{index-lowering}, \tag{1.87a}$$

$$B^j := B_i\,g^{ij} = g^{ij}\,B_i = g^{ji}\,B_i, \qquad \textit{index-raising}, \tag{1.87b}$$

having used, respectively, the "musical isomorphisms" (1.79) and (1.82).

Remark 1.9: Whereas "being able to do geometry analytically" presumes the knowledge of the metric and so warrants the extended definition 1.8 and (1.86), the explicit writing of the metric in these formulae reminds us of this. In turn, no metric was required in the definition 1.4, as motivated by the exterior derivative WoE 1.1. Therefore, the scalar product $\mathrm{d}\vec{r}\cdot\vec{\nabla} = \mathrm{d}\xi^i \frac{\partial}{\partial\xi^i}$ is simpler

[8] This then is referred to as the *musical isomorphism* ***sharp***, and denoted: $(\vec{B}^{\,\sharp})|_i = (g^{ij}B_j)$.

(more primitive) than, say, $\mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r} = \mathrm{d}\xi^i\, g_{ij}\,\mathrm{d}\xi^j$, which requires the additional knowledge of a metric. Scalar products such as $\mathrm{d}\vec{r}\cdot\vec{\nabla}$ are often referred to as *natural* or *canonical*—foremost because the two factors *naturally* transform oppositely, and no knowledge of any metric is needed. In turn, upon providing the additional information of a chosen metric, the "musical isomorphisms" (1.87) permit the components of any vector to be expressed as covariant as easily as contravariant, thus blurring the distinction.

— ⋆ —

We pause to point out a singular opportunity in $n = 3$: Since it now takes three vectors to form an volume product a derivative of this product by the components of one of the factors defines a product of the remaining two vectors such that this product must transform inversely to the vector with respect to which we took the derivative! Or, put differently, we may define the k^{th} component of this vector product:

$$\textbf{\textit{n} = 3 vector product:}\quad \begin{cases} (\vec{A}\times\vec{B})_k := \langle\vec{A},\vec{B},\underline{\ \ }\rangle_k = \sqrt{g}\,\varepsilon_{ijk}\,A^i\,B^j, \\ (\vec{A}\times\vec{B})^k := \langle\vec{A},\vec{B},\underline{\ \ }\rangle^k = \frac{1}{\sqrt{g}}\,\varepsilon^{ijk}\,A_i\,B_j, \end{cases} \tag{1.88}$$

Remark 1.10: The contravariant components of $\vec{A},\vec{B}$ used in this formula are easily obtained by (1.82), should the covariant components have been given instead. Also, the intermediate, defining, expression (1.88) shows that $\vec{A}\times\vec{B}$ may be thought of as the *functional* which when evaluated on the vector $\vec{C}$ produces the (ground field $\Bbbk$-valued) volume product, $\langle\vec{A},\vec{B},\vec{C}\rangle$. Finally, comparison of (1.88), (1.86) and (1.75) implies straightforwardly that

$$\langle\vec{A},\vec{B},\vec{C}\rangle = \big(\vec{A}\times\vec{B}\big)\cdot\vec{C}, \tag{1.89}$$

from which follows the "infinitesimal volume element" formula given in most other texts.

Remark 1.11: More generally (not requiring $n = 3$), the quantity

$$\langle\vec{A}_1,\vec{A}_2,\cdots,\vec{A}_{n-1},\underline{\ \ }\rangle \tag{1.90}$$

still defines a (volume-dual) vector product of $n{-}1$ vectors $\vec{A}_1,\cdots,\vec{A}_{n-1}$. As the so-implied, so-called "$n{-}1$-ary multiplication structures" turn out to be rather both less well studied in mathematics and less applied in physics and engineering, we will not devote this any more time.

While we're at it, we might as well define the extreme opposite of (1.88):

$$\textbf{the Hodge-dual}\text{ of a vector:}\quad \begin{cases} *\vec{A}|_{i_1,\cdots,i_{n-1}} = \sqrt{g}\,\varepsilon_{i_1,\cdots,i_{n-1},i_n}\,A^{i_n}, \\ *\vec{A}|^{i_1,\cdots,i_{n-1}} = \frac{1}{\sqrt{g}}\,\varepsilon^{i_1,\cdots,i_{n-1},i_n}\,A_{i_n}. \end{cases} \tag{1.91}$$

1.2 More Building Blocks

Comparing the formula (1.72) with those in definition (1.2), we see that the formula (1.72) provides a generalization of the notion of a co- and contra-variant vector: whereas the transformation of a vector involves the occurrence of a single factor $\frac{\partial\xi}{\partial\eta}$ or its inverse—as specified in (1.6a)–(1.6b), the metric requires two such factors (1.72). It is then clear how to generalize this:

Definition 1.9 *(Tensor) An array of quantities, $T^{i_1\cdots i_p}_{j_1\cdots j_q}(\xi)$, specified as functions in the ξ-coordinate system, are elements of a $(\boldsymbol{p},\boldsymbol{q})$**-tensor**, $\mathbb{T}$, precisely if they transform like:*

$$\boxed{T^{i_1\cdots i_p}_{j_1\cdots j_q}(\xi) \;\mapsto\; T^{i_1\cdots i_p}_{j_1\cdots j_q}(\eta) \;=\; \frac{\partial\eta^{i_1}}{\partial\xi^{k_1}}\cdots\frac{\partial\eta^{i_p}}{\partial\xi^{k_p}}\frac{\partial\xi^{l_1}}{\partial\eta^{j_1}}\cdots\frac{\partial\xi^{l_p}}{\partial\eta^{j_q}}\,T^{k_1\cdots k_p}_{l_1\cdots l_q}(\xi).} \tag{1.92}$$

*We say that its **rank** is $p{+}q$, and its **type** is (p,q).*

Remark 1.12: A covariant vector is thus a $(0,1)$-tensor, a contravariant vector is a $(1,0)$-tensor, a scalar is a $(0,0)$-tensor, the metric, $g_{ij}(\xi)$, is a $(0,2)$-tensor, and δ^i_j is a $(1,1)$-tensor.

It is also true, completely generally, that a Cartesian product of a (p,q)-tensor and an (r,s)-tensor is a $((p{+}r),(q{+}s))$-tensor; see the Lexicon B. Thus, from a contravariant vector, *i.e.*, a $(1,0)$-tensor and the metric $(0,2)$-tensor, we can construct the Cartesian product

$$\{\,A^i,\ i=1,\cdots,n\,\}\times\{\,g_{jk},\ j,k=1,\cdots,n\,\}\ \longrightarrow\ \{\,A^i\,g_{jk},\ i,j,k=1,\cdots,n\,\}, \tag{1.93}$$

of which Eq. (1.80) considers the *contraction:*

$$g_{ij}\,A^j := g_{ij}\,A^j = \delta^j_k\,g_{ij}\,A^k, \quad = A_i. \tag{1.94}$$

Similarly, Eq. (1.83) considers the *contraction:*

$$g^{ij}\,B_j := g^{ij}\,B_j = \delta^k_j\,g^{ij}\,B_k, \quad = B^i. \tag{1.95}$$

We have already seen that δ^i_j does not change at all when we change from the initial, general ξ-coordinates to any other, general η-coordinates. It was also used in defining contractions of tensors: see Eqs. (1.94)–(1.95) and the Lexicon B entry. And, since δ^i_j is a $(1,1)$-tensor, the contraction maps a (p,q)-tensor into a $((p{-}1),(q{-}1))$-tensor. That is, to contract a (p,q)-tensor, it must be that $p,q>0$.

A few more examples are as follows:

WoE 1.13 *(Contractions)*: Let $A_{ij}, B_{ij}{}^k, C^{ijkl}$ be tensors of rank-2, -3 and -4, respectively, and type-$(0,2)$, -$(1,2)$ and -$(4,0)$ as indicated by their indices. Let g_{ij} be the metric tensor and g^{ij} its inverse, as usual. Then

Contraction	Rank	Type
$g^{ij}A_{ij}$	0	$(0,0)$
$g^{ij}B_{ij}{}^k$	1	$(1,0)$
$\delta^i_k B_{ij}{}^k := B_{ij}{}^i$	1	$(0,1)$
$\delta^j_k B_{ij}{}^k := B_{ij}{}^j$	1	$(0,1)$
$g_{ij}C^{ijkl}$	2	$(2,0)$
$g_{ij}g_{kl}C^{ijkl}$	0	$(0,0)$

Contraction	Rank	Type
$A_{ij}\,\delta^i_m\,B_{kl}{}^m = A_{ij}\,B_{kl}{}^i$	3	$(0,3)$
$A_{ij}\,g^{ik}\,B_{kl}{}^m$	3	$(1,2)$
$A_{ij}\,\delta^i_m\,g^{jk}\,B_{kl}{}^m = A_{ij}\,g^{jk}\,B_{kl}{}^i$	1	$(0,1)$
$A_{ij}\,g^{il}\,g^{jk}\,B_{kl}{}^m$	1	$(1,0)$
$A_{ij}\,\delta^i_m\,g^{jk}\,B_{kl}{}^m\,g^{ln} = A_{ij}\,g^{jk}\,B_{kl}{}^i\,g^{ln}$	1	$(1,0)$
$A_{ij}\,g^{il}\,g^{jk}\,B_{kl}{}^m\,g_{mn}$	1	$(0,1)$

are some of the possible contractions. The Reader should have no difficulty verifying all the table entries and creating many more.

However, if we restrict somehow the class of coordinate transformations, there may well exist tensors that happen to be invariant with respect to such a restricted class of coordinate transformations. In fact, we can turn this around: select a particular tensor and then restrict to those coordinate transformations which leave the chosen tensor invariant.

— ⋆ —

Yet more general than tensors—but still quite "orderly" in their transformation properties are quantities such as the determinant product (1.55) and the determinant of the metric (1.74): their transformation includes also a power of the Jacobian (determinant) of the transformation:

> ***Definition 1.10** (Tensor Density)* *An array of quantities, $T^{i_1\cdots i_p}_{j_1\cdots j_q}(\xi)$, specified as functions in the ξ-coordinate system, are elements of **(p,q)-tensor density** of **weight w**, $\mathbb{T}$, precisely if they transform like:*
>
> $$\boxed{T^{i_1\cdots i_p}_{j_1\cdots j_q}(\xi) \;\mapsto\; T^{i_1\cdots i_p}_{j_1\cdots j_q}(\eta) \;=\left|\frac{\partial\eta}{\partial\xi}\right|^w \frac{\partial\eta^{i_1}}{\partial\xi^{k_1}}\cdots\frac{\partial\eta^{i_p}}{\partial\xi^{k_p}}\frac{\partial\xi^{l_1}}{\partial\eta^{j_1}}\cdots\frac{\partial\xi^{l_q}}{\partial\eta^{j_q}}\,T^{k_1\cdots k_p}_{l_1\cdots l_q}(\xi).} \tag{1.96}$$
>
> *We say that its **rank** is $p{+}q$, its **type** is (p,q) and its **weight** is w.*

1.2.1 Structure-Preserving Transformations

In various considerations, we may wish to restrict the changes of variables so as to preserve a defined quantity or structure.

Volume-Preserving Transformations: These are the special class of coordinate transformations that satisfy[9]:

$$\boxed{\textbf{volume-preserving:}\quad (\xi^1,\cdots,\xi^n)\to(\eta^1,\cdots,\eta^n)\;:\quad \det\left[\frac{\partial(\xi^1,\cdots,\xi^n)}{\partial(\eta^1,\cdots,\eta^n)}\right]\overset{!}{=}1.} \tag{1.97}$$

Rotations: The metric being of special interest in many considerations, we may wish to restrict to allowing only coordinate changes that leave a given metric invariant. To this end, we'd restrict:

$$(\xi^1,\cdots,\xi^n)\to(\eta^1,\cdots,\eta^n)\;:\quad g_{ij}(\eta)=\frac{\partial\xi^k}{\partial\eta^i}\frac{\partial\xi^l}{\partial\eta^j}\,g_{kl}(\xi)\overset{!}{=}g_{kl}(\xi) \tag{1.98}$$

In this case, g_{ij} is also an invariant tensor (trivially, with respect to g_{ij}-preserving transformations), but so are then also all the tensors one can construct from products of δ^i_j, g_{ij}, g^{ij}.

In the special case of such a restriction, when the original ξ-coordinates may in fact be transformed into Cartesian and

$$g_{ij}(\xi)=\delta_{ij}=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases} \tag{1.99}$$

the condition (1.62) becomes

$$(x^1,\cdots,x^n)\to(\xi^1,\cdots,\xi^n)\;:\quad \frac{\partial x^k}{\partial\xi^i}\frac{\partial x^l}{\partial\xi^j}\,\delta_{kl}\overset{!}{=}\delta_{ij}. \tag{1.100}$$

---

[9] The exclaimed equality symbol "$\overset{!}{=}$" denotes that we demand the indicated equality to hold. By the same token, "$\overset{?}{=}$" would question the so-indicated equality.

Rewritten in matrix notation, this is

$$\left[\frac{\partial x}{\partial \xi}\right]\left[\frac{\partial x}{\partial \xi}\right]^T = \mathbb{1}, \qquad \textit{i.e.} \qquad \left[\frac{\partial x}{\partial \xi}\right]^T = \left[\frac{\partial x}{\partial \xi}\right]^{-1}, \tag{1.101}$$

which defines *orthogonal* matrices. It is not hard to show that the product of two orthogonal matrices is again orthogonal, that the identity matrix, $\mathbb{1}$, is in fact orthogonal, and that every orthogonal matrix has a unique inverse which is again orthogonal. Such $n \times n$ matrices—and coordinate transformations—therefore form a group, denoted $O(n)$. Restricting in addition to volume-preserving transformations restricts $O(n) \to SO(n)$, and these are well-known to represent familiar rotations.

1.3 More on Bases

1.3.1 Bases

The example WoE 1.1 indicates the possibility of constructing—from co- and contra-variantly defined components of vectors—objects that are invariant with respect to general invertible transformations of coordinates. To specify such constructions, we may select a basis (having the maximal number of linearly independent vectors) for any given vector space—one from many.

Covariant Basis: Having chosen a particular coordinate system $(\xi^1, \cdots, \xi^n)$, we may introduce "coordinate vectors", $\vec{\xi}^i$, to serve as a place-holder or order-counter for the i^{th} component of vectors[10]:

$$\boxed{\textbf{covariant basis vectors:} \quad \vec{\xi}_i := \frac{\partial \vec{r}}{\partial \xi^i}} \tag{1.102}$$

where $\vec{r}$ is the "position vector," specifying the considered point in the ξ-space. Using the basis elements $\vec{\xi}_i$ within the given vector space, a generic linear combination:

$$\vec{A} := A^i(\xi)\,\vec{\xi}_i, \qquad A^i(\eta) = \frac{\partial \eta^i}{\partial \xi^j}\,A^j(\xi), \tag{1.103}$$

specifies $\vec{A}$ in terms of its contravariant vector components.

Contravariant Basis: Given the covariant basis (1.102) and the metric (1.62), we define:

$$\boxed{\textbf{contravariant basis vectors:} \quad \vec{\xi}^i := g^{ij}(\xi)\,\vec{\xi}_j} \tag{1.104}$$

indicating basis vectors that transform contravariantly, the way the $\mathrm{d}\xi^i$ do. Using the basis elements $\vec{\xi}_i$ within the given vector space, a generic linear combination:

$$\vec{A} := A_i(\xi)\,\vec{\xi}^i, \qquad A_i(\eta) = \frac{\partial \xi^j}{\partial \eta^i}\,A_j(\xi), \tag{1.105}$$

specifies $\vec{A}$ in terms of its covariant vector components.

---

[10] The n-tuple $(\xi^1, \cdots, \xi^n)$ is a vector only for Cartesian coordinates: only therein do such n-tuples of coordinates satisfy the definition 1.1. This is another way of seeing how special Cartesian coordinates are.

Comparing (1.103) with (1.105) reveals that

$$\vec{A} = A^i(\xi)\,\vec{\xi}_i \;= A_j(\xi)\,g^{ji}(\xi)\,\vec{\xi}_i \;= A_j(\xi)\,\vec{\xi}^j \tag{1.106}$$

the same vector has both covariant components (when expressed in a frame of contravariant coordinate vector basis) and contravariant components (with respect to a covariant basis) and one can transform covariant quantities into contravariant ones using the metric.

On Transformations and Invariance: There is a notion of *active* and *passive* transformations in the literature. The former actually changes the physical state of the system being described, and so can be represented if *either* the components *or* the basis vectors transform, but not both. No vector can be invariant with respect to such a "one-sided" transformation; that manifestly is *not* what we are discussing here. In turn, the latter are transformations in the description of a physical system, with respect to which the described system itself indeed should be invariant—all the various descriptions being assumed to be faithful. In this sense, we are herein discussing these latter, passive transformations.

Note that neither the $\vec{\xi}_i$ nor the $\vec{\xi}^i$ are normalized in any sense—since no norm has been defined so far for these vectors.

The vector $\vec{A}$ itself is an invariant quantity:

$$\vec{A} = A^i\vec{\xi}_i \mapsto \Big(\frac{\partial\xi^i}{\partial\eta^j}A^j\Big)\Big(\frac{\partial\eta^k}{\partial\xi^i}\vec{\eta}_k\Big) = A^j\Big(\frac{\partial\eta^k}{\partial\xi^i}\frac{\partial\xi^i}{\partial\eta^j}\Big)\vec{\eta}_k = A^j\Big(\frac{\partial\eta^k}{\partial\eta^j}\Big)\vec{\eta}_k = A^j\Big(\delta^k_j\Big)\vec{\eta}_k = A^j\vec{\eta}_j. \tag{1.107}$$

Indeed, the vector (possibly representing a physical quantity) should not depend on how we choose our frame of unit vectors, *i.e.*, coordinates.

Remark 1.13: Since the vector $\vec{r}$ is invariant with respect to arbitrary (passive) transformations, the definition (1.102) and the computation (1.103)–(1.107) make it clear that $\vec{\xi}_i$ transforms the same as $\frac{\partial}{\partial\xi^i}$ does. Indeed, the partial derivative operators $\frac{\partial}{\partial\xi^i}$ may well be regarded as covariant vector basis elements. Herein, we avoid using differential operators as basis elements, but the Reader should note this oft-used possibility.

1.3.2 Coordinate Vectors, Ortho-Normalized

The sequence of equalities $\vec{A} = A^i\vec{\xi}_i = A^i(g_{ij}\vec{\xi}^j) = (A^i g_{ij})\vec{\xi}^j = A_i\vec{\xi}^i$ demonstrates that a vector itself is neither covariant nor contravariant, but that its components may be specified as contravariant or covariant, depending on whether we choose to specify them with respect to covariant or contravariant basis elements.

Caution: since the $\vec{\xi}_i$ and the $\vec{\xi}^i$ are not unit-normalized, the coefficients A^i and A_i in (1.103)–(1.105) are in the expressions (1.103)–(1.107) specified not with respect to unit-normalized basis vectors and so differ from the conventions adopted by most texts at the start.

We thus turn to this normalization.

Returning to the square of the line element, it must be expressible as the scalar product of the differential of the position vector, $\mathrm{d}\vec{r}$:

$$\mathrm{d}s^2 = \mathrm{d}\vec{r}\cdot\mathrm{d}\vec{r} = \big(\mathrm{d}\xi^i\,\vec{\xi}_i\big)\cdot\big(\mathrm{d}\xi^j\,\vec{\xi}_j\big) = \mathrm{d}\xi^i\,\mathrm{d}\xi^j\,\big(\vec{\xi}_i\cdot\vec{\xi}_j\big), \tag{1.108a}$$
$$= g_{ij}(\xi)\,\mathrm{d}\xi^i\,\mathrm{d}\xi^j. \tag{1.108b}$$

Comparing the two results (1.108a)–(1.108b) implies that

$$\vec{\xi}_i\cdot\vec{\xi}_j = g_{ij}(\xi), \tag{1.109}$$

thus *defining* what the $\vec{\xi}_i\cdot\vec{\xi}_j$ product should mean. Then,

$$\vec{\xi^i}\cdot\vec{\xi^j} = \big(g^{ik}\vec{\xi}_k\big)\cdot\big(g^{jl}\vec{\xi}_l\big) = g^{ik}\,g^{jl}\big(\vec{\xi}_k\cdot\vec{\xi}_l\big) = g^{ik}\,g^{jl}\,g_{kl} = g^{ik}\,\delta^j_k = g^{ij}. \tag{1.110}$$

Using this, we can use the coordinate vectors to *project* components of a vector:

$$\vec{\xi}_i\cdot\vec{A} = \vec{\xi}_i\cdot\big(\,A^j\,\vec{\xi}_j\,\big) = (\vec{\xi}_i\cdot\vec{\xi}_j)\,A^j = g_{ij}\,A^j =: A_i\ :\quad \text{covariant(ized) component,} \tag{1.111a}$$
$$\vec{\xi^i}\cdot\vec{A} = \vec{\xi^i}\cdot\big(\,A^j\,\vec{\xi}_j\,\big) = (\vec{\xi^i}\cdot\vec{\xi}_j)\,A^j = \delta^i_j\,A^j = A^i\ \ :\quad \text{contravariant(ized) component,} \tag{1.111b}$$

This demonstrates again that the transformation properties of the components of a vector depend on how—with respect to which basis, $\{\vec{\xi}_1,\cdots,\vec{\xi}_n\}$ or $\{\vec{\xi^1},\cdots,\vec{\xi^n}\}$—the components have been specified, echoing (yet again) an earlier conclusion.

Since $g_{ij}(\xi)$ and $g^{ij}(\xi)$ are nontrivial and non-constant functions in general coordinate systems, we see that even the norm of $\vec{\xi}_i$ is not constant:

$$\left.\begin{array}{l} \|\vec{\xi}_i\| := \sqrt{\vec{\xi}_i\cdot\vec{\xi}_i} \;=\sqrt{g_{ii}(\xi)} \\[1ex] \|\vec{\xi^i}\| := \sqrt{\vec{\xi^i}\cdot\vec{\xi^i}} \;=\sqrt{g^{ii}(\xi)} \end{array}\right\}\qquad \begin{array}{l}\text{for all } i=1,\cdots,n,\\ \text{with no summation.}\end{array} \tag{1.112}$$

In general, $g_{ii}(\xi) \neq \big(g^{ii}(\xi)\big)^{-1}$, and the proper scaling factors, $\|\vec{\xi}_i\|^{-1}$ and $\|\vec{\xi^i}\|^{-1}$ differ.

However, in orthogonal coordinate systems, $g^{\perp}_{ii}(\xi) = \big(g^{ii}_{\perp}(\xi)\big)^{-1} = {h_i}^2(\xi)$, where the indices "$\perp$" are meant to remind of the orthogonality of the considered coordinate system. The coefficients $h_i(\xi) := \sqrt{g^{\perp}_{ii}(\xi)} = 1\Big/\sqrt{g^{ii}_{\perp}(\xi)}$ are called scaling factors.

Restrict to orthogonal coordinates for the rest of this section.

To distinguish components in a unit-normalized orthogonal basis (used by Ref. [1]) from those that are not (as used in the preceding several sections), we will hereafter write

$$\vec{A} = A_i\,\vec{\xi^i} = A_{\hat{\imath}}\,\hat{\mathrm{e}}^i = A^{\hat{\imath}}\,\hat{\mathrm{e}}_i = A^i\,\vec{\xi}_i, \tag{1.113}$$

where (no summation on any indices):

$$\hat{\mathrm{e}}_i := \|\vec{\xi}_i\|^{-1}\vec{\xi}_i = \frac{1}{\sqrt{g_{ii}}}\,\vec{\xi}_i = {h_i}^{-1}\,\vec{\xi}_i, \qquad \hat{\mathrm{e}}^i := \|\vec{\xi^i}\|^{-1}\vec{\xi^i} = \frac{1}{\sqrt{g^{ii}}}\,\vec{\xi^i} = h_i\vec{\xi^i}, \tag{1.114}$$

$$\vec{\xi}_i = \sqrt{g_{ii}}\,\hat{e}_i = h_i\,\hat{e}_i, \qquad\qquad \vec{\xi}^i = \sqrt{g^{ii}}\,\hat{e}^i = h_i{}^{-1}\,\hat{e}^i. \tag{1.115}$$

Therefore, whereas

$$\vec{\xi}^i = g^{ij}\,\vec{\xi}_j \tag{1.116}$$

is a simple relationship, the general relationship between unit-normalized vectors

$$\sqrt{g^{ii}}\,\hat{e}^i = \sum_j g^{ij}\sqrt{g_{jj}}\,\hat{e}_j, \qquad \text{no sum on } i, \tag{1.117}$$

is not. However, for orthogonal coordinate systems, where $g_{ij} = h_i{}^2\,\delta_{ij}$ and $g^{ij} = h_j{}^{-2}\,\delta^{ij}$ with no summation over i, this simplifies into:

$$h_i^{-1}\,\hat{e}^i = \sum_j \big(h_j{}^{-2}\,\delta^{ij}\big)h_j\,\hat{e}_j = h_i{}^{-1}\,\hat{e}_i, \qquad \Rightarrow \qquad \boxed{\hat{\mathbf{e}}^{\boldsymbol{i}}_{\perp} = \hat{\mathbf{e}}^{\perp}_{\boldsymbol{i}}}. \tag{1.118}$$

The simplifications due to this result are made manifest in Table 1 by labeling the simplifying equations as "$\stackrel{\perp}{=}$." It is fairly clear from Table 1 that the definition of vector components with

$\vec{\xi}_i$-basis	$\vec{\xi}^i$-basis	$\hat{e}^i$-basis	$\hat{e}_i$-basis
A^i	$A_i = \sum_j g_{ij} A^j$	$A_{\hat{\imath}} = \sum_j \sqrt{g^{ii}} g_{ij} A^j \stackrel{\perp}{=} h_i\, A^i$	$A^{\hat{\imath}} = \sqrt{g_{ii}} A^i \stackrel{\perp}{=} h_i\, A^i$
$A^i = \sum_j g^{ij} A_j$	A_i	$A_{\hat{\imath}} = \sqrt{g^{ii}} A_i \stackrel{\perp}{=} h_i{}^{-1}\, A_i$	$A^{\hat{\imath}} = \sum_j \sqrt{g_{ii}}\, g^{ij} A_j \stackrel{\perp}{=} h_i{}^{-1}\, A_i$

Table 1: The conversion relations among the four possible definitions of components of a vector, with respect to the $\vec{\xi}_i$-basis, the $\vec{\xi}^i$-basis, the $\hat{e}^i$-basis and the $\hat{e}_i$-basis. Throughout the table, only explicitly written summations are implied. Also, the conversion relations involving the $\hat{e}^i$- and $\hat{e}_i$-bases tacitly assume orthogonality of the coordinate system.

respect to the unit-normalized $\hat{e}_i$- and $\hat{e}^i$-basis turns out indistinguishable and so "half-way" between the $\vec{\xi}_i$- and $\vec{\xi}^i$-basis. While this does turn out to be economical—when it can be applied—it equivocates between all the diverse quantities discussed above, the differences between which stem solely from the difference of the transformation of natural components of $\mathrm{d}\vec{r}$ and $\vec{\nabla}$. Also, the thoroughly consistent introduction and use of unit-normalized basis vectors is less elegant (to be modest) in generic, non-orthogonal coordinate systems.

1.4 Derivatives of Vectors

As a warm-up, note that the $\vec{\nabla}$-derivative of a scalar function is straightforward, since there is only one type of product between the vector $\vec{\nabla}$ and the scalar, $f(\vec{r})$. Then,

$$\vec{\nabla} f \equiv \mathbf{grad}(f) = \vec{\xi}^i\,\frac{\partial f}{\partial \xi^i}. \tag{1.119}$$

1.4.1 Variations of Coordinate Vectors

Being that vectors are linear combinations $A^i(\xi)\,\vec{\xi_i}$, a derivative of vectors will necessarily have to include both a derivative of the components, $A^i(\xi)$ and of the basis vectors $\vec{\xi_i}$.

We will thus need to know how does the basis vector $\vec{\xi_i}$ change as coordinates ξ^j vary. Since the derivative of a vector—computed by definition from the difference of the vector at a nearby point and the vector at the original point—must itself be a vector and so a linear combination of the coordinate vectors, $\vec{\xi_k}$:

$$\frac{\partial\vec{\xi_i}}{\partial\xi^j} := \lim_{\epsilon^j\to 0}\frac{\vec{\xi_i}(\xi+\epsilon)-\vec{\xi_i}(\xi)}{\epsilon^j} = \Gamma^k_{ij}\,\vec{\xi_k}, \tag{1.120}$$

where Γ^k_{ij} are the coefficients in this linear combination, and we show below that they are far from zero in general (A.7).

As regards the rate of change in $\vec{\xi^i}$, we use the fact that

$$\vec{\xi_i}\cdot\vec{\xi_j} = g_{ij} \qquad \text{and} \qquad \vec{\xi_i}\cdot\vec{\xi^k} = \delta^k_i, \tag{1.121}$$

so obtained by contracting the left-hand side equality, Eq. (1.109), with g^{jk}. Next, we compute the rate of change of both sides of the right-hand side equality (1.121): first, the right-hand side is an invariant *constant*, so

$$\frac{\partial}{\partial\xi^j}\Big(\,\vec{\xi_i}\cdot\vec{\xi^k}\,\Big) = \frac{\partial}{\partial\xi^j}\Big(\,\delta^k_i\,\Big) = 0, \tag{1.122}$$

on the other hand, the left-hand side consists of two factors, for one of which we know the transformation to be

$$\begin{aligned}\frac{\partial}{\partial\xi^j}\Big(\,\vec{\xi_i}\cdot\vec{\xi^k}\,\Big) &= \frac{\partial\vec{\xi_i}}{\partial\xi^j}\cdot\vec{\xi^k} + \vec{\xi_i}\cdot\frac{\partial\vec{\xi^k}}{\partial\xi^j} = \Gamma^l_{ij}\,\vec{\xi_l}\cdot\vec{\xi^k} + \vec{\xi_i}\cdot\frac{\partial\vec{\xi^k}}{\partial\xi^j} = \Gamma^l_{ij}\,\delta^k_l + \vec{\xi_i}\cdot\frac{\partial\vec{\xi^k}}{\partial\xi^j},\\ &= \Gamma^k_{ij} + \vec{\xi_i}\cdot\frac{\partial\vec{\xi^k}}{\partial\xi^j}.\end{aligned} \tag{1.123}$$

Thus, we have obtained that

$$\Gamma^k_{ij} + \vec{\xi_i}\cdot\frac{\partial\vec{\xi^k}}{\partial\xi^j} = 0, \qquad \textit{i.e.} \qquad \vec{\xi_i}\cdot\frac{\partial\vec{\xi^k}}{\partial\xi^j} = -\Gamma^k_{ij}. \tag{1.124}$$

By Eqs (1.111),

$$-\Gamma^k_{ij} = \frac{\partial\vec{\xi^k}}{\partial\xi^j}\Big|_{i\ \text{(covariant)}}, \qquad \textit{i.e.} \qquad \frac{\partial\vec{\xi^k}}{\partial\xi^j} = -\Gamma^k_{ij}\,\vec{\xi^i}. \tag{1.125}$$

Notice, by the way, that simply preserving the free index (here j,k) positions from left-hand side to right-hand side, this is the only way to write the formula—which could have been used as a derivation shortcut.

Armed with these observations and the results the derivation of which we defer to the Appendix A, we now turn to compute derivatives of vectors.

1.4.2 The General Derivative of a Vector

Since the coordinate vectors, $\vec{\xi}_i$ and $\vec{\xi}^i$, are not constant in general coordinates (A.7), this must be taken into account when differentiating either vector (1.103)–(1.105). We'll start with the contravariant vector:

$$\frac{\partial \vec{A}}{\partial \xi^i} = \frac{\partial}{\partial \xi^i}\, A^j\, \vec{\xi}_j = \frac{\partial A^j}{\partial \xi^i}\, \vec{\xi}_j + A^j\, \frac{\partial \vec{\xi}_j}{\partial \xi^i} = \frac{\partial A^j}{\partial \xi^i}\, \delta_j^k\, \vec{\xi}_k + A^j\, \Gamma^k_{ji}\, \vec{\xi}_k, \tag{1.126}$$

$$= \Big[\, \frac{\partial A^k}{\partial \xi^i} + \Gamma^k_{ji}\, A^j \Big]\, \vec{\xi}_k \;\; =: \;\; A^k{}_{;i}\, \vec{\xi}_k. \tag{1.127}$$

Similarly,

$$\frac{\partial \vec{B}}{\partial \xi^i} = \frac{\partial}{\partial \xi^i}\, B_j\, \vec{\xi}^j = \frac{\partial B_j}{\partial \xi^i}\, \vec{\xi}^j + B_j\, \frac{\partial \vec{\xi}^j}{\partial \xi^i} = \frac{\partial B_j}{\partial \xi^i}\, \delta_k^j\, \vec{\xi}^k - B_j\, \Gamma^j_{ki}\, \vec{\xi}^k, \tag{1.128}$$

$$= \Big[\, \frac{\partial B_k}{\partial \xi^i} - \Gamma^j_{ki}\, B_j \Big]\, \vec{\xi}^k \;\; =: \;\; B_{k;i}\, \vec{\xi}^k. \tag{1.129}$$

We have introduced the component notation for this *covariant derivative*, $A^k{}_{;i}$ and $B_{k;i}$, whereby the derivative by ξ^i is indicated to have been performed not only on the component, but also on the component vectors—representing thus the derivative of the *whole* vector.

1.4.3 Divergence

The *divergence* of a vector is now obtained as the *contraction*

$$\vec{\nabla} \cdot \vec{A} \equiv \mathbf{div}(\vec{A}) := \vec{\xi}^i \cdot \frac{\partial \vec{A}}{\partial \xi^i} = \vec{\xi}^i \cdot \big(\, A^k{}_{;i}\, \vec{\xi}_k \,\big) = A^k{}_{;i}\, \big(\, \vec{\xi}^i \cdot \vec{\xi}_k \,\big) = A^k{}_{;i}\, \delta^i_k = A^i{}_{;i},$$

$$= \Big[\, \frac{\partial A^i}{\partial \xi^i} + \Gamma^i_{ji}\, A^j \Big], \qquad \text{for a contravariant vector,} \tag{1.130}$$

$$\overset{(A.11)}{=} \Big[\, \frac{\sqrt{g}}{\sqrt{g}} \frac{\partial A^i}{\partial \xi^i} + \Big(\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \xi^j} \Big)\, A^j \Big] = \frac{1}{\sqrt{g}} \frac{\partial \big(\sqrt{g}\, A^i \big)}{\partial \xi^i}. \tag{1.131}$$

For a vector of which we are given the covariant components, $B_i(\xi)$, we compute the divergence by converting its components into contravariant ones:

$$\vec{\nabla} \cdot \vec{B} \equiv \mathbf{div}(\vec{B}) = \frac{1}{\sqrt{g}} \frac{\partial \big(\sqrt{g}\, g^{ij}\, B_j \big)}{\partial \xi^i}, \tag{1.132}$$

and note that the lack of symmetry (perhaps expected, naïvely) owes to the fact that the partial derivative operators, $\frac{\partial}{\partial \xi^i}$, themselves transform as components of a covariant vector, thus manifestly breaking whatever symmetry one might have expected between the two types of vectors.

1.4.4 Curl

The *curl* of a vector in three dimensions, $n = 3$, may now be defined using the above-introduced notion of the cross-product (1.88), noting that the $\vec{\nabla}$ operator is naturally covariant:

$$\vec{\nabla} \times \vec{B} \equiv \mathbf{curl}(\vec{B}) := \frac{1}{\sqrt{g}} \Big[\Big(\frac{\partial B_j}{\partial \xi^i} - \Gamma^l_{ji}\, B_l \Big)\, \varepsilon^{ijk} \Big]\, \vec{\xi}_k = \frac{1}{\sqrt{g}} \Big(\frac{\partial B_j}{\partial \xi^i} \Big)\, \varepsilon^{ijk}\, \vec{\xi}_k. \tag{1.133}$$

The second term, involving the Christoffel symbol does not contribute since:

$$\Gamma^l_{ji}\,\varepsilon^{ijk} \stackrel{1}{=} \Gamma^l_{ij}\,\varepsilon^{jik} \stackrel{2}{=} (+\Gamma^l_{ji})\,(-\varepsilon^{ijk}) = -\Gamma^l_{ji}\,\varepsilon^{ijk} \;\equiv\; 0, \tag{1.134}$$

where the first equality follows upon simply renaming the indices $i \leftrightarrow j$; the second follows on using both that $\Gamma^l_{ij} = +\Gamma^l_{ji}$ and that $\varepsilon^{jik} = -\varepsilon^{ijk}$. Finally, the last equality follows on realizing that we have just proved that this quantity equals the negative of itself.

Again, the curl of a vector of which contravariant components were given is computed by first lowering the component index:

$$\mathbf{curl}(\vec{A}) = \frac{1}{\sqrt{g}}\left(\frac{\partial\big(A^l\,g_{lj}\big)}{\partial\xi^i}\right)\varepsilon^{ijk}\,\vec{\xi}_k. \tag{1.135}$$

Note that in n-dimensional spaces

$$\mathbf{curl}(\vec{B}) = \frac{1}{\sqrt{g}}\Big(\frac{\partial B_j}{\partial\xi^i}\Big)\,\varepsilon^{ijk_1\cdots k_{n-2}}\,\vec{\xi}_{k_1}\cdots\vec{\xi}_{k_{n-2}} = \frac{1}{\sqrt{g}}\Big(\frac{\partial(B^l g_{lj})}{\partial\xi^i}\Big)\,\varepsilon^{ijk_1\cdots k_{n-2}}\,\vec{\xi}_{k_1}\cdots\vec{\xi}_{k_{n-2}} \tag{1.136}$$

is a totally antisymmetric rank-$(n{-}2)$ tensor.

1.4.5 Laplacian

The Laplacian (Laplace-Beltrami operator) of a scalar is fairly straightforward, since

$$\vec{\nabla}^2 f(\vec{r}) = \vec{\nabla}\cdot\big(\vec{\nabla}f(\vec{r})\big) = \frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial\xi^i}\Big(\sqrt{g}\,g^{ij}\,\frac{\partial f}{\partial\xi^j}\Big)\Big], \tag{1.137}$$

where we used Eq. (1.131), with $B_j = \frac{\partial f}{\partial\xi^j}$.

For the Laplacian of a vector $\vec{A}$, we use the identity[11] $\vec{\nabla}^2\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$:

$$\vec{\nabla}^2\vec{A} = \vec{\xi}^i\frac{\partial}{\partial\xi^i}\left(\frac{1}{\sqrt{g}}\frac{\partial\big(\sqrt{g}\,A^j\big)}{\partial\xi^j}\right) - \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial\xi^i}\Big(\frac{1}{\sqrt{g}}\frac{\partial(A^p\,g_{pn})}{\partial\xi^m}\,\varepsilon^{mnl}\,g_{lj}\Big)\right)\varepsilon^{ijk}\,\vec{\xi}_k, \tag{1.138}$$

$$= \left[g^{ik}\frac{\partial}{\partial\xi^i}\left(\frac{1}{\sqrt{g}}\frac{\partial\big(\sqrt{g}\,A^j\big)}{\partial\xi^j}\right) - \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial\xi^i}\Big(\frac{1}{\sqrt{g}}\frac{\partial(A^p\,g_{pn})}{\partial\xi^m}\,\varepsilon^{mnl\cdots}\,g_{lj}\cdots\Big)\right)\varepsilon^{ij\cdots k}\right]\vec{\xi}_k. \tag{1.139}$$

The ellipses in the index span of each Levi-Civita symbol indicate additional $n{-}3$ indices for the n-dimensional generalization; these are contracted with additional metric tensors, replacing the ellipses inside the innermost large parentheses, like so: $\cdots\varepsilon^{mnlp\cdots r}\,g_{lj}g_{pq}\cdots g_{rs}))\,\varepsilon^{ijq\cdots sk}\cdots$.

1.5 All Together

We collect the first and second derivatives:

$$\vec{\nabla}f = \Big(\frac{\partial f}{\partial\xi^i}\Big)\,\vec{\xi}^i; \tag{1.140}$$

[11] This identity is dimension-independent for $n \geq 2$, owing to the result (1.135): whereas the "inside" $\times$-product results in a rank-$(n{-}2)$ tensor, the "outside" $\times$-product results again in a vector.

$$\vec{\nabla}^2 f(\vec{r}) = \frac{1}{\sqrt{g}}\Big[\frac{\partial}{\partial \xi^i}\Big(\sqrt{g}\, g^{ij}\,\frac{\partial f}{\partial \xi^j}\Big)\Big]; \tag{1.141}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\sqrt{g}}\frac{\partial\big(\sqrt{g}\,A^i\big)}{\partial \xi^i}; \tag{1.142}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{\sqrt{g}}\Big(\frac{\partial A_j}{\partial \xi^i}\Big)\,\varepsilon^{ijk_1\cdots k_{n-2}}\,\vec{\xi}_{k_1}\cdots\vec{\xi}_{k_{n-2}}; \qquad \begin{cases}\text{a totally antisymmetric}\\ \text{rank-}(n{-}2)\text{ tensor}\end{cases} \tag{1.143}$$

$$\vec{\nabla}^2\vec{A} = \Big[g^{ik}\frac{\partial}{\partial \xi^i}\Big(\frac{1}{\sqrt{g}}\frac{\partial\big(\sqrt{g}\,A^j\big)}{\partial \xi^j}\Big) - \frac{1}{\sqrt{g}}\Big(\frac{\partial}{\partial \xi^i}\Big(\frac{1}{\sqrt{g}}\frac{\partial A_n}{\partial \xi^m}\,\varepsilon^{mnl\cdots}\,g_{lj}\cdots\Big)\Big)\,\varepsilon^{ij\cdots k}\Big]\,\vec{\xi}_k. \tag{1.144}$$

The Reader should note the relative simplicity of these expressions—while applicable in any consistent coordinate system! In fact, we have also expressed the curl of a vector for all n, as well as the Laplacian of a vector. For $n=3$, the factors set in orange ink are absent and the more familiar expressions emerge.

2 Rounding Up the Usual Suspects

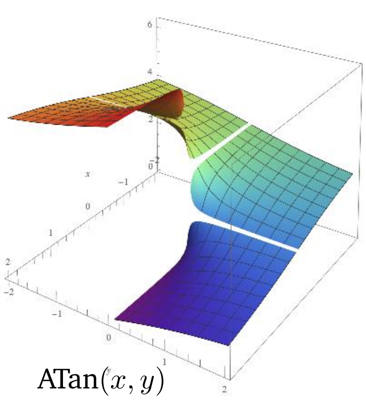
2.1 Cylindrical Coordinates

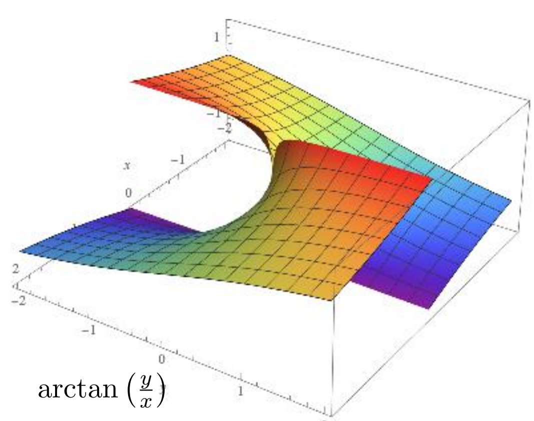
The well-known cylindrical coordinates may be defined by way of referring to the Cartesian coordinates, where z is common, while:

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \qquad \text{and} \qquad \rho = \sqrt{x^2+y^2}, \quad \phi = \text{ATan}(x,y) \tag{2.1}$$

where

$$\text{ATan}(x,y) := \begin{cases} \arctan(y/x) & \text{for} \quad x,y>0,\\ \pi+\arctan(y/x) & \text{for} \quad x\le 0,\\ 2\pi+\arctan(y/x) & \text{for} \quad y\le 0<x.\end{cases} \tag{2.2a}$$





(2.2b)

corrects arctan-function, which—uncorrected—does not return the full $[0,2\pi]$ range of the angle. Now compute:

$$[g_{ij}] = \begin{bmatrix} \frac{\partial x}{\partial\rho}\frac{\partial x}{\partial\rho}+\frac{\partial y}{\partial\rho}\frac{\partial y}{\partial\rho}+\frac{\partial z}{\partial\rho}\frac{\partial z}{\partial\rho} & \frac{\partial x}{\partial\rho}\frac{\partial x}{\partial\phi}+\frac{\partial y}{\partial\rho}\frac{\partial y}{\partial\phi}+\frac{\partial z}{\partial\rho}\frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial\rho}\frac{\partial x}{\partial z}+\frac{\partial y}{\partial\rho}\frac{\partial y}{\partial z}+\frac{\partial z}{\partial\rho}\frac{\partial z}{\partial z}\\ \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\rho}+\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\rho}+\frac{\partial z}{\partial\phi}\frac{\partial z}{\partial\rho} & \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\phi}+\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\phi}+\frac{\partial z}{\partial\phi}\frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial z}+\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial z}+\frac{\partial z}{\partial\phi}\frac{\partial z}{\partial z}\\ \frac{\partial x}{\partial z}\frac{\partial x}{\partial\rho}+\frac{\partial y}{\partial z}\frac{\partial y}{\partial\rho}+\frac{\partial z}{\partial z}\frac{\partial z}{\partial\rho} & \frac{\partial x}{\partial z}\frac{\partial x}{\partial\phi}+\frac{\partial y}{\partial z}\frac{\partial y}{\partial\phi}+\frac{\partial z}{\partial z}\frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial z}\frac{\partial x}{\partial z}+\frac{\partial y}{\partial z}\frac{\partial y}{\partial z}+\frac{\partial z}{\partial z}\frac{\partial z}{\partial z}\end{bmatrix},$$

$$= \begin{bmatrix} (\cos\phi)^2{+}(\sin\phi)^2{+}0 & (\cos\phi)(-\rho\sin\phi){+}(\sin\phi)(\rho\cos\phi){+}0 & 0{+}0{+}0 \\ (-\rho\sin\phi)(\cos\phi){+}(\rho\cos\phi)(\sin\phi){+}0 & (-\rho\sin\phi)^2{+}(\rho\cos\phi)^2{+}0 & 0{+}0{+}0 \\ 0{+}0{+}0 & 0{+}0{+}0 & 0{+}0{+}1 \end{bmatrix},$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Rightarrow \qquad [g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

Then

$$g := \det\,[g_{ij}] = \rho^2, \qquad \mathrm{d}^3\vec{r} = \sqrt{g}\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z = \rho\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z, \tag{2.4}$$

and

$$\|\vec{\xi}_\rho\| = \sqrt{g_{\rho\rho}} = 1, \quad \text{so} \quad \vec{\xi}_\rho = \hat{\mathrm{e}}_\rho; \qquad \|\vec{\xi}^\rho\| = \sqrt{g^{\rho\rho}} = 1, \quad \text{so} \quad \vec{\xi}^\rho = \hat{\mathrm{e}}^\rho; \tag{2.5}$$

$$\|\vec{\xi}_\phi\| = \sqrt{g_{\phi\phi}} = \rho, \quad \text{so} \quad \vec{\xi}_\phi = \rho\hat{\mathrm{e}}_\phi; \qquad \|\vec{\xi}^\phi\| = \sqrt{g^{\phi\phi}} = \rho^{-1}, \quad \text{so} \quad \vec{\xi}^\phi = \rho^{-1}\hat{\mathrm{e}}^\phi; \tag{2.6}$$

$$\|\vec{\xi}_z| = \sqrt{g_{zz}} = 1, \quad \text{so} \quad \vec{\xi}_z = \hat{\mathrm{e}}_z; \qquad \|\vec{\xi}^z\| = \sqrt{g^{zz}} = 1, \quad \text{so} \quad \vec{\xi}^z = \hat{\mathrm{e}}^z. \tag{2.7}$$

Also, since (ρ, ϕ, z) is orthogonal, $\hat{\mathrm{e}}_\rho = \hat{\mathrm{e}}^\rho$, $\hat{\mathrm{e}}_\phi = \hat{\mathrm{e}}^\phi$, $\hat{\mathrm{e}}_z = \hat{\mathrm{e}}^z$. These results imply, as in Eqs. (1.113)–(1.118), that

$$\begin{matrix} A_\rho & = A_{\hat\rho} & = A^{\hat\rho} & = A^\rho, \\ A_z & = A_{\hat z} & = A^{\hat z} & = A^z, \end{matrix} \qquad \text{but} \qquad A_\phi = \rho^2 A^\phi, \quad \text{and} \quad \begin{matrix} A_\phi & = \rho A_{\hat\phi}, \\ A^\phi & = \rho^{-1} A_{\hat\phi}. \end{matrix} \tag{2.8}$$

Then

$$\begin{aligned} \vec{\nabla} f &= \vec{\xi}^\rho \frac{\partial f}{\partial \rho} + \vec{\xi}^\phi \frac{\partial f}{\partial \phi} + \vec{\xi}^z \frac{\partial f}{\partial z}, \\ &= \hat{\mathrm{e}}_\rho \frac{\partial f}{\partial \rho} + \hat{\mathrm{e}}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\mathrm{e}}_z \frac{\partial f}{\partial z}; \end{aligned} \tag{2.9}$$

$$\begin{aligned} \vec{\nabla}^2 f &= \frac{1}{\rho}\Big[\frac{\partial}{\partial\rho}\Big(\rho\, g^{\rho\rho}\frac{\partial f}{\partial\rho}\Big) + \frac{\partial}{\partial\phi}\Big(\rho\, g^{\phi\phi}\frac{\partial f}{\partial\phi}\Big) + \frac{\partial}{\partial z}\Big(\rho\, g^{zz}\frac{\partial f}{\partial z}\Big)\Big], \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial f}{\partial\rho}\Big) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}; \end{aligned} \tag{2.10}$$

$$\begin{aligned} \vec{\nabla}\cdot\vec{A} &= \frac{1}{\rho}\left[\frac{\partial(\rho A^\rho)}{\partial\rho} + \frac{\partial(\rho A^\phi)}{\partial\phi} + \frac{\partial(\rho A^z)}{\partial z}\right] = \frac{1}{\rho}\frac{\partial(\rho A^\rho)}{\partial\rho} + \frac{\partial A^\phi}{\partial\phi} + \frac{\partial A^z}{\partial z}, \\ &= \frac{1}{\rho}\frac{\partial(\rho A_{\hat\rho})}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_{\hat\phi}}{\partial\phi} + \frac{\partial A_{\hat z}}{\partial z}; \end{aligned} \tag{2.11}$$

$$\begin{aligned} \vec{\nabla}\times\vec{A} &= \frac{1}{\rho}\Big[\vec{\xi}_\rho\,\varepsilon^{\rho\phi z}\Big(\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\Big) + \vec{\xi}_\phi\,\varepsilon^{\phi z\rho}\Big(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\Big) + \vec{\xi}_z\,\varepsilon^{z\rho\phi}\Big(\frac{\partial A_\phi}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\Big)\Big], \\ &= \frac{1}{\rho}\Big[\hat{\mathrm{e}}_\rho\Big(\frac{\partial A_{\hat z}}{\partial\phi} - \frac{\partial(\rho A_{\hat\phi})}{\partial z}\Big) + \rho\hat{\mathrm{e}}_\phi\Big(\frac{\partial A_{\hat\rho}}{\partial z} - \frac{\partial A_{\hat z}}{\partial\rho}\Big) + \hat{\mathrm{e}}_z\Big(\frac{\partial(\rho A_{\hat\phi})}{\partial\rho} - \frac{\partial A_{\hat\rho}}{\partial\phi}\Big)\Big], \\ &= \Big[\hat{\mathrm{e}}_\rho\Big(\frac{1}{\rho}\frac{\partial A_{\hat z}}{\partial\phi} - \frac{\partial A_{\hat\phi}}{\partial z}\Big) + \hat{\mathrm{e}}_\phi\Big(\frac{\partial A_{\hat\rho}}{\partial z} - \frac{\partial A_{\hat z}}{\partial\rho}\Big) + \hat{\mathrm{e}}_z\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat\phi})}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_{\hat\rho}}{\partial\phi}\Big)\Big]; \end{aligned} \tag{2.12}$$

$$\begin{aligned} \vec{\nabla}^2\vec{A} = \vec{\xi}_\rho\Big\{g^{\rho\rho}\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\rho} - \varepsilon^{\rho\phi z}\frac{1}{\rho}\Big[\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\phi}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\Big)\varepsilon^{\rho\phi z}g_{zz}\Big) - \frac{\partial}{\partial z}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\Big)\varepsilon^{z\rho\phi}g_{\phi\phi}\Big)\Big]\Big\} \\ + \vec{\xi}_\phi\Big\{g^{\phi\phi}\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\phi} - \varepsilon^{\phi z\rho}\frac{1}{\rho}\Big[\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\Big(\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\Big)\varepsilon^{\phi z\rho}g_{\rho\rho}\Big) - \frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\phi}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\Big)\varepsilon^{\rho\phi z}g_{zz}\Big)\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
&+\vec{\xi}_z\Big\{g^{zz}\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial z}-\varepsilon^{\rho\phi z}\frac{1}{\rho}\Big[\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\Big)\varepsilon^{z\rho\phi}g_{\phi\phi}\Big)-\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\Big(\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\Big)\varepsilon^{\phi z\rho}g_{\rho\rho}\Big)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
=\ &\vec{\xi}_\rho\Big\{\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\rho}-\frac{1}{\rho}\Big[\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\phi}{\partial\rho}-\frac{\partial A_\rho}{\partial\phi}\Big)\Big)-\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\Big)\rho^2\Big)\Big]\Big\}\\
&+\vec{\xi}_\phi\Big\{\rho^{-2}\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\phi}-\frac{1}{\rho}\Big[\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\Big(\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\Big)\Big)-\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\phi}{\partial\rho}-\frac{\partial A_\rho}{\partial\phi}\Big)\Big)\Big]\Big\}\\
&+\vec{\xi}_z\Big\{\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial z}-\frac{1}{\rho}\Big[\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\Big(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\Big)\rho^2\Big)-\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\Big(\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\Big)\Big)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
=\ &\vec{\xi}_\rho\Big\{\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\rho}-\Big[\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\frac{\partial A_\phi}{\partial\rho}-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\Big)-\frac{\partial}{\partial z}\Big(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\Big)\Big]\Big\}\\
&+\frac{1}{\rho}\vec{\xi}_\phi\Big\{\frac{1}{\rho}\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\phi}-\Big[\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{1}{\rho}\frac{\partial A_\phi}{\partial z}\Big)-\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\frac{\partial A_\phi}{\partial\rho}-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\Big)\Big]\Big\}\\
&+\vec{\xi}_z\Big\{\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial z}-\Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial A_\rho}{\partial z}-\rho\frac{\partial A_z}{\partial\rho}\Big)-\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{1}{\rho}\frac{\partial A_\phi}{\partial z}\Big)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
=\ &\hat{\mathrm{e}}_\rho\Big\{\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\rho}-\Big[\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\phi}})}{\partial\rho}-\frac{1}{\rho}\frac{\partial A_{\hat{\rho}}}{\partial\phi}\Big)-\frac{\partial}{\partial z}\Big(\frac{\partial A_{\hat{\rho}}}{\partial z}-\frac{\partial A_{\hat{z}}}{\partial\rho}\Big)\Big]\Big\}\\
&+\hat{\mathrm{e}}_\phi\Big\{\frac{1}{\rho}\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial\phi}-\Big[\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\frac{\partial A_{\hat{z}}}{\partial\hat{\phi}}-\frac{\partial A_{\hat{\phi}}}{\partial z}\Big)-\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\phi}})}{\partial\rho}-\frac{1}{\rho}\frac{\partial A_{\hat{\rho}}}{\partial\phi}\Big)\Big]\Big\}\\
&+\hat{\mathrm{e}}_z\Big\{\frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial z}-\Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial A_{\hat{\rho}}}{\partial z}-\rho\frac{\partial A_{\hat{z}}}{\partial\rho}\Big)-\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\frac{\partial A_{\hat{z}}}{\partial\phi}-\frac{\partial A_{\hat{\phi}}}{\partial z}\Big)\Big]\Big\},
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
=\ &\hat{\mathrm{e}}_\rho\Big\{\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\rho}})}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_{\hat{\phi}}}{\partial\phi}+\frac{\partial A_{\hat{z}}}{\partial z}\Big)-\Big[\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\phi}})}{\partial\rho}-\frac{1}{\rho}\frac{\partial A_{\hat{\rho}}}{\partial\phi}\Big)-\frac{\partial}{\partial z}\Big(\frac{\partial A_{\hat{\rho}}}{\partial z}-\frac{\partial A_{\hat{z}}}{\partial\rho}\Big)\Big]\Big\}\\
&+\hat{\mathrm{e}}_\phi\Big\{\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\rho}})}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_{\hat{\phi}}}{\partial\phi}+\frac{\partial A_{\hat{z}}}{\partial z}\Big)-\Big[\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\frac{\partial A_{\hat{z}}}{\partial\hat{\phi}}-\frac{\partial A_{\hat{\phi}}}{\partial z}\Big)-\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\phi}})}{\partial\rho}-\frac{1}{\rho}\frac{\partial A_{\hat{\rho}}}{\partial\phi}\Big)\Big]\Big\}\\
&+\hat{\mathrm{e}}_z\Big\{\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\rho}})}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_{\hat{\phi}}}{\partial\phi}+\frac{\partial A_{\hat{z}}}{\partial z}\Big)-\Big[\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial A_{\hat{\rho}}}{\partial z}-\rho\frac{\partial A_{\hat{z}}}{\partial\rho}\Big)-\frac{1}{\rho}\frac{\partial}{\partial\phi}\Big(\frac{1}{\rho}\frac{\partial A_{\hat{z}}}{\partial\phi}-\frac{\partial A_{\hat{\phi}}}{\partial z}\Big)\Big]\Big\},
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
=\ &\hat{\mathrm{e}}_\rho\Big\{\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\rho}})}{\partial\rho}+\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial A_{\hat{\phi}}}{\partial\phi}+\frac{\partial^2 A_{\hat{z}}}{\partial\rho\partial z}\\
&\qquad\qquad-\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\Big(\rho\frac{\partial A_{\hat{\phi}}}{\partial\rho}+A_{\hat{\phi}}\Big)+\frac{1}{\rho^2}\frac{\partial^2 A_{\hat{\rho}}}{\partial\phi^2}+\frac{\partial^2 A_{\hat{\rho}}}{\partial z^2}-\frac{\partial^2 A_{\hat{z}}}{\partial z\partial\rho}\Big]\Big\}\\
&+\hat{\mathrm{e}}_\phi\Big\{\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\Big(\rho\frac{\partial A_{\hat{\rho}}}{\partial\rho}+A_{\hat{\rho}}\Big)+\frac{1}{\rho^2}\frac{\partial^2 A_{\hat{\phi}}}{\partial\phi^2}+\frac{1}{\rho}\frac{\partial^2 A_{\hat{z}}}{\partial\phi\partial z}\\
&\qquad\qquad-\frac{1}{\rho}\frac{\partial^2 A_{\hat{z}}}{\partial z\partial\phi}+\frac{\partial^2 A_{\hat{\phi}}}{\partial z^2}+\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial(\rho A_{\hat{\phi}})}{\partial\rho}-\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial A_{\hat{\rho}}}{\partial\phi}\Big\}\\
&+\hat{\mathrm{e}}_z\Big\{\frac{1}{\rho}\frac{\partial}{\partial z}\Big(\rho\frac{\partial A_{\hat{\rho}}}{\partial\rho}+A_{\hat{\rho}}\Big)+\frac{1}{\rho}\frac{\partial^2 A_{\hat{\phi}}}{\partial z\partial\phi}+\frac{\partial^2 A_{\hat{z}}}{\partial z^2}\\
&\qquad\qquad-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_{\hat{\rho}}}{\partial z}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_{\hat{z}}}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 A_{\hat{z}}}{\partial\phi^2}-\frac{1}{\rho}\frac{\partial^2 A_{\hat{\phi}}}{\partial\phi\partial z}\Big\},
\end{aligned}
\tag{2.15}
$$

$$= \hat{e}_\rho\Big\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial A_{\hat\rho}}{\partial\rho}\Big) - \frac{1}{\rho^2}A_{\hat\rho} + \frac{1}{\rho}\frac{\partial^2 A_{\hat\phi}}{\partial\rho\partial\phi} - \frac{1}{\rho^2}\frac{\partial A_{\hat\phi}}{\partial\phi} + \frac{\partial^2 A_{\hat z}}{\partial\rho\partial z}$$
$$- \frac{1}{\rho}\frac{\partial^2 A_{\hat\phi}}{\partial\phi\partial\rho} - \frac{1}{\rho^2}\frac{\partial A_{\hat\phi}}{\partial\phi} + \frac{1}{\rho^2}\frac{\partial^2 A_{\hat\rho}}{\partial\phi^2} + \frac{\partial^2 A_{\hat\rho}}{\partial z^2} - \frac{\partial^2 A_{\hat z}}{\partial z\partial\rho}\Big]\Big\}$$
$$+ \hat{e}_\phi\Big\{\frac{1}{\rho}\frac{\partial^2 A_{\hat\rho}}{\partial\phi\partial\rho} + \frac{1}{\rho^2}\frac{\partial A_{\hat\rho}}{\partial\phi} + \frac{1}{\rho^2}\frac{\partial^2 A_{\hat\phi}}{\partial\phi^2} + \frac{1}{\rho}\frac{\partial^2 A_{\hat z}}{\partial\phi\partial z}$$
$$- \frac{1}{\rho}\frac{\partial^2 A_{\hat z}}{\partial z\partial\phi} + \frac{\partial^2 A_{\hat\phi}}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_{\hat\phi}}{\partial\rho} - \frac{1}{\rho^2}A_{\hat\phi} - \frac{1}{\rho}\frac{\partial^2 A_{\hat\rho}}{\partial\rho\partial\phi} + \frac{1}{\rho^2}\frac{\partial A_{\hat\rho}}{\partial\phi}\Big\}$$
$$+ \hat{e}_z\Big\{\frac{\partial^2 A_{\hat\rho}}{\partial z\partial\rho} + \frac{1}{\rho}\frac{\partial A_{\hat\rho}}{\partial z} + \frac{1}{\rho}\frac{\partial^2 A_{\hat\phi}}{\partial z\partial\phi} + \frac{\partial^2 A_{\hat z}}{\partial z^2}$$
$$- \frac{\partial^2 A_{\hat\rho}}{\partial\rho\partial z} - \frac{1}{\rho}\frac{\partial A_{\hat\rho}}{\partial z} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_{\hat z}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 A_{\hat z}}{\partial\phi^2} - \frac{1}{\rho}\frac{\partial^2 A_{\hat\phi}}{\partial\phi\partial z}\Big\}, \tag{2.16}$$
$$= \hat{e}_\rho\Big[\vec\nabla^2(A_{\hat\rho}) - \frac{1}{\rho^2}A_{\hat\rho} - \frac{2}{\rho^2}\frac{\partial A_{\hat\phi}}{\partial\phi}\Big] + \hat{e}_\phi\Big[\vec\nabla^2(A_{\hat\phi}) - \frac{1}{\rho^2}A_{\hat\phi} + \frac{2}{\rho^2}\frac{\partial A_{\hat\rho}}{\partial\phi}\Big] + \hat{e}_z\vec\nabla^2(A_{\hat z}), \tag{2.17}$$

where the last, blue-inked expressions (referring to the unit-normalized basis) are the ones to be compared with the corresponding ones in Ref. [1].

WoE 2.1 *(The Flip-ρ-Identities)*: We have also used the identities:

$$[\frac{\partial}{\partial\rho}, \rho]f = \frac{\partial}{\partial\rho}\rho f - \rho\frac{\partial f}{\partial\rho} = \rho\frac{\partial f}{\partial\rho} + f - \rho\frac{\partial f}{\partial\rho} = (\frac{\partial}{\partial\rho}\rho)\, f = +f, \tag{2.18}$$
$$[\frac{\partial}{\partial\rho}, \frac{1}{\rho}]f = \frac{\partial}{\partial\rho}\frac{1}{\rho}f - \frac{1}{\rho}\frac{\partial f}{\partial\rho} = \frac{1}{\rho}\frac{\partial f}{\partial\rho} - \frac{1}{\rho^2}f - \frac{1}{\rho}\frac{\partial f}{\partial\rho} = (\frac{\partial}{\partial\rho}\frac{1}{\rho})\, f = -\frac{1}{\rho^2}f, \tag{2.19}$$

and

$$[\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho - \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}]f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial f}{\partial\rho} + f\Big) - \frac{1}{\rho^2}\Big(\rho\frac{\partial f}{\partial\rho} + f\Big) - \Big(\frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho}\Big),$$
$$= \frac{\partial^2 f}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial\rho} - \frac{1}{\rho}\frac{\partial f}{\partial\rho} - \frac{1}{\rho^2}f - \frac{\partial^2 f}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial f}{\partial\rho} = -\frac{1}{\rho^2}f. \tag{2.20}$$

2.2 Spherical Coordinates

The diligent Reader is invited to do the same for the spherical coordinates, where:

$$x = r\,\sin\theta\,\cos\phi, \qquad y = r\,\sin\theta\,\sin\phi, \qquad z = r\,\cos\theta, \tag{2.21}$$
$$r = \sqrt{x^2+y^2+z^2}, \qquad \theta = \mathrm{ATan}\Big(\sqrt{x^2+y^2}, z\Big), \qquad \phi = \mathrm{ATan}(y,x), \tag{2.22}$$

so that

$$[g_{ij}] = \begin{bmatrix}1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta\end{bmatrix}, \qquad [g^{ij}] = \begin{bmatrix}1 & 0 & 0\\ 0 & r^{-2} & 0\\ 0 & 0 & r^{-2}\sin^{-2}\theta\end{bmatrix}, \tag{2.23}$$
$$g = r^4\sin^2\theta, \qquad \mathrm{d}^3\vec r = r^2\mathrm{d}r\,\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi; \tag{2.24}$$
$$\vec\xi_r = \hat{e}_r, \quad \vec\xi_\theta = r\,\hat{e}_\theta, \quad \vec\xi_\phi = r\sin\theta\,\hat{e}_\phi, \qquad A_r = A_{\hat r}, \quad A_\theta = r\,A_{\hat\theta}, \quad A_\phi = r\sin\theta\,A_{\hat\phi}. \tag{2.25}$$

2.3 Hyper-Spherical Coordinates

Following the pattern of the polar coordinates in the plane, the spherical coordinates in Euclidean 4d-space, we can introduce:

$$
\begin{aligned}
x &= r\ \sin\theta_1\ \sin\theta_2\ \cos\phi, & r &= \sqrt{x^2+y^2+z^2+w^2}, && \text{(2.26a)}\\
y &= r\ \sin\theta_1\ \sin\theta_2\ \sin\phi, & \phi &= \mathrm{ATan}(y,x), && \text{(2.26b)}\\
z &= r\ \sin\theta_1\ \cos\theta_2, & \theta_2 &= \mathrm{ATan}(\sqrt{x^2+y^2},z), && \text{(2.26c)}\\
w &= r\ \cos\theta_1, & \theta_1 &= \mathrm{ATan}(\sqrt{x^2+y^2+z^2},w). && \text{(2.26d)}
\end{aligned}
$$

so that

$$
[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta_1 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta_1\sin^2\theta_2 \end{bmatrix}, \qquad \begin{aligned} g,,&= r^6\sin^4\theta_1\sin^2\theta_2, \\ \mathrm{d}^4\vec{r},,&= r^3\mathrm{d}r\ \sin^2\theta_1\mathrm{d}\theta_1\ \sin\theta_2\mathrm{d}\theta_2\,\mathrm{d}\phi, \end{aligned} \tag{2.27}
$$

and so on: the pattern is easy to continue for all n.

Acknowledgement: I should like to thank Philip Kurian for an exceptionally diligent reading of this dense text, and uncovering scads of typos. I am quite certain, however, that plenty remain for you to find and report to me, which I wholeheartedly encourage and entreat. Thank you!

A The Christoffel Symbol

We now turn to exploring the Γ^k_{ij} which is evidently necessary when taking derivatives of vectors. The choice of the coefficient Γ^k_{ij} does depend on the choice of all three of i, j, k, but we will show that it is not a tensor.

A.1 Computing the Christoffel Symbol

First, we notice that

$$\Gamma^k_{ij}\, \vec{\xi}_k = \frac{\partial \vec{\xi}_i}{\partial \xi^j} = \frac{\partial^2 \vec{r}}{\partial \xi^i \partial \xi^j} = \frac{\partial^2 \vec{r}}{\partial \xi^j \partial \xi^i} = \frac{\partial \vec{\xi}_j}{\partial \xi^i} = \Gamma^k_{ji}\, \vec{\xi}_k, \tag{A.1}$$

so that $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Next, we compute the Γ^k_{ij}. We start with:

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \xi^k} &= \frac{\partial}{\partial \xi^k}\, \vec{\xi}_i \cdot \vec{\xi}_j = \frac{\partial \vec{\xi}_i}{\partial \xi^k} \cdot \vec{\xi}_j + \vec{\xi}_i \cdot \frac{\partial \vec{\xi}_j}{\partial \xi^k}, \\ &= \Gamma^l_{ik}\, \vec{\xi}_l \cdot \vec{\xi}_j + \vec{\xi}_i \cdot \Gamma^l_{jk}\, \vec{\xi}_l = g_{lj}\, \Gamma^l_{ik} + g_{li}\, \Gamma^l_{jk}, \end{aligned} \tag{A.2}$$

where we used that $g_{ij} = g_{ji}$. It is not possible to solve for $\Gamma^{\cdot}_{\cdot\cdot}$ from here, since we'd have to contract with the inverse metric, but in the two terms on the right-hand side, the metrics have different free indices. For example, contracting with g^{jm} to "free up" the Γ^l_{ij} results in

$$\begin{aligned} g^{jm}\, \frac{\partial g_{ij}}{\partial \xi^k} &= g^{jm}\, g_{lj}\, \Gamma^l_{ij} + g^{jm}\, g_{li}\, \Gamma^l_{jk} = \delta^m_l\, \Gamma^l_{ij} + g^{jm}\, g_{li}\, \Gamma^l_{jk}, \\ &= \Gamma^m_{ij} + g^{jm}\, g_{li}\, \Gamma^l_{jk}, \end{aligned} \tag{A.3}$$

from which only one of the two instances of $\Gamma^{\cdot}_{\cdot\cdot}$ has been "freed up".

We thus write out this same expression, but cyclicly renaming the (i, j, k) indices:

$$\frac{\partial g_{ij}}{\partial \xi^k} = g_{lj}\, \Gamma^l_{ik} + g_{li}\, \Gamma^l_{jk}, \tag{A.4a}$$

$$\frac{\partial g_{jk}}{\partial \xi^i} = g_{lk}\, \Gamma^l_{ji} + g_{lj}\, \Gamma^l_{ki}, \tag{A.4b}$$

$$\frac{\partial g_{ki}}{\partial \xi^j} = g_{li}\, \Gamma^l_{kj} + g_{lk}\, \Gamma^l_{ij}. \tag{A.4c}$$

Recalling that $g_{ij} = g_{ji}$, we see that every term occurs twice in the right-hand sides of this system. In particular, $g_{lj}\, \Gamma^l_{ki}$ is common to the last two, and the rest of the sum of the last two expressions is the first. Therefore:

$$\frac{\partial g_{ik}}{\partial \xi^j} + \frac{\partial g_{jk}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^k} = 2\, g_{lk}\, \Gamma^l_{ij}, \tag{A.5}$$

which *can be solved* for Γ^l_{ij} via contracting with $\frac{1}{2}\, g^{km}$ (and then renaming $m \to k$):

$$\Gamma^k_{ij} = \tfrac{1}{2}\, g^{kl} \Big(\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l} \Big). \tag{A.6}$$

As promised, we have proved that

$$\frac{\partial \vec{\xi}_i}{\partial \xi^j} = \Gamma^k_{ij}\, \vec{\xi}_k = \tfrac{1}{2}\, g^{kl} \Big(\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l} \Big)\, \vec{\xi}_k, \tag{A.7}$$

is far from zero in general coordinate systems: the coordinate vectors $\vec{\xi}_i$ indeed vary in general.

A.2 Two Trace Formulae

We will need:

$$\Gamma^i_{ij} = \tfrac{1}{2}\, g^{il}\Big(\frac{\partial g_{il}}{\partial\xi^j} + \frac{\partial g_{jl}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^l}\Big) = \tfrac{1}{2}\, g^{il}\frac{\partial g_{il}}{\partial\xi^j}, \tag{A.8}$$

where the last two terms cancel, since the second equals the third:

$$g^{il}\,\frac{\partial g_{jl}}{\partial\xi^i} \;\overset{1}{=}\; g^{li}\,\frac{\partial g_{ji}}{\partial\xi^l} \;\overset{2}{=}\; g^{il}\,\frac{\partial g_{ij}}{\partial\xi^l}, \tag{A.9}$$

where the first equality follows on simultaneously renaming $(i,l)\to(l,i)$, and the second on using the symmetry $g_{ij}=g_{ji}$.

Now we need a result about determinants:

$$\frac{\partial g(\xi)}{\partial\xi^j} = g(\xi)\, g^{il}(\xi)\,\frac{\partial g_{il}(\xi)}{\partial\xi^j}, \qquad \textit{i.e.} \qquad g^{il}(\xi)\,\frac{\partial g_{il}(\xi)}{\partial\xi^j} = \frac{1}{g(\xi)}\,\frac{\partial g(\xi)}{\partial\xi^j} = \frac{\partial\ln\big(g(\xi)\big)}{\partial\xi^j}. \tag{A.10}$$

Identifying this factor in the expression (A.8) lets us simplify it:

$$\Gamma^i_{ij} = \tfrac{1}{2}\frac{\partial\ln(g(\xi))}{\partial\xi^j} = \frac{\partial\ln\sqrt{g(\xi)}}{\partial\xi^j} = \frac{1}{\sqrt{g(\xi)}}\frac{\partial\sqrt{g(\xi)}}{\partial\xi^j}. \tag{A.11}$$

On the other hand,

$$g^{ij}\,\Gamma^k_{ij} = \tfrac{1}{2}\, g^{ij}\, g^{kl}\Big(\frac{\partial g_{il}}{\partial\xi^j} + \frac{\partial g_{jl}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^l}\Big) = \tfrac{1}{2}\, g^{kl}\Big(2g^{ij}\frac{\partial g_{il}}{\partial\xi^j} - g^{ij}\,\frac{\partial g_{ij}}{\partial\xi^l}\Big), \tag{A.12}$$

since

$$g^{ij}\frac{\partial g_{jl}}{\partial\xi^i} \;\overset{1}{=}\; g^{ji}\frac{\partial g_{il}}{\partial\xi^j} \;\overset{2}{=}\; g^{ij}\frac{\partial g_{il}}{\partial\xi^j} \tag{A.13}$$

where the first equality follows on simultaneously renaming $(i,j)\to(j,i)$, and the second on using the symmetry $g^{ij}=g^{ji}$.

A.3 The Transformation of the Christoffel Symbol

Given the explicit formula (A.6), we can in fact compute straightforwardly how Γ^k_{ij} transforms!

To this end, we note first that

$$\begin{aligned}
\frac{\partial g_{ij}(\xi)}{\partial\xi^k} &= \frac{\partial\eta^l}{\partial\xi^k}\frac{\partial}{\partial\eta^l}\Big(\frac{\partial\eta^m}{\partial\xi^i}\frac{\partial\eta^n}{\partial\xi^j}\, g_{mn}(\eta)\Big) \;=\; \frac{\partial}{\partial\xi^k}\Big(\frac{\partial\eta^m}{\partial\xi^i}\frac{\partial\eta^n}{\partial\xi^j}\, g_{mn}(\eta)\Big),\\
&= \frac{\partial^2\eta^m}{\partial\xi^k\partial\xi^i}\frac{\partial\eta^n}{\partial\xi^j}\, g_{mn}(\eta) + \frac{\partial\eta^m}{\partial\xi^i}\frac{\partial^2\eta^n}{\partial\xi^k\partial\xi^j}\, g_{mn}(\eta) + \frac{\partial\eta^m}{\partial\xi^i}\frac{\partial\eta^n}{\partial\xi^j}\frac{\partial g_{mn}(\eta)}{\partial\xi^k},\\
&= \frac{\partial\eta^m}{\partial\xi^i}\frac{\partial\eta^n}{\partial\xi^j}\frac{\partial\eta^l}{\partial\xi^k}\frac{\partial g_{mn}(\eta)}{\partial\eta^l} + \frac{\partial^2\eta^m}{\partial\xi^k\partial\xi^i}\frac{\partial\eta^n}{\partial\xi^j}\, g_{mn}(\eta) + \frac{\partial\eta^m}{\partial\xi^i}\frac{\partial^2\eta^n}{\partial\xi^k\partial\xi^j}\, g_{mn}(\eta).
\end{aligned} \tag{A.14}$$

The occurrence of the mixed, 2nd derivatives manifestly *does not conform* to the definition 1.9, and—in general—those are non-zero: $\frac{\partial g_{ij}}{\partial\xi^k}$ is not a tensor, and neither is Γ^k_{ij}.

Substituting the result (A.14), the tensorial transformation rule (1.72) and the inverse of that for g^{kl} into (A.6), (A.8) and (A.12) is now straightforward, albeit a bit tedious; the result is

$$\Gamma^i_{jk}(\xi) = \underbrace{\frac{\partial\xi^i}{\partial\eta^\ell}\frac{\partial\eta^m}{\partial\xi^j}\frac{\partial\eta^n}{\partial\xi^k}\Gamma^\ell_{mn}(\eta)}_{\text{tensorial}} + \underbrace{\frac{\partial\xi^i}{\partial\eta^\ell}\frac{\partial^2\eta^\ell}{\partial\xi^j\partial\xi^k}}_{\text{inhomogeneous}}. \tag{A.15}$$

Its trace, for example, is:

$$
\begin{aligned}
\Gamma^i_{ij}(\xi) &= \tfrac{1}{2}\, g^{il}(\xi)\, \frac{\partial g_{il}(\xi)}{\partial \xi^j}, \qquad \text{(A.16)}\\
&= \frac{1}{2}\Big(\frac{\partial \xi^i}{\partial \eta^k} g^{km}(\eta) \frac{\partial \xi^l}{\partial \eta^m}\Big)\Big(\frac{\partial \eta^n}{\partial \xi^i}\frac{\partial \eta^p}{\partial \xi^l}\frac{\partial \eta^q}{\partial \xi^j}\,\frac{\partial g_{np}(\eta)}{\partial \eta^q} + \frac{\partial^2 \eta^n}{\partial \xi^j \partial \xi^i}\,\frac{\partial \eta^p}{\partial \xi^l}\, g_{np}(\eta) + \frac{\partial \eta^n}{\partial \xi^i}\,\frac{\partial^2 \eta^p}{\partial \xi^j \partial \xi^l}\, g_{np}(\eta)\Big),\\
&= \frac{1}{2}\frac{\partial \xi^i}{\partial \eta^k} g^{km}(\eta) \frac{\partial \xi^l}{\partial \eta^m}\frac{\partial \eta^n}{\partial \xi^i}\frac{\partial \eta^p}{\partial \xi^l}\frac{\partial \eta^q}{\partial \xi^j}\,\frac{\partial g_{np}(\eta)}{\partial \eta^q} + \frac{1}{2}\frac{\partial \xi^i}{\partial \eta^k} g^{km}(\eta) \frac{\partial \xi^l}{\partial \eta^m}\,\frac{\partial^2 \eta^n}{\partial \xi^j \partial \xi^i}\,\frac{\partial \eta^p}{\partial \xi^l}\, g_{np}(\eta)\\
&\quad + \frac{1}{2}\frac{\partial \xi^i}{\partial \eta^k} g^{km}(\eta) \frac{\partial \xi^l}{\partial \eta^m}\,\frac{\partial \eta^n}{\partial \xi^i}\,\frac{\partial^2 \eta^p}{\partial \xi^j \partial \xi^l}\, g_{np}(\eta),\\
&= \frac{1}{2}\frac{\partial \eta^n}{\partial \eta^k}\frac{\partial \eta^p}{\partial \eta^m}\frac{\partial \eta^q}{\partial \xi^j}\, g^{km}(\eta)\frac{\partial g_{np}(\eta)}{\partial \eta^q} + \frac{1}{2}\frac{\partial \xi^i}{\partial \eta^k}\frac{\partial^2 \eta^n}{\partial \xi^j \partial \xi^i}\frac{\partial \eta^p}{\partial \eta^m}\, g^{km}(\eta) g_{np}(\eta) + \frac{1}{2}\frac{\partial \xi^l}{\partial \eta^m}\frac{\partial \eta^n}{\partial \eta^k}\frac{\partial^2 \eta^p}{\partial \xi^j \partial \xi^l}\, g^{km}(\eta) g_{np}(\eta),\\
&= \frac{\partial \eta^q}{\partial \xi^j}\left[\tfrac{1}{2} g^{km}(\eta)\frac{\partial g_{km}(\eta)}{\partial \eta^q}\right] + \frac{1}{2}\frac{\partial \xi^i}{\partial \eta^k}\frac{\partial^2 \eta^n}{\partial \xi^j \partial \xi^i}\, g^{km}(\eta) g_{nm}(\eta) + \frac{1}{2}\frac{\partial \xi^l}{\partial \eta^m}\frac{\partial^2 \eta^p}{\partial \xi^j \partial \xi^l}\, g^{km}(\eta) g_{kp}(\eta),\\
&= \frac{\partial \eta^q}{\partial \xi^j}\left[\tfrac{1}{2} g^{km}(\eta)\frac{\partial g_{km}(\eta)}{\partial \eta^q}\right] + \frac{1}{2}\frac{\partial \xi^i}{\partial \eta^k}\frac{\partial^2 \eta^n}{\partial \xi^j \partial \xi^i}\,\delta^k_n + \frac{1}{2}\frac{\partial \xi^l}{\partial \eta^m}\frac{\partial^2 \eta^p}{\partial \xi^j \partial \xi^l}\,\delta^m_p,\\
&= \underbrace{\frac{\partial \eta^q}{\partial \xi^j}\,\Gamma^k_{kq}(\eta)}_{\text{tensorial}} + \underbrace{\frac{\partial \xi^i}{\partial \eta^k}\frac{\partial^2 \eta^k}{\partial \xi^j \partial \xi^i}}_{\text{inhomogeneous}} \qquad \text{(A.17)}
\end{aligned}
$$

is one of the simplest expressions. It shows that, although it has one single free index—as the components of a covariant vector would have—Γ^i_{ij} is not a vector: besides the leading (tensorial) transformation term, it also picks up an *inhomogeneous* term. The property of transforming *inhomogeneously* is characteristic for Γ^k_{ij} in its rôle as a "connexion."

B Lexicon

The following is a swift reminder of the definitions of some of the terms used herein.

Basis: Given a vector space, a basis is a maximal collection of its linearly independent vectors; the number of basis elements is the dimension of the vector space.

Cartesian product: Given two tensors (all indices take on values $1, \cdots, n$),

$$\mathbb{T} := \{ T^{i_1,\cdots,i_p}_{j_1,\cdots,j_q} \} \quad \text{and} \quad \mathbb{U} := \{ U^{k_1,\cdots,k_r}_{l_1,\cdots,l_s} \}, \tag{B.1}$$

the *Cartesian product* (or *tensor product*) is the tensor with components

$$\mathbb{V} := \{ V^{i_1,\cdots,i_p,k_1,\cdots,k_r}_{j_1,\cdots,j_q,l_1,\cdots,l_s} := (T^{i_1,\cdots,i_p}_{j_1,\cdots,j_q} U^{k_1,\cdots,k_r}_{l_1,\cdots,l_s}) \}. \tag{B.2}$$

Contraction: Given a (p, q)-tensor $\mathbb{T}$, any $(p-1, q-1)$-tensor the components of which are representable as a sum of the form

$$\delta^{j_n}_{i_m} T^{i_1,\cdots,i_m,\cdots,i_p}_{j_1,\cdots,j_n,\cdots,j_q} = T^{i_1,\cdots,k,\cdots,i_p}_{j_1,\cdots,k,\cdots,j_q} = U^{i_1,\cdots,\widehat{i_m},\cdots,i_p}_{j_1,\cdots,\widehat{j_n},\cdots,j_q} \tag{B.3}$$

is called *a contraction of* $\mathbb{T}$; "$\widehat{i_m}$" indicates that i_m is omitted from the sequence.

Coordinate system: For our purpose, it is a system of formal variables assigned to completely, unambiguously and irredundantly describe the configurations of a (physical) system.

Linear operation: Any operation, $\mathscr{O}(\cdots)$, defined on (linear superpositions of) vectors, $\sum_i c_i\vec{v}_i$, such that the following holds:

$$\mathscr{O}\big(\sum_i c_i\vec{v}_i\big) = \sum_i c_i\, \mathscr{O}(\vec{v}_i), \qquad \text{all vectors } \vec{v}_i \text{ and all scalars } c_i. \tag{B.4}$$

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