1 Integrating via Residues

Cauchy's integral formula[1]

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{\mathrm{d}\zeta\; f(\zeta)}{(\zeta - z_0)^{n+1}} \tag{1}$$

and the associated Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)\,(z-z_0)^n, \qquad a_n(z_0) = \frac{1}{2\pi i} \oint_C \frac{\mathrm{d}\zeta\; f(\zeta)}{(\zeta - z_0)^{n+1}} \tag{2}$$

have numerous applications. In particular, if we integrate (2) along a contour that encircles z_0 once, counter-clockwise,

$$\oint_C \mathrm{d}z\; f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0) \oint_C \mathrm{d}z (z-z_0)^n, \qquad \text{change var's } z - z_0 =: \zeta = re^{i\theta}. \tag{3}$$

We modify the contour C to a circular one, centered at z_0 and of radius r:

$$\begin{aligned} &= \sum_{n=-\infty}^{\infty} a_n(z_0) \oint_{C_r'} \mathrm{d}\zeta\; \zeta^n = \sum_{n=-\infty}^{\infty} a_n(z_0) \int_0^{2\pi} (re^{i\theta}\, i\mathrm{d}\theta)\,(r^n\, e^{in\theta}) = \sum_{n=-\infty}^{\infty} a_n(z_0)\, i\, r^{n+1} \int_0^{2\pi} \mathrm{d}\theta\; e^{i(n+1)\theta}, \\ &= \sum_{\substack{n=-\infty \\ n\neq -1}}^{\infty} a_n(z_0)\, i\, r^{n+1} \Big[\underbrace{\frac{e^{i(n+1)2\pi} - e^0}{i(n+1)}}_{=0} \Big] + a_{-1}(z_0)\, i \int_0^{2\pi} \mathrm{d}\theta \;=\; 2\pi i\, a_{-1}(z_0). \end{aligned} \tag{4}$$

This being the only contribution, it is called a *residue* and denoted

$$\operatorname*{Res}_{z\to z_0}[f(z)] \stackrel{\text{def}}{=} a_{-1}(z_0) \;= \frac{1}{2\pi i} \oint_C \mathrm{d}z\; f(z), \tag{5}$$

where C is a sufficiently small contour that it encloses only one singularity of $f(z)$, the one at z_0. The key result is that —if $f(z)$ has a pole of order m at z_0— this *residue* can just as well computed as the limit

$$\boxed{\operatorname*{Res}_{z\to z_0}[f(z)] = \frac{1}{(m-1)!} \lim_{z\to z_0} \Big[\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \Big((z-z_0)^m\, f(z) \Big) \Big],} \tag{6}$$

which is easily proven by using Cauchy's integral formula (1) to write the derivative as a contour-integral:

$$\operatorname*{Res}_{z\to z_0}[f(z)] = \frac{1}{(m-1)!} \Big[\frac{(m-1)!}{2\pi i} \oint_C \frac{\mathrm{d}\zeta\; \big((\zeta-z_0)^m\, f(\zeta)\big)}{(\zeta-z_0)^{(m-1)+1}} \Big] = \frac{1}{2\pi i} \oint_C \frac{\mathrm{d}\zeta\; f(\zeta)}{(\zeta-z_0)}, \tag{7}$$

which precisely reproduces the definition (5).

For a larger contour that encloses more than one singularity, the definition (5) generalizes to

$$\boxed{\oint_C \mathrm{d}z\; f(z) = 2\pi i \sum_{z_i \circlearrowleft C} \operatorname*{Res}_{z\to z_i}[f(z)]} \tag{8}$$

where "$z_i \circlearrowleft C$" means that the summation runs over all "z_i that are encircled (once, CCW) by the contour C."

We now explore some cases where these results can be used to compute definite integrals.

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[1] Unless otherwise specified, the contour C encircles z_0 once, counter-clockwise.

1.1 Circular Integrals

Given an integral of the general form $I = \int_0^{2\pi} \mathrm{d}\theta\; f(\sin\theta, \cos\theta)$, where the integrand can be written entirely in terms of the trigonometric functions $\sin\theta$ and $\cos\theta$, we use that

$$z = e^{i\theta} \quad\Rightarrow\quad \mathrm{d}\theta = -i\frac{\mathrm{d}z}{z}, \quad \sin\theta = \frac{z-1/z}{2i}, \quad \cos\theta = \frac{z+1/z}{2}, \tag{9}$$

so that

$$I = \int_0^{2\pi} \mathrm{d}\theta\; f(\sin\theta, \cos\theta) = -i\oint_{|z|=1} \frac{\mathrm{d}z}{z}\, f\Big(\frac{z-1/z}{2i}, \frac{z+1/z}{2}\Big). \tag{10}$$

In addition to Examples 11.8.1 and 11.8.2 in [1, p. 523–524], consider a rather more complicated-looking integral

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta\; (\cos(\theta) + i\sin(\theta) - 1)^2}{4\cos(4\theta) + 4i\sin(4\theta) + 15\cos(2\theta) + 15i\sin(2\theta) - 4}, \tag{11}$$

and note that the suggested change of variables $z = e^{i\theta}$ implies that $z^k = \cos(k\theta) + i\sin(k\theta)$. Therefore

$$\cos(k\theta) = \frac{z^k + z^{-k}}{2} \qquad \sin(k\theta) = \frac{z^k - z^{-k}}{2i}, \quad \text{and} \quad \mathrm{d}\theta = -i\frac{\mathrm{d}z}{z}. \tag{12}$$

Substituting, we obtain

$$I = -i\oint_{|z|=1} \frac{\mathrm{d}z}{z}\, \frac{\big((z{+}1/z)/2 + (z{-}1/z)/2 - 1\big)^2}{2(z^4{+}1/z^4) + 2(z^4{-}1/z^4) + 15(z^2{+}1/z^2)/2 + 15(z^2{-}1/z^2)/2 - 4)}, \tag{13}$$

which looks perhaps even more intimidating... until we note that the $1/z^k$-terms cancel upon expanding the numerator and the denominator separately, while the z^k-terms add up:

$$I = -i\oint \mathrm{d}z\, F(z), \qquad F(z) \stackrel{\text{def}}{=} \frac{(z{-}1)^2}{z(z{+}2i)(z{-}2i)(2z{+}1)(2z{-}1)}, \tag{14}$$

which makes it clear that the integrand has simple poles at $z = 0$, $z = \pm\frac{1}{2}$ and $z = \pm 2i$ (which is where the denominator goes to zero), the first three of which are inside the unit circle and contribute in the residue formula:

$$I = -i\Big\{2\pi i\Big(\operatorname*{Res}_{z\to 0}[F(z)] + \operatorname*{Res}_{z\to\frac{1}{2}}[F(z)] + \operatorname*{Res}_{z\to-\frac{1}{2}}[F(z)]\Big)\Big\}, \tag{15}$$

$$= 2\pi\Big(\lim_{z\to 0}[z\,F(z)] + \lim_{z\to\frac{1}{2}}[(z{-}\tfrac{1}{2})F(z)] + \lim_{z\to-\frac{1}{2}}[(z{-}\tfrac{1}{2})F(z)]\Big), \tag{16}$$

$$= 2\pi\Big(-\frac{1}{4} + \frac{1}{34} + \frac{9}{34} = \frac{3}{68}\Big) = \frac{3\pi}{34} = 0.277199\ldots \tag{17}$$

1.2 Completing the Contour

Consider comoputing a real-valued integral $I = \int_a^b \mathrm{d}x\; f(x)$ as if it was a complex-valued integral, drawing the straight-interval "contour" $x \in [a, b]$ in the complex plane, $z = x + iy$. The general idea is then:

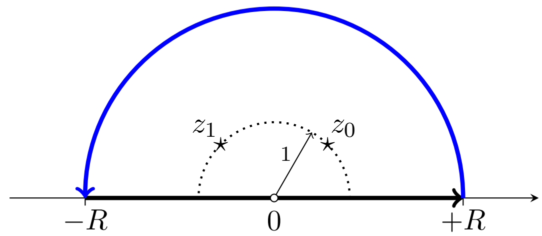
1. Complete the interval $z \in [a{+}i0, b{+}i0]$ by connecting segments to it to form a closed contour $C = [a, b] + C_1 + C_2 + \ldots,$
2. such that the complex integral $\int_{C_i} \mathrm{d}z\; f(z)$ on each additional segment is:
   (a) either easy to compute
   (b) or proportional to I.
3. Find the poles of $f(z)$ enclosed by the so-formed combined contour, and employ the residue formula:

$$\int_a^b \mathrm{d}x\; f(x) + \sum_k \oint_{C_k} \mathrm{d}z\; f(z) = \oint_C \mathrm{d}z\; f(z) \;=\; \sum_{z_i \circlearrowleft C} \operatorname*{Res}_{z\to z_i}[f(z)] \tag{18}$$

A. The integral $I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4}$ is well-defined all along the real axis, and we define it as

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4} \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}z}{1+z^4}. \tag{19}$$

The "contour" $x \in [-R, R]$ is depicted at right, together with a possible completion of the contour, the blue-ink semi-circle of radius R. The enclosed poles of the integrand are shown as "$\star$".



The poles of the complex-valued integrand $\frac{1}{1+z^4}$ are the locations where the denominator vanishes:

$$1 + z^4 = 0 \quad \Rightarrow \quad z_k^4 = -1 = e^{i\pi} = e^{i(\pi+2k\pi)}, \quad z_k = e^{i(1+2k)\pi/4}, \;\; k = 0, 1, 2, 3, \tag{20}$$

of which the first two, $z_0 = e^{i\pi/4}$ and $z_1 = e^{3i\pi/4}$ are the enclosed poles, and $z_2 = e^{5i\pi/4}$ and $z_3 = e^{7i\pi/4}$ are excluded.

The circular arc contribution is easy to compute in the limit $r \to \infty$, since this

$$\int_{\text{arc}} \frac{\mathrm{d}z}{1+z^4} = \lim_{R\to\infty} \int_0^{\pi} \frac{iRe^{i\theta}\mathrm{d}\theta}{1+(Re^{i\theta})^4} = \lim_{R\to\infty} \int_0^{\pi} \frac{iRe^{i\theta}\mathrm{d}\theta}{R^4 e^{4i\theta}} = \underbrace{\lim_{R\to\infty} \frac{i}{R^3}}_{\to 0} \int_0^{\pi} \mathrm{d}\theta\; e^{-3i\theta} = 0. \tag{21}$$

Finally, we are in the position to employ the residue formula:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4} + \int_{\text{arc}} \frac{\mathrm{d}z}{1+z^4} = \oint_C \frac{\mathrm{d}z}{1+z^4} = 2\pi i \Big\{ \operatorname*{Res}_{z\to z_0} \big[\frac{1}{1+z^4}\big] + \operatorname*{Res}_{z\to z_1} \big[\frac{1}{1+z^4}\big] \Big\}, \tag{22}$$

$$= 2\pi i \Big\{ \lim_{z\to z_0} \big[(z-z_0)\frac{1}{1+z^4}\big] + \lim_{z\to z_1} \big[(z-z_1)\frac{1}{1+z^4}\big] \Big\}. \tag{23}$$

Although it is easy to factorize $(1+z^4) = (z-z_0)(z-z_1)(z-z_2)(z-z_3)$, we proceed instead by employing L'Hospital's rule (you show that the factorization produces the same result):

$$z_0 = e^{i\pi/4}: \qquad \lim_{z\to e^{i\pi/4}} \frac{z - e^{i\pi/4}}{1+z^4} = \lim_{z\to e^{i\pi/4}} \frac{\frac{\mathrm{d}}{\mathrm{d}z}(z - e^{i\pi/4})}{\frac{\mathrm{d}}{\mathrm{d}z}(1+z^4)} = \lim_{z\to e^{i\pi/4}} \frac{1}{4z^3} = \frac{1}{4e^{3i\pi/4}} = \frac{e^{-3i\pi/4}}{4}; \tag{24}$$

$$z_1 = e^{3i\pi/4}: \quad \lim_{z\to e^{3i\pi/4}} \frac{z - e^{3i\pi/4}}{1+z^4} = \lim_{z\to e^{3i\pi/4}} \frac{\frac{\mathrm{d}}{\mathrm{d}z}(z - e^{3i\pi/4})}{\frac{\mathrm{d}}{\mathrm{d}z}(1+z^4)} = \lim_{z\to e^{3i\pi/4}} \frac{1}{4z^3} = \frac{1}{4e^{9i\pi/4}} = \frac{e^{-9i\pi/4}}{4}. \tag{25}$$

Therefore,

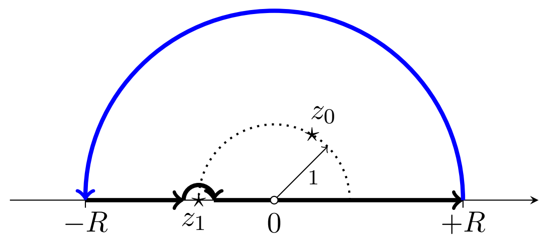
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^4} + \underbrace{\int_{\text{arc}} \frac{\mathrm{d}z}{1+z^4}}_{\to 0} = 2\pi i \Big\{ \frac{e^{-3i\pi/4}}{4} + \frac{e^{-9i\pi/4}}{4} \Big\} = 2\pi i \Big\{ -\frac{i}{2\sqrt{2}} \Big\} = \frac{\pi}{\sqrt{2}}. \tag{26}$$

Incidentally, one would obtain the same end-result if one completed the contour by adding the arc in the lower half-plane. That arc-contribution would also vanish, just like (21). This time however, the combined contour would close clockwise, thus acquiring an overall (-1) factor on the left-hand side of the analogue of (23). The so-completed contour would however enclose the poles z_2 and z_3, and exclude z_0 and z_1. But, don't take my word for it: do the math.

B. Consider now the similar integral $I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^3}$, which is *not* well-defined all along the real axis: the path of integration $x \in (-\infty, +\infty)$ encounters the singularity $x = -1$, where the integrand blows up. We then need to define the principal part as

$$I = \wp \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^3} \stackrel{\text{def}}{=} \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \Big[\int_{-R}^{-1-\epsilon} \frac{\mathrm{d}z}{1+z^3} + \int_{-1+\epsilon}^{+R} \frac{\mathrm{d}z}{1+z^3} \Big]. \tag{27}$$

Completing the contour is now a little more involved, since we need to add two arcs: one of radius R as before, and now also an ϵ-radius semicircle that connects the $(-R, -1-\epsilon]$ and $[-1+\epsilon, +R)$ segments. The result is depicted at right, where the big semicircle is depicted in blue-ink, while the ϵ-semicircle is indicated by red ink. The relevant poles of the integrand are shown as "$\star$".



The poles of the complex-valued integrand $\frac{1}{1+z^3}$ are the locations where the denominator vanishes:

$$1+z^3=0 \quad \Rightarrow \quad z_k^3=-1=e^{i\pi}=e^{i(\pi+2k\pi)}, \quad z_k=e^{i(1+2k)\pi/3}, \;\; k=0,1,2, \tag{28}$$

of which the first one, $z_0=e^{i\pi/3}$is enclosed by the above choice for the completed contour, while $z_1=e^{3i\pi/3}=-1$ and $z_2=e^{5i\pi/3}$ are excluded.

The R-radius arc contribution (blue) is again zero

$$\int_{R\text{-arc}}\frac{\mathrm{d}z}{1+z^3}=\lim_{R\to\infty}\int_0^\pi\frac{iRe^{i\theta}\mathrm{d}\theta}{1+(Re^{i\theta})^3}=\lim_{R\to\infty}\int_0^\pi\frac{iRe^{i\theta}\mathrm{d}\theta}{R^3e^{3i\theta}}=\underbrace{\lim_{R\to\infty}\frac{i}{R^2}}_{\to 0}\int_0^\pi\mathrm{d}\theta\;e^{-2i\theta}=0. \tag{29}$$

For the little contour, we change variables $z\to\zeta=z{+}1$ to re-center at $z=-1$, and then we use polar coordinates $\zeta=\epsilon e^{i\theta}$. That is, we use $z=\epsilon e^{i\theta}{-}1$, where θ varies from π to 0, clockwise:

$$\int_{\epsilon\text{-arc}}\frac{\mathrm{d}z}{1+z^3}=\underbrace{\lim_{\epsilon\to0}\int_\pi^0\frac{i\epsilon e^{i\theta}\mathrm{d}\theta}{1+(\epsilon e^{i\theta}{-}1)^3}}_{\text{indeterminate, of the } 0/0 \text{ form}}=\lim_{\epsilon\to0}\int_\pi^0\frac{\frac{\mathrm{d}}{\mathrm{d}\epsilon}i\epsilon e^{i\theta}\mathrm{d}\theta}{\frac{\mathrm{d}}{\mathrm{d}\epsilon}(1+(\epsilon e^{i\theta}{-}1)^3)}=\lim_{\epsilon\to0}\int_\pi^0\frac{ie^{i\theta}\mathrm{d}\theta}{3(\epsilon e^{i\theta}{-}1)^2(e^{i\theta})}, \tag{30}$$

$$=\lim_{\epsilon\to0}\int_\pi^0\frac{i\mathrm{d}\theta}{3(-1)^2}=\frac{i}{3}\int_\pi^0\mathrm{d}\theta=\frac{i}{3}(-\pi)=\frac{\pi}{3i}. \tag{31}$$

Notice that this infinitesimal semicircle-integral equals $\frac{1}{2}$ of the full-circle integral, and that it is computed *clockwise,* so that its residue-formula (alternative) computation is (notice the $-\pi i=(-1)\frac{1}{2}(2\pi i)$ factor multiplying the residue contributions):

$$\int_{\epsilon\text{-arc}}\frac{\mathrm{d}z}{1+z^3}=-i\pi\operatorname*{Res}_{z\to-1}\Big[\frac{1}{1+z^3}\Big]=-i\pi\lim_{z\to-1}\Big[(z-(-1))\frac{1}{1+z^3}\Big],$$

$$=-i\pi\lim_{z\to-1}\Big[\frac{\frac{\mathrm{d}}{\mathrm{d}z}(z+1)}{\frac{\mathrm{d}}{\mathrm{d}z}(1+z^3)}\Big]=-i\pi\lim_{z\to-1}\Big[\frac{1}{3z^2}\Big]=-i\pi\frac{1}{3(-1)^2}=\frac{\pi}{3i}. \tag{32}$$

Combining the segment contributions, the residue formula provides

$$\wp\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^3}+\int_{\epsilon\text{-arc}}\frac{\mathrm{d}z}{1+z^3}+\int_{R\text{-arc}}\frac{\mathrm{d}z}{1+z^3}=2\pi i\operatorname*{Res}_{z\to z_0}\Big[\frac{1}{1+z^3}\Big], \tag{33}$$

$$I+\frac{\pi}{3i}+0=2\pi i\lim_{z\to e^{i\pi/3}}\Big[\frac{(z-e^{i\pi/3})}{1+z^3}\Big]=2\pi i\lim_{z\to e^{i\pi/3}}\Big[\frac{\frac{\mathrm{d}}{\mathrm{d}z}(z-e^{i\pi/3})}{\frac{\mathrm{d}}{\mathrm{d}z}(1+z^3)}\Big]=2\pi i\lim_{z\to e^{i\pi/3}}\Big[\frac{1}{3z^2}\Big], \tag{34}$$

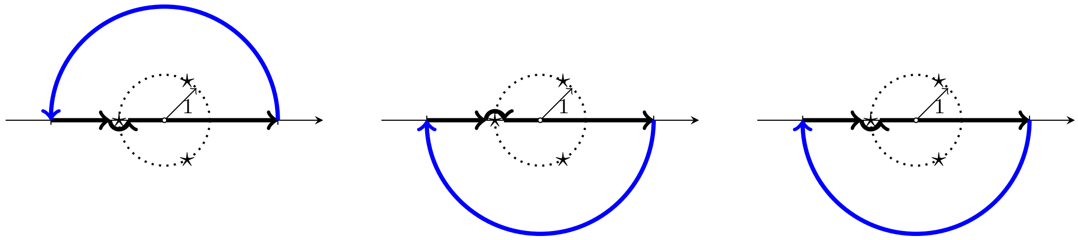
$$I+\frac{\pi}{3i}=2\pi i\frac{1}{3}(e^{-2i\pi/3}) \quad \Rightarrow \quad I=\frac{2\pi i}{3}e^{-2i\pi/3}-\frac{\pi}{3i}=\frac{2\pi i}{3}e^{-2i\pi/3}+\frac{\pi i}{3} \tag{35}$$

This may be simplifed:

$$I=\frac{\pi i}{3}\big(2e^{-2i\pi/3}+1\big)=\frac{\pi i}{3}\big(2\cos(2\pi/3)-2i\sin(2\pi/3)+1\big)=\frac{\pi i}{3}\big(2(-\tfrac{1}{2})-2i(\tfrac{\sqrt{3}}{2})+1\big), \tag{36}$$

$$= \frac{\pi i}{3}\big(-i(\sqrt{3})\big) = \frac{\pi}{3}(\sqrt{3}) = \frac{\pi}{\sqrt{3}}. \tag{37}$$

This time, both semicircles could have been drawn in the opposing way, resulting in three additional candidate contours:



(38)

each of which leads to the same end-result (37). Again, don't take my word for it: do the math.

C. In the above two examples, the integrand was such that the R-arc integral vanished in the $R \to \infty$ limit both in the upper and the lower half-plane. Obviously, this need not always be the case, and one or the other may have to be chosen. This typically hinges on analyzing what happens to the complex-extended integrand $f(z)$ in the limit $\lim_{R\to\infty} f(x+iR)$. Some of the most typical examples of this sort are analyzed in [1, p. 527–531]; they all rely on the fact that:

$$\lim_{R\to\infty} e^{iz} = \lim_{R\to\infty} e^{iRe^{i\theta}} = \lim_{R\to\infty} e^{iR\cos\theta - R\sin\theta} = \lim_{R\to\infty} \underbrace{e^{iR\cos\theta}}_{\text{bounded}} \underbrace{e^{-R\sin\theta}}_{\to 0} = 0, \qquad \theta \in [0,\pi]; \tag{39}$$

$$\lim_{R\to\infty} e^{-iz} = \lim_{R\to\infty} e^{-iRe^{i\theta}} = \lim_{R\to\infty} e^{-iR\cos\theta + R\sin\theta} = \lim_{R\to\infty} \underbrace{e^{-iR\cos\theta}}_{\text{bounded}} \underbrace{e^{R\sin\theta}}_{\to \infty} = \infty, \qquad \theta \in [0,\pi]; \tag{40}$$

while in the lower half-plane:

$$\lim_{R\to\infty} e^{iz} = \lim_{R\to\infty} e^{iRe^{i\theta}} = \lim_{R\to\infty} e^{iR\cos\theta - R\sin\theta} = \lim_{R\to\infty} \underbrace{e^{iR\cos\theta}}_{\text{bounded}} \underbrace{e^{-R\sin\theta}}_{\to \infty} = \infty, \qquad \theta \in [\pi,2\pi]; \tag{41}$$

$$\lim_{R\to\infty} e^{-iz} = \lim_{R\to\infty} e^{-iRe^{i\theta}} = \lim_{R\to\infty} e^{-iR\cos\theta + R\sin\theta} = \lim_{R\to\infty} \underbrace{e^{-iR\cos\theta}}_{\text{bounded}} \underbrace{e^{R\sin\theta}}_{\to 0} = 0, \qquad \theta \in [\pi,2\pi]. \tag{42}$$

This extends to any other function for which the same type of limiting behavior can be established.

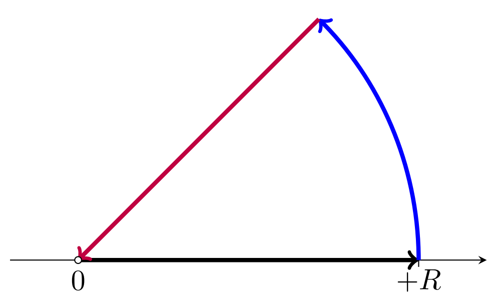
To this end, consider the behavior e^{iz^2}. Upon writing $z = Re^{i\theta}$, we have that

$$iz^2 = iR^2\cos(2\theta) - R^2\sin(2\theta), \tag{43}$$

where the sign of $\sin(2\theta)$ changes twice as fast, so that an integral involving e^{iz^2} would have a vanishing R-arc contribution only in the 1st and 3rd quadrant; those in the 2nd and 4th quadrant are divergent. Next, noting that along the bisector of the 1st and 3rd quadrant, $e^{iz^2}|_{z=re^{i\pi/4}} = e^{i(r\sqrt{i})^2} = e^{-r^2}$, we can evaluate the so-called Fresnell integral,

$$I = \int_0^\infty \mathrm{d}x\ e^{ix^2} \tag{44}$$

by completing the contour as shown to the right.



The R-arc (blue) contribution, $\int_0^{\pi/4} \mathrm{d}\theta\ e^{i(Re^{i\theta})^2} = \int_0^{\pi/4} \mathrm{d}\theta\ e^{iR^2\cos(2\theta) - R^2\sin(2\theta)} = \int_0^{\pi/4} \mathrm{d}\theta\ e^{iR^2\cos(2\theta)}\ e^{-R^2\sin(2\theta)}$ vanishes in the $R \to \infty$ limit, since $\sin(2\theta) \geq 0$ for $\theta \in [0,\pi/4]$, and $|e^{iR^2\cos(2\theta)}| \leq 1$ is bounded. The slanted (maroon) contribution, $\int_R^0 \mathrm{d}(re^{i\pi/4})\; e^{i(re^{i\pi/4})^2} = -e^{i\pi/4}\int_0^R \mathrm{d}r\; e^{ir^2(e^{i\pi/2}=i)} = -e^{i\pi/4}\int_0^R \mathrm{d}r\; e^{-r^2}$ should be familiar: in the $R\to\infty$ limit, it involves the known[2] result $\int_0^\infty \mathrm{d}r\; e^{-r^2} = \sqrt{\pi}/2$. In addition, the complex-valued function e^{iz^2} has no poles in the finite $\mathbb{C}$-plane, so that we have:

$$\Big(I = \int_0^\infty \mathrm{d}x\; e^{ix^2}\Big) + \underbrace{\int_0^{\pi/4} \mathrm{d}\theta\; e^{i(Re^{i\theta})^2}}_{\to 0} + \underbrace{\int_R^0 \mathrm{d}(re^{i\pi/4})\; e^{i(re^{i\pi/4})^2}}_{\to -e^{i\pi/4}\sqrt{\pi}} = \oint \mathrm{d}z\; e^{iz^2} = 0, \tag{45}$$

producing the result

$$I = \int_0^\infty \mathrm{d}x\; e^{ix^2} = e^{i\pi/4}\sqrt{\pi} \tag{46}$$

Including a ratio of polynomials in z in the integrand will not change the vanishing of the R-arc segment contribution, since the exponential decays faster than any polynomial. However, this *will* change the slanted segment contribution. To that end, recall formula (2.15) from the "Know Thy Math" handout:

$$\int_0^\infty \mathrm{d}t\; e^{-(\alpha t)^\beta}\, t^\gamma = \frac{\Gamma\big(\frac{\gamma+1}{\beta}\big)}{\beta\,\alpha^{(\gamma+1)}} \tag{47}$$

which can be used to evaluate, term-by-term, any slanted-segment integral of the form

$$\lim_{R\to\infty}\int_R^0 \mathrm{d}(te^{i\pi/4})\;\Big[\sum_k c_k (te^{i\pi/4})^{\gamma_k}\Big]\, e^{i(te^{i\pi/4})^2} = -\sum_k c_k\, e^{i\pi(\gamma_k+1)/4}\frac{1}{2}\Gamma\Big(\frac{\gamma_k+1}{2}\Big). \tag{48}$$

where we identified $\alpha = 1$ and $\beta = 2$. If all γ_k are non-negative integers, the integrand is analytic, and right-hand side of the analogue of Eq. (45) continues to vanish, as there are no poles to enclose.

If the integrand needs to contain a ratio of polynomials in the square brackets, the reciprocal of the denominator can be expanded into a power-series using $\frac{1}{1-\xi} = \sum_{k=0}^\infty \xi^k$, turning the desired integral into an integral over a Taylor series. Wherever this series converges absolutely, it can be integrated term-by-term, and the result re-summed. The ratio of polynomials inserted within the square-brackets in the template (48) may well have poles, and those that are enclosed within the pie-slice contour (44) will now contribute $2\pi i$-multiples of the corresponding residues. Thus, as long as this slanted (maroon) contribution can be evaluated (by using (47), for example), so can then all portions of the template (48), producing a value for integrals of the form $\int_0^\infty \mathrm{d}x\; \frac{\mathrm{poly}_1(x)}{\mathrm{poly}_2(x)}\, e^{ix^2}$.

D. The type of contour used in (44) may also be useful in another type of integral:

$$\int_0^\infty \mathrm{d}x\; f(x), \qquad \text{where} \quad f(\omega x) = \omega^n f(x), \;\; \text{with} \;\; |\omega| = 1. \tag{49}$$

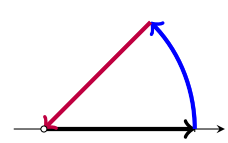
In that case, the integrals along the "spokes" $z = r\omega^k$ for $k = 0, 1, \ldots$ are all proportional to each other:

$$\int_0^\infty \mathrm{d}(\omega^k x)\; f(\omega^k x) = \omega^{(n+1)k}\int_0^\infty \mathrm{d}x\; f(x), \tag{50}$$

and we only need to connect the $k = 0$ (original) "spoke" with another one by means of an arc at infinity, choosing this other "spoke" so that the connecting arc would be easy to evaluate, and possibly zero.

To this end, consider the integral

$$I = \int_0^\infty \mathrm{d}z\; f(x), \qquad f(z) = \frac{x^8}{6 - 7x^8 + x^{24}}, \tag{51}$$



[2]This is half of the integral computed on p. 4 of "Know Thy Math" handout, and shows up in many other places.

using the 45° pie-slice contour (44). For the R-arc (blue),

$$\lim_{R\to\infty}\int_0^{\pi/4}\mathrm{d}(Re^{i\theta})\;f\big(Re^{i\theta}\big)=\lim_{R\to\infty}\int_0^{\pi/4}\frac{iRe^{i\theta}\mathrm{d}\theta\;R^8e^{8i\theta}}{6-7R^8e^{8i\theta}+R^{24}e^{24i\theta}}=\lim_{R\to\infty}\int_0^{\pi/4}\frac{iR^9e^{9i\theta}\mathrm{d}\theta}{R^{24}e^{24i\theta}},$$
$$=\underbrace{\lim_{R\to\infty}\frac{i}{R^{13}}}_{\to 0}\underbrace{\int_0^{\pi/4}\mathrm{d}\theta\;e^{-13i\theta}}_{\text{bounded}}=0. \tag{52}$$

whereas for the slanted (maroon) segment

$$\lim_{R\to\infty}\int_R^0\mathrm{d}(re^{i\pi/4})\;f\big(re^{i\pi/4}\big)=\lim_{R\to\infty}\int_R^0\frac{\mathrm{d}r\;r^8\;e^{9i\pi/4}}{6-7r^8\,e^{8i\pi/4}+r^{24}e^{24i\pi/4}}=-\lim_{R\to\infty}\int_0^R\frac{\mathrm{d}r\;r^8\;e^{i\pi/4}}{6-7r^8+r^{24}},$$
$$=-e^{i\pi/4}\lim_{R\to\infty}\int_0^R\frac{\mathrm{d}r\;r^8}{6-7r^8+r^{24}},\quad=-e^{i\pi/4}\,I. \tag{53}$$

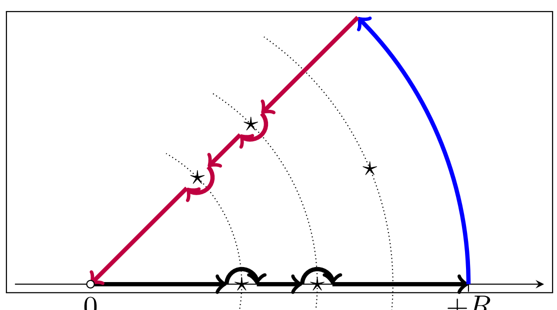
Now, the complex-valued integrand $f(z)$ has poles where the denominator vanishes:

$$6-7z^8+z^{24}=(z^8-1)(z^8-2)(z^8+3) \tag{54}$$

vanishes at the 24 points:

1. $z^8=1$: $z_{1,k}=\sqrt[8]{1{\cdot}e^{2ki\pi}}=e^{2ki\pi/8}$, $k=0,1,\dots,7$;
2. $z^8=2$: $z_{2,k}=\sqrt[8]{2{\cdot}e^{2ki\pi}}=\sqrt[8]{2}\,e^{2ki\pi/8}$, $k=0,1,\dots,7$;
3. $z^8=-3$: $z_{3,k}=\sqrt[8]{3{\cdot}e^{(2k+1)i\pi}}=\sqrt[8]{3}\,e^{(2k+1)i\pi/8}$, $k=0,1,\dots,7$.

Of these the contour encloses only $z_{3,0}$. However, note that $z_{1,0},z_{2,0}$ are in the path of the original (horizontal "spoke") integral, while $z_{1,1},z_{2,1}$ are in the path of the slanted "spoke" integral — these will modify the otherwise straight contour segments with the ϵ-semicircle detours. Among the various options, consider:



(55)

The straight-line portions of the $[0,R]$-segment provide, in the double limit ($R\to\infty$, $\epsilon\to 0$) the principal value of the original integral (51), and the principal value of the integral along the slanted "spoke" is defined accordingly, so it is equal to $-e^{i\pi/4}\,I$ as shown in (53). The ϵ-semicircular detours add, respectively $\pm\pi i\,\mathrm{Res}_{z\to z_i}[f(z)]$ for each of the singular points encountered on the path, with the signs chosen: "+" for CCW detours, and "$-$" for CW detours. Putting these together along the so-chosen contour, which encloses only the $z_{3,0}$ singularity, produces:

$$\Big(\wp I-i\pi\,\mathop{\mathrm{Res}}_{z\to 1}[f(z)]-i\pi\,\mathop{\mathrm{Res}}_{z\to\sqrt[8]{2}}[f(z)]\Big)+0+\Big(-e^{i\pi/4}\,\wp I+i\pi\mathop{\mathrm{Res}}_{z\to\sqrt[8]{2}e^{ki\pi/4}}[f(z)]+i\pi\mathop{\mathrm{Res}}_{z\to 1e^{ki\pi/4}}[f(z)]\Big)$$
$$=2i\pi\mathop{\mathrm{Res}}_{z\to\sqrt[8]{3}e^{i\pi/8}}[f(z)], \tag{56}$$

which we solve for

$$\wp I=\frac{1}{1-e^{i\pi/4}}\Big(i\pi\,\mathop{\mathrm{Res}}_{z\to 1}[f(z)]+i\pi\,\mathop{\mathrm{Res}}_{z\to\sqrt[8]{2}}[f(z)]-i\pi\mathop{\mathrm{Res}}_{z\to\sqrt[8]{2}e^{ki\pi/4}}[f(z)]-i\pi\mathop{\mathrm{Res}}_{z\to 1e^{ki\pi/4}}[f(z)]+2i\pi\mathop{\mathrm{Res}}_{z\to\sqrt[8]{3}e^{i\pi/8}}[f(z)]\Big). \tag{57}$$

Whereas this is tedious to evaluate, it is *straightforward*: all the poles are simple ($m = 1$), and the limit-formula (6) insures that each evaluation is a limit. For example,

$$\operatorname*{Res}_{z\to1}[f(z)] = \operatorname*{Res}_{z\to1}[f(z)] = \lim_{z\to1}\big[(z-1)\frac{z^8}{6-7z^8+z^{24}}\big], \tag{58}$$

which may be easier to evaluate using L'Hospital's rule than utilizing the factorization of the denominator:

$$\begin{aligned}(z-1)f(z) &= \frac{z^8(z-1)}{6-7z^8+z^{24}} = \frac{z^8(z-1)}{(z^8-1)(z^8-2)(z^8+3)},\\ &= \frac{z^8\cancel{(z-1)}}{\big(\cancel{(z-z_{1,0})}(z-z_{1,1})(z-z_{1,2})\cdots(z-z_{1,7})\big)(z^8-2)(z^8+3)},.\end{aligned} \tag{59}$$

since $z_{1,0} = e^{2(0)i\pi/8} = 1$. Either way, the final evaluation (57) is left as an exercise.

1.3 Exploiting Multi-Valuedness

In this group, read carefully the examples worked out in detail on pp. 534–537.

1.4 Exploiting Periodicity

In this group, read carefully the examples worked out in detail on pp. 537–538.

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*. Academic Press, 7 ed., 2012. 2, 5