Orthogonal Coordinate Systems

We generalize what we've learned from Cartesian coordinates to arbitrary orthogonal coordinate systems, and then specialize to cylindrical and to spherical coordinate systems.

1 Orthogonal Coordinate Systems

In Cartesian coordinates, Pythagoras' theorem guarantees that (remember Einstein's summation convention):

$$\mathrm{d}s^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 = \mathrm{d}x^i\,\delta_{ij}\,\mathrm{d}x^j \tag{1.1}$$

is the square of the infinitesimal line element along any continuous curve. This will of course look very differently in an *arbitrary* "curvilinear" coordinate system:

$$\mathrm{d}s^2 = \mathrm{d}\xi^i\,g_{ij}(\xi)\,\mathrm{d}\xi^j, \tag{1.2}$$

and the matrix of functions $g_{ij}(\xi)$ that specifies how to compute this fundamental quantity in geometry is called the *metric* of the coordinate system $(\xi^1, \xi^2, \dots)$. In retrospect, comparing (1.1) with (1.2), we see that the Kronecker symbol, δ_{ij} (which represents the identity matrix), is the metric of the Cartesian coordinate system—that's where the simplicity of the Cartesian coordinate system comes from.

The metric of the ξ-coordinate system is always related to the Cartesian metric by the change of coordinates:

$$\mathrm{d}^2s = \mathrm{d}x^i\,\delta_{ij}\,\mathrm{d}x^j = \Big(\mathrm{d}\xi^k\frac{\partial x^i}{\partial \xi^k}\Big)\delta_{ij}\Big(\frac{\partial x^j}{\partial \xi^l}\mathrm{d}\xi^l\Big) = \mathrm{d}\xi^k\Big(\underbrace{\frac{\partial x^i}{\partial \xi^k}\delta_{ij}\frac{\partial x^j}{\partial \xi^l}}_{g_{kl}(\xi)}\Big)\mathrm{d}\xi^l, \tag{1.3a}$$

$$\Rightarrow \quad g_{kl}(\xi) = \frac{\partial x^i}{\partial \xi^k}\delta_{ij}\frac{\partial x^j}{\partial \xi^l}, \quad \text{where } g_{kl}(\xi) \equiv g_{lk}(\xi) \text{ by definition.} \tag{1.3b}$$

This *metric* was used to define the scalar product, and scalar products can be used to define angles:

$$\vec{A}\cdot\vec{B} = A^i(\xi)\,g_{ij}(\xi)\,B^j(\xi) = |\vec{A}||\vec{B}|\cos(\theta_{AB}), \tag{1.4}$$

where $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A^i(\xi)\,g_{ij}(\xi)\,A^j(\xi)}$, so that

$$\theta_{AB} = \cos^{-1}\left(\frac{A^i(\xi)\,g_{ij}(\xi)\,B^j(\xi)}{\sqrt{A^i(\xi)\,g_{ij}(\xi)\,A^j(\xi)}\sqrt{B^i(\xi)\,g_{ij}(\xi)\,B^j(\xi)}}\right) \tag{1.5}$$

defines angles between vectors in terms of the scalar product of vectors. Therefore, a coordinate system (and its geometry) is fully specified by giving the list of the coordinates $\xi^1, \xi^2, \dots$) and providing the metric $g_{ij}(\xi)$, i.e., the generalization of Pythagoras' theorem.

At the end of this semester, we will see how vector calculus generalizes to all "curvilinear" coordinates. For now, we specialize to that subset, wherein the expansion (1.2) does not have any mixed terms, where

$$\mathrm{d}s^2 = \sum_{i=1}^{3}(\mathrm{d}\xi^i)^2\,g_{ii}(\xi) = \sum_{i=1}^{3}\big(h_i(\xi)\,\mathrm{d}\xi^i\big)^2, \qquad h_i(\xi) := \sqrt{g_{ii}(\xi)}. \tag{1.6}$$

That is: the metric $g_{ij}(\xi)$ in these coordinate systems is *diagonal*, i.e., $g_{ij}(\xi) = 0$ if $i \neq j$. Square-roots of the diagonal elements (the $h_i(\xi)$-coefficients) can then be used to rescale the coordinate differentials, and since ds has physical units (dimensions) of length, so do the products

$$h_1(\xi)\,\mathrm{d}\xi^1, \quad h_2(\xi)\,\mathrm{d}\xi^2, \quad h_3(\xi)\,\mathrm{d}\xi^3. \tag{1.7}$$

In this sense, these rescaled differentials are *similar* to the Cartesian differentials, straightforwardly generalizing the volume integration differential element: $\mathrm{d}^3\vec{r} = h_1 h_2 h_3 \mathrm{d}\xi^1 \mathrm{d}\xi^2 \mathrm{d}\xi^3$. Similarly, the vector-derivative operator becomes

$$\vec{\nabla} = \hat{\mathrm{e}}_1 \frac{1}{h_1}\frac{\partial}{\partial \xi^1} + \hat{\mathrm{e}}_1 \frac{1}{h_2}\frac{\partial}{\partial \xi^2} + \hat{\mathrm{e}}_1 \frac{1}{h_3}\frac{\partial}{\partial \xi^3}, \tag{1.8}$$

and its action on a scalar function is straightforward:

$$\mathbf{grad}(f) = \hat{\mathrm{e}}_1 \frac{1}{h_1}\frac{\partial f}{\partial \xi^1} + \hat{\mathrm{e}}_1 \frac{1}{h_2}\frac{\partial f}{\partial \xi^2} + \hat{\mathrm{e}}_1 \frac{1}{h_3}\frac{\partial f}{\partial \xi^3}. \tag{1.9}$$

Its action on vectors, however, is a bit trickier, since we have to take into account that not only are the components of a vector ξ-dependent, but so are the scaling factors, as well as the unit vectors: $\hat{\mathrm{e}}_{\xi^i}$ are constant in magnitude, but their directions *do* change in general.

The textbook [1, § 3.10, pp. 182–187] provides also a geometrical derivation and explanation of the following formulae:

$$\mathbf{div}(\vec{A}) = \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \left[\frac{\partial}{\partial \xi^i} \Big(\frac{A_i\, h_1 h_2 h_3}{h_i} \Big) \right], \tag{1.10}$$

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \xi^1}\big(A_1 h_2 h_3\big) + \frac{\partial}{\partial \xi^2}\big(h_1 A_2 h_3\big) + \frac{\partial}{\partial \xi^3}\big(h_1 h_2 A_3\big) \right], \tag{1.11}$$

$$\mathbf{curl}(\vec{A}) = \vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathrm{e}}_1 & h_2 \hat{\mathrm{e}}_2 & h_3 \hat{\mathrm{e}}_3 \\ \frac{\partial}{\partial \xi^1} & \frac{\partial}{\partial \xi^2} & \frac{\partial}{\partial \xi^3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{h_1 h_2 h_3} \sum_{i,j,k=1}^{3} \epsilon^{ijk} (h_i \hat{\mathrm{e}}_i) \Big[\frac{\partial}{\partial \xi^j} (h_k A_k) \Big] \tag{1.12}$$

Combining the above, we can compute

$$\mathbf{div}\big(\mathbf{grad}(f)\big) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \xi^1} \Big(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial \xi^1} \Big) + \frac{\partial}{\partial \xi^2} \Big(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial \xi^2} \Big) + \frac{\partial}{\partial \xi^3} \Big(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial \xi^3} \Big) \right] \tag{1.13}$$

as well as

$$\vec{\nabla}^2 \vec{A} = \mathbf{grad}\big(\mathbf{div}(\vec{A})\big) - \mathbf{curl}\big(\mathbf{curl}(\vec{A})\big), \tag{1.14}$$

the expansion of which is tedious in general; see below for special cases.

1.1 Cylindrical Coordinates

These coordinates (ρ, ϕ, z) may be specified by relating them to Cartesian coordinates:

$$\rho = \sqrt{x^2 + y^2}, \qquad x = \rho\, \cos(\phi), \tag{1.15a}$$

$$\phi = \mathrm{Arctan}(y, x), \qquad y = \rho\, \sin(\phi), \tag{1.15b}$$

and the cylindtical coordinate z is identical to the Cartesian z. The function $\mathrm{Arctan}(y/x)$ is defined:

$$\mathrm{Arctan}(y,x) := \tan^{-1}\left(\frac{y}{x}\right) + \vartheta(-x)\pi + \vartheta(x)\vartheta(-y)2\pi. \tag{1.16}$$

this renders the numerical value of $\mathrm{Arctan}(y,x) \in [0, 2\pi]$, which is the usual choice of the range for the polar angle ϕ. Here,

$$\vartheta(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0, \end{cases} \tag{1.17}$$

is the Heaviside step-function.

We can use the relation (1.3) to compute the metric of the cylindrical system:

$$\frac{\partial x}{\partial \rho} = \cos(\phi), \qquad \frac{\partial y}{\partial \rho} = \sin(\phi), \qquad \frac{\partial z}{\partial \rho} = 0, \tag{1.18a}$$

$$\frac{\partial x}{\partial \phi} = -\rho\sin(\phi), \qquad \frac{\partial y}{\partial \phi} = \rho\cos(\phi), \qquad \frac{\partial z}{\partial \phi} = 0, \tag{1.18b}$$

$$\frac{\partial x}{\partial z} = 0, \qquad \frac{\partial y}{\partial z} = 0, \qquad \frac{\partial z}{\partial z} = 1. \tag{1.18c}$$

Substituting these into (1.3), we have:

$$g_{\rho\rho} = \sum_{i=1}^{3}\left(\frac{\partial x^i}{\partial \rho}\right)^2 = \cos^2(\phi) + \sin^2(\phi) + (0)^2 = 1, \tag{1.19a}$$

$$g_{\rho\phi} = \sum_{i=1}^{3}\frac{\partial x^i}{\partial \rho}\frac{\partial x^i}{\partial \phi} = \big(\cos(\phi)\big)\big(-\rho\sin(\phi)\big) + \big(\sin(\phi)\big)\big(\rho\cos(\phi)\big) + (0)(0) = 0, \tag{1.19b}$$

$$g_{\rho z} = \sum_{i=1}^{3}\frac{\partial x^i}{\partial \rho}\frac{\partial x^i}{\partial z} = \big(\cos(\phi)\big)\big(-\rho\sin(\phi)\big) + \big(\sin(\phi)\big)\big(\rho\cos(\phi)\big) + (0)(0) = 0, \tag{1.19c}$$

$$g_{\phi\phi} = \sum_{i=1}^{3}\left(\frac{\partial x^i}{\partial \phi}\right)^2 = \big(-\rho\sin(\phi)\big)^2 + \big(\rho\cos(\phi)\big)^2 + (0)^2 = \rho^2, \tag{1.19d}$$

$$g_{\phi z} = \sum_{i=1}^{3}\frac{\partial x^i}{\partial \phi}\frac{\partial x^i}{\partial z} = \big(-\rho\sin(\phi)\big)(0) + \big(\rho\cos(\phi)\big)(0) + (0)(1) = 0, \tag{1.19e}$$

$$g_{zz} = \sum_{i=1}^{3}\left(\frac{\partial x^i}{\partial z}\right) = (0)(0) + (0)(0) + (1)^2 = 1, \tag{1.19f}$$

so that the (ρ, ϕ, z)-coordinate system orthogonal:

$$\big[g_{ij}(\rho,\phi,z)\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad h_\rho = 1, \;\; h_\phi = \rho, \;\; h_z = 1. \tag{1.20}$$

We thus have [1, (3.147)–(3.151), pp. 189–190]:

$$\mathbf{grad}(f) = (\vec{\nabla} f) = \hat{\mathrm{e}}_\rho\frac{\partial f}{\partial \rho} + \hat{\mathrm{e}}_\phi\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\mathrm{e}}_z\frac{\partial f}{\partial z}, \tag{1.21}$$

$$\mathbf{div}(\vec{A}) = (\vec{\nabla}\cdot\vec{A}) = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}, \tag{1.22}$$

$$\mathbf{curl}(\vec{A}) = (\vec{\nabla}\times\vec{A}) = \frac{1}{\rho}\begin{vmatrix} \hat{\mathrm{e}}_\rho & \rho\,\hat{\mathrm{e}}_\phi & \hat{\mathrm{e}}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho\,A_\phi & A_z \end{vmatrix}, \tag{1.23}$$

$$(\vec{\nabla}^2 f) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big(\rho\frac{\partial f}{\partial\rho}\Big) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}, \tag{1.24}$$

$$(\vec{\nabla}^2\vec{A})_\rho = (\vec{\nabla}^2 A_\rho) - \frac{1}{\rho^2}A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi}, \tag{1.25}$$

$$(\vec{\nabla}^2\vec{A})_\phi = (\vec{\nabla}^2 A_\phi) - \frac{1}{\rho^2}A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi}, \tag{1.26}$$

$$(\vec{\nabla}^2\vec{A})_z = (\vec{\nabla}^2 A_z). \tag{1.27}$$

1.2 Spherical Coordinates

These coordinates (r, θ, ϕ) may be specified by relating them to Cartesian coordinates:

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad x = r\ \sin(\theta)\cos(\phi), \tag{1.28a}$$

$$\phi = \mathrm{Arctan}\left(z, \sqrt{x^2 + y^2 + z^2}\right), \qquad y = r\ \sin(\theta)\sin(\phi), \tag{1.28b}$$

$$\phi = \mathrm{Arctan}(y, x), \qquad z = r\ \cos(\theta). \tag{1.28c}$$

We can use the relation (1.3) to compute the metric of the cylindrical system:

$$\frac{\partial x}{\partial r} = \sin(\theta)\cos(\phi), \qquad \frac{\partial y}{\partial r} = \sin(\theta)\sin(\phi), \qquad \frac{\partial z}{\partial r} = \cos(\theta), \tag{1.29a}$$

$$\frac{\partial x}{\partial\theta} = r\cos(\theta)\cos(\phi), \qquad \frac{\partial y}{\partial\theta} = r\cos(\theta)\sin(\phi), \qquad \frac{\partial z}{\partial\theta} = -r\ \sin(\theta), \tag{1.29b}$$

$$\frac{\partial x}{\partial\phi} = -r\ \sin(\theta)\sin(\phi), \qquad \frac{\partial y}{\partial\phi} = r\ \sin(\theta)\cos(\phi), \qquad \frac{\partial z}{\partial\phi} = 0. \tag{1.29c}$$

Substituting these into (1.3), we have:

$$g_{rr} = \sum_{i=1}^{3}\left(\frac{\partial x^i}{\partial r}\right)^2 = \underbrace{\sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi)}_{=\sin^2(\theta)} + \cos^2(\theta) \qquad = 1, \tag{1.30a}$$

$$g_{r\theta} = \sum_{i=1}^{3}\frac{\partial x^i}{\partial r}\frac{\partial x^i}{\partial\theta} = \underbrace{\Big(\sin(\theta)\cos(\phi)\Big)\Big(r\cos(\theta)\cos(\phi)\Big) + \Big(\sin(\theta)\sin(\phi)\Big)\Big(r\cos(\theta)\sin(\phi)\Big)}_{=\ r\sin(\theta)\cos(\theta)} + \Big(\cos(\theta)\Big)\Big(-r\sin(\theta)\Big) \qquad = 0, \tag{1.30b}$$

$$g_{r\phi} = \sum_{i=1}^{3}\frac{\partial x^i}{\partial r}\frac{\partial x^i}{\partial\phi} = \underbrace{\Big(\sin(\theta)\cos(\phi)\Big)\Big(-r\sin(\theta)\sin(\phi)\Big) + \Big(\sin(\theta)\sin(\phi)\Big)\Big(r\sin(\theta)\cos(\phi)\Big)}_{=\ 0} + \Big(\cos(\theta)\Big)\Big(0\Big) \qquad = 0, \tag{1.30c}$$

$$g_{\theta\theta} = \sum_{i=1}^{3} \left(\frac{\partial x^i}{\partial\theta}\right)^2 = \underbrace{\big(r\cos(\theta)\cos(\phi)\big)^2 + \big(r\cos(\theta)\sin(\phi)\big)^2}_{=\ r^2\cos^2(\theta)} + \big(-r\ \sin(\theta)\big)^2 \quad = r^2, \tag{1.30d}$$

$$g_{\theta\phi} = \sum_{i=1}^{3} \frac{\partial x^i}{\partial\theta}\frac{\partial x^i}{\partial\phi} = \underbrace{\big(r\cos(\theta)\cos(\phi)\big)\big(-r\sin(\theta)\sin(\phi)\big) + \big(r\cos(\theta)\sin(\phi)\big)\big(r\sin(\theta)\cos(\phi)\big)}_{=0}$$

$$+ \big(-r\ \sin(\theta)\big)\big(0\big) \qquad = 0, \tag{1.30e}$$

$$g_{\phi\phi} = \sum_{i=1}^{3} \left(\frac{\partial x^i}{\partial\phi}\right) = \big(-r\sin(\theta)\sin(\phi)\big)^2 + \big(r\sin(\theta)\cos(\phi)\big)^2 + \big(0\big)^2 \qquad = r^2\sin^2(\theta), \tag{1.30f}$$

so that the (r, θ, ϕ)-coordinate system orthogonal:

$$\big[g_{ij}(r,\theta,\phi)\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{bmatrix} \qquad \text{and} \qquad h_\rho = 1,\ \ h_\theta = r,\ \ h_\phi = r\sin(\theta). \tag{1.31}$$

We thus have [1, (3.156)–(3.160), p. 192]:

$$\mathbf{grad}(f) = (\vec{\nabla} f) = \hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin(\theta)}\frac{\partial f}{\partial\phi}, \tag{1.32}$$

$$\mathbf{div}(\vec{A}) = (\vec{\nabla}\cdot\vec{A}) = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial\rho} + \frac{1}{r\sin(\theta)}\frac{\partial(\sin(\theta)A_\theta)}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial(A_\phi)}{\partial\phi}, \tag{1.33}$$

$$\mathbf{curl}(\vec{A}) = (\vec{\nabla}\times\vec{A}) = \frac{1}{r\sin(\theta)}\begin{vmatrix} \hat{e}_\rho & r\,\hat{e}_\theta & r\sin(\theta)\,\hat{e}_\phi \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_\rho & r\,A_\theta & r\sin(\theta)A_\phi \end{vmatrix}, \tag{1.34}$$

$$(\vec{\nabla}^2 f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}, \tag{1.35}$$

$$(\vec{\nabla}^2\vec{A})_r = (\vec{\nabla}^2 A_r) - \frac{2}{r^2}A_r - \frac{2}{r^2\sin(\theta)}\frac{\partial(\sin(\theta)A_\theta)}{\partial\theta} - \frac{2}{r^2\sin(\theta)}\frac{\partial A_\phi}{\partial\phi}, \tag{1.36}$$

$$(\vec{\nabla}^2\vec{A})_\theta = (\vec{\nabla}^2 A_\theta) - \frac{1}{r^2\sin^2(\theta)}A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2(\theta)}\frac{\partial A_\phi}{\partial\phi}, \tag{1.37}$$

$$(\vec{\nabla}^2\vec{A})_\phi = (\vec{\nabla}^2 A_\phi) - \frac{1}{r^2\sin^2(\theta)}A_\phi + \frac{2}{r^2\sin(\theta)}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2(\theta)}\frac{\partial A_\phi}{\partial\phi}. \tag{1.38}$$

There is also the oft-used identity

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) \ \equiv\ \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} \ \equiv\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r} + f\right) \ \equiv\ \frac{1}{r}\frac{\partial^2(rf)}{\partial r^2}. \tag{1.39}$$

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*. Academic Press, 7 ed., 2012.