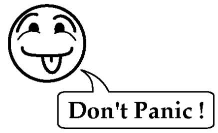
HOWARD UNIVERSITY
WASHINGTON, D.C. 20059

DEPARTMENT OF PHYSICS AND ASTRONOMY
(202)-806-6245 (Main Office)
(202)-806-5830 (FAX)

2355 Sixth Str., NW, TKH Rm.215
thubsch@howard.edu
(202)-806-6257



Mathematical Methods I
Quizz Solution

3rd Dec. '97.
by T. Hübsch

1. Calculate the integral $\int_0^\infty \mathrm{d}x\ e^{-x^2}\ x^4$ by transforming it into the Gamma-function integral. [=10pt]

Comparing to $\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}t\ e^{-t}t^{z-1}$, we notice that the integrand of the required integral involves an exponential, but that the exponent there is quadratic, rather than linear. This prompts the change of variables $x = \sqrt{t} = t^{\frac{1}{2}}$. Then, $\mathrm{d}x = \frac{1}{2}\mathrm{d}t\ t^{\frac{1}{2}-1}$. Substituting these into the required integral produces

$$\int_0^\infty \mathrm{d}x\ e^{-x^2}\ x^4 = \int_0^\infty \left(\tfrac{1}{2}\mathrm{d}t\ t^{\frac{1}{2}-1}\right)\ e^{-t}\ t^2 = \tfrac{1}{2}\int_0^\infty \mathrm{d}t\ e^{-t}\ t^{\frac{5}{2}-1} = \tfrac{1}{2}\Gamma(\tfrac{5}{2})\ .$$

Next, we use iteratively the recursion relation, $\Gamma(z+1) = z\Gamma(z)$:

$$\int_0^\infty \mathrm{d}x\ e^{-x^2}\ x^4 = \tfrac{1}{2}\Gamma(\tfrac{5}{2}) = \tfrac{1}{2}\tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\tfrac{3}{2}\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{3}{8}\sqrt{\pi}\ .$$

2. Calculate the integral $\int_0^1 \mathrm{d}x\ \big(\ln x\big)^5$ by transforming it into the Gamma-function integral. [=10pt]

As in the previous problem, comparing to $\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{d}t\ e^{-t}t^{z-1}$, we notice that the integrand of the required integral involves no exponential, but a logarithm which does not appear in the defining integral of $\Gamma(z)$. This prompts the change of variables $x = e^t$. Then, $\mathrm{d}x = \mathrm{d}t\ e^t$. Substituting these into the required integral prouces

$$\begin{aligned}\int_0^1 \mathrm{d}x\ \big(\ln x\big)^5 &= \int_{-\infty}^0 \big(\mathrm{d}t\ e^t\big)\ t^5 \stackrel{\tau=-t}{=} \int_\infty^0 \big(-\mathrm{d}\tau\ e^{-\tau}\big)\ (-\tau)^5\ ,\\ &= -\int_0^\infty \mathrm{d}\tau\ e^{-\tau}\ \tau^5\ = -\Gamma(6)\ = -5!\ =\ -120\ .\end{aligned}$$

3. Show that the equation $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = k^2\vec{A}$, where k is a constant, implies that both $\vec{\nabla}\cdot\vec{A} = 0$ and also that $\vec{\nabla}^2\vec{A} = -k^2\vec{A}$. (Hint: apply $\vec{\nabla}\cdot$ to the original equation.) [10pt.]

FOLLOW THE HINT!!!
Following the hint, we obtain:

$$\vec{\nabla}\cdot\big(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\big)\ =\ \vec{\nabla}\cdot(k^2\vec{A})\ =\ k^2\vec{\nabla}\cdot\vec{A}\ ,$$

since k is a constant. Next recall that $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{X})\equiv 0$ for **any** vector, and so also in particular for $\vec{X}=(\vec{\nabla}\times\vec{A})$. Thus, the far left-hand side in our above calculation equation vanishes, and we have

$$0 \;=\; k^2\vec{\nabla}\cdot\vec{A} \qquad \Rightarrow \qquad \vec{\nabla}\cdot\vec{A} \;=\; 0\;. \tag{1}$$

Next we use the identity (1.86) from Arfken's p. 50, to rewrite the left-hand side of the original equation and obtain:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) \;=\; \vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A} \;\stackrel{(1)}{=}\; -\vec{\nabla}^2\vec{A}\;. \tag{2}$$

The left-hand side being equal to $k^2\vec{A}$ from the original equation, we have that

$$-\vec{\nabla}^2\vec{A} \;=\; k^2\vec{A} \qquad i.e. \qquad \vec{\nabla}^2\vec{A} \;=\; -k^2\;, \tag{3}$$

which is the required second result.

Comment 1: A result such as (1) is often called the integrability or consistency condition. It is not a solution of the original equation, but a condition that must be true of the original equation is to hold. Typically, such conditions are obtained from the original one, after some further differentiation and the use of some identity. Equation (3) is then in fact the original one, rewritten by using **(a)** an identity, and **(b)** the consistency condition. Indeed, Eq. (3) is a rather simpler version of the original equation.

Comment 2: Working out something in components is quaranteed to be the longest way, and is to be done **only when all else fails**!

Comment 3: FOLLOW THE HINTS!!! ... unles you are convinced that the instructors give them for maliciously derailing purposes.

4. Circle the equation number for each generalized coordinate system (ξ,η,ζ) which is valid but not orthogonal:

$$x \;=\xi+\eta\;, \qquad y \;=\xi-\eta\;, \qquad z \;=\zeta\;; \tag{4a}$$

$$x \;=\xi+\eta\;, \qquad y \;=\xi+\eta\;, \qquad z \;=\zeta\;; \tag{4b}$$

$$x \;=\xi+\eta\;, \qquad y \;=\xi-2\eta\;, \qquad z \;=\zeta\;; \tag{4c}$$

$$x \;=\xi+\eta\;, \qquad y \;=\xi-\eta\;, \qquad z \;=2\zeta\;; \tag{4d}$$

$$x \;=\xi+\eta\;, \qquad y \;=\xi-2\eta\;, \qquad z \;=2\zeta\;; \tag{4e}$$

(Show all work below this line; use overleaf if necessary.)

As defined in class, the metric tensor is calculated as:

$$g_{jk} \stackrel{\text{def}}{=} \sum_{i=1}^{3} \frac{\partial x^i}{\partial q^j}\frac{\partial x^i}{\partial q^k} = \frac{\partial x}{\partial q^j}\frac{\partial x}{\partial q^k} + \frac{\partial y}{\partial q^j}\frac{\partial y}{\partial q^k} + \frac{\partial z}{\partial q^j}\frac{\partial z}{\partial q^k} . \tag{5}$$

Looking at the equations (4), it is fairly clear that only the third contribution to the metric is nonzero if either q^j or q^k is chosen to be ζ. Moreover, since z is in all cases only a function of ζ, both $g_{\xi\zeta}$ and $g_{\eta\zeta}$ vanish. On the other hand,

$$g_{\xi\eta} \stackrel{\text{def}}{=} \sum_{i=1}^{3} \frac{\partial x^i}{\partial \xi}\frac{\partial x^i}{\partial \eta} = \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \xi}\frac{\partial z}{\partial \eta} \tag{6}$$

is possibly nonzero and is an off-diagonal term. If non-zero, this will indicate non-orthogonality. Note also that the third contribution vanishes in all five cases, since z is only a function of ζ, not ξ, η. Now, $\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \xi} = \frac{\partial x}{\partial \eta} = 1$ in all five cases, and $\frac{\partial y}{\partial \eta}$ equals, respectively, $-1, +1, -2, -1, -2$ in the five cases. Thus, we have

$$g_{\xi,\eta} = 0\ , \quad 2\ , \quad -1\ , \quad 0\ , \quad -1\ ,$$

respectively for the five cases. Thus, the coordinate systems ($4a$) and ($4d$) are orthogonal, and ($4b$), ($4c$) and ($4e$) are not. However, note that Eqs. ($4b$) also specifies that $x = y$, which cannot possibly be true in any valid 3-dimensional system of coordinates, and we remain with ($4c$) and ($4e$) as specifying valid but not orthogonal coordinate systems (ξ, η, ζ).

5. If A^i, $i = 1, 2, 3$, are components of a contravariant vector and B_i of a covariant one, prove that $\vec{A}\cdot\vec{B} \stackrel{\text{def}}{=} \sum_{i=1}^{3} A^i B_i$ is a general scalar, *i.e.*, invariant under any change of coordinates, but that $\vec{A}^2 \stackrel{\text{def}}{=} \sum_{i=1}^{3} A^i A^i$ is a scalar only with respect to a subclass of specific transformations.

(Show all work below this line; use overleaf if necessary.)

Since $\vec{A}$ is a contravariant vector and $\vec{B}$ a covariant one, we have that, with respect to a general change of coordinates:

$$\widetilde{A}^i = \sum_{j=1}^{3} \frac{\partial \widetilde{x}^i}{\partial x^j} A^j\ , \qquad \text{and} \qquad \widetilde{B}_i = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \widetilde{x}^i} B_k\ .$$

Then

$$\sum_{i=1}^{3} \widetilde{A}^i \widetilde{B}_i = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial \widetilde{x}^i}{\partial x^j} A^j \sum_{k=1}^{3} \frac{\partial x^k}{\partial \widetilde{x}^i} B_k\ , \tag{7a}$$

$$= \sum_{j,k=1}^{3} A^j \Big[\sum_{i=1}^{3} \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{\partial x^k}{\partial \widetilde{x}^i}\Big] B_k = \sum_{j,k=1}^{3} A^j \Big[\frac{\partial x^k}{\partial x^j}\Big] B_k\ , \tag{7b}$$

$$= \sum_{j,k=1}^{3} A^j \delta_j^k B_k = \sum_{j=1}^{3} A^j B_j\ , \tag{7c}$$

which proves that $\vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A^i B_i$ is a general scalar. On the other hand,

$$\sum_{i=1}^{3} \widetilde{A}^i \widetilde{A}^i = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial\widetilde{x}^i}{\partial x^j} A^j \sum_{k=1}^{3} \frac{\partial\widetilde{x}^i}{\partial x^k} A^k \;=\; \sum_{j,k=1}^{3} A^j \Big[\sum_{i=1}^{3} \frac{\partial\widetilde{x}^i}{\partial x^j}\frac{\partial\widetilde{x}^i}{\partial x^k}\Big] A^k \tag{8}$$

does not further simplify along the lines in Eqs. (7*b*)–(7*c*), because the matrix $\frac{\partial\widetilde{x}^i}{\partial x^j}$ is not the inverse of $\frac{\partial\widetilde{x}^i}{\partial x^k}$ in general. However, in the special case when the change of coordinates $(x^1, x^2, x^3) \mapsto (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$ is a (uniform, constant) rotation—$\frac{\partial\widetilde{x}^i}{\partial x^j}$ *is* the inverse of $\frac{\partial\widetilde{x}^i}{\partial x^k}$, they contract to the identity matrix and $|\vec{A}|^2 \stackrel{\text{def}}{=} \sum_{i=1}^{3} A^i A^i$ *is* a scalar.

6. For the matrix $\mathbf{M} = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$, the Cayley-Hamilton theorem implies that $\mathbf{M}^2 = \alpha\mathbf{M} + \beta\mathbf{1}$.

a. Write down the equation satisfied by $\mathbf{M}$ owing to the Cayley-Hamilton theorem. [=5pt]

b. Calculate γ and δ in $\mathbf{M}^3 = \gamma\mathbf{M} + \delta\mathbf{1}$. [=5pt]

(Show all work below this line; use overleaf if necessary.)

a. The Cayley-Hamilton theorem says that every matrix satisfies its own secular equation, so this is what we need for the matrix $\mathbf{M}$:

$$\det\big[\mathbf{M} - \lambda\mathbf{1}\big] = \det\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \det\begin{bmatrix}-\lambda & 1\\ 1 & -\lambda\end{bmatrix} \;=\; \lambda^2 - 1\,, \tag{9}$$

so $\lambda^2 - 1 = 0$ or $\lambda^2 = 1$ is the secular equation for $\mathbf{M} = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$. The Cayley-Hamilton theorem than states that $\mathbf{M}^2 = \mathbf{1}$ (which, for this small and simple matrix is easy to check exlicitly). That is, $\alpha = 0$ and $\beta = 1$.

b. Since $\mathbf{M}^2 = \mathbf{1}$, $\mathbf{M}^3 = \mathbf{M}$. Comparison with the given equation, $\mathbf{M}^3 = \gamma\mathbf{M} + \delta\mathbf{1}$, immediately yields $\gamma = 1$ and $\delta = 0$.

—○—

In fact, it is easy to deduce that $\mathbf{M}^n = \mathbf{1}$ for any even n, and $\mathbf{M}^n = \mathbf{M}$ for odd n. Remarkably then, we can write down that any (finite or infinite) series can be simplified through this simple replacement, *i.e.*, through the use of the Cayley-Hamilton theorem. For example, $(\mathbf{1} + \mathbf{M})^4 = \mathbf{1}^4 + 4\,\mathbf{1}^3\mathbf{M} + 6\,\mathbf{1}^2\mathbf{M}^2 + 4\,\mathbf{1}\mathbf{M}^3 + \mathbf{M}^4$, which then becomes $\mathbf{1} + 4\,\mathbf{1}\mathbf{M} + 6\,\mathbf{1}\,\mathbf{1} + 4\,\mathbf{1}\mathbf{M} + \mathbf{1} = 8\mathbf{1} + 8\,\mathbf{M} = \left[\begin{smallmatrix}8 & 8\\ 8 & 8\end{smallmatrix}\right]$.

7. With $\cdot$ denoting multiplication, show that $\big(\{1, -1\}, \cdot\}\big)$ is a group.

(Show all work below this line; use overleaf if necessary.)

Closure: $(1)\cdot(1) = (1)$, $(1)\cdot(-1) = (-1)$, $(-1)\cdot(1) = (-1)$, $(-1)\cdot(-1) = (1)$; the result of the product of all pairs is either 1 or -1, both in the group.

Unit: $(1)\cdot(1) = (1)$ and $(1)\cdot(-1) = (-1)\cdot(1) = (-1)$, so 1 is the unit (identity) element.

Inverse: $(1){\cdot}(1) = (1)$ so 1 is its own inverse. Similarly, $(-1){\cdot}(-1) = (1)$, -1 is its own inverse.

Associativity: Must show that $[a{\cdot}b]{\cdot}c = a{\cdot}[b{\cdot}c]$ for all (eight!) combinations, letting a, b, c be each 1 or -1. (Left as an excercise.)

8. Consider the rotations (in the plane) by (multiples of) 90° and reflections about the horzontal and the vertical axis.

a. Find the letter(s) of the English alphabet which are (is) symmetric with respect to all of these. [=5pt]

b. Find the letter(s) which are (is) symmetric with respect to rotations by 180°. [=5pt]

(More than just the solutions to the above questions are shown below.)

a. The letters O and X remain looking recognizably same after (multiple) rotation(s) by 90° and reflections.

b. H, I, N, O, S, X and Z remain looking the same after rotation by 180°.

c. A, H, I, M, O, T, U, V, W, X, Y and Z remain looking the same after a reflection about the vertical axis.

d. B, C, D, E, H, I, K, O, X and Z remain looking the same after a reflection about the horizontal axis.

e. We construct the multiplication table for the group formed by rotatons[1]: R_k (rotation counter-clockwise by $k\pi/2$), H and V (reflections about a horizontal and about a vertical axis). As it turns out, R_1H produces an operation not in the list thus far: reflection about the NE–SW axis, which we must include as a new element, and call it, say, S. Similarly, R_1V produces a new reflection: about the WE–SE axis, which we must include as a new element, and call it, say, Z. Now this 8-element group is complete:

	$\mathbf{1}$	R_1	R_2	R_3	H	V	S	Z
$R_0 = \mathbf{1}$	$\mathbf{1}$	R_1	R_2	R_3	H	V	S	Z
R_1	R_1	R_2	R_3	$\mathbf{1}$	S	Z	V	H
R_2	R_2	R_3	$\mathbf{1}$	R_1	V	H	Z	S
R_3	R_3	$\mathbf{1}$	R_1	R_2	Z	S	H	V
H	H	Z	V	S	$\mathbf{1}$	R_2	R_3	R_1
V	V	S	H	Z	R_2	$\mathbf{1}$	R_1	R_3
S	S	H	Z	V	R_1	R_3	$\mathbf{1}$	R_2
Z	Z	V	S	H	R_3	R_1	R_2	$\mathbf{1}$

[1] Note that $R_0 = R_4 = 1$ is the 'do nothing' identify element.

9. Using that $\sin(ax) = \frac{1}{2i}\big(e^{iax} - e^{-iax}\big)$, obtain a series for $\sin(ax)$ and check the first 3 terms against the Taylor expansion. [=5+5pt]

(More than just the solutions to the above questions are shown below.)

a. As obtained in a previous class,

$$e^{iax} = \sum_{k=0}^{\infty} \frac{(iax)^k}{k!} ,$$

so, using the hint:

$$\begin{aligned}
\sin(ax) &= \frac{1}{2i}\Big[\sum_{k=0}^{\infty} \frac{(iax)^k}{k!} - \sum_{k=0}^{\infty} \frac{(-iax)^k}{k!}\Big] = \frac{1}{2i}\sum_{k=0}^{\infty}\Big[1-(-1)^k\Big]\frac{(iax)^k}{k!} ,\\
&= \frac{1}{2i}\sum_{\substack{k=0\\ k\ \text{odd}}}^{\infty} 2\frac{(iax)^k}{k!} = \sum_{n=0}^{\infty}\frac{(iax)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{i^{2n}(ax)^{2n+1}}{(2n+1)!} ,\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(ax)^{2n+1}}{(2n+1)!} = ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} + \dots
\end{aligned}$$

b. Comparison with the Taylor series is straightforward:

$$\begin{aligned}
\sin(ax) &= \sum_{k=0}^{\infty}\frac{(ax)^k}{k!}\Big[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin(ax)\Big]_{x=0} ,\\
&= 0 + ax\ \cos(0) + 0 + \frac{(ax)^3}{3!}\big[-\cos(0)\big] + 0 + \frac{(ax)^5}{5!}\big[\cos(0)\big] + \dots\\
&= ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} + \dots
\end{aligned}$$

10. Consider the function $f(z) = \sqrt[3]{z}$.

a. How many values does $f(z)$ have? [=3pt]

b. List the different values. [=3pt]

c. Determine where is $f(z)$ analytic. [=4pt]

a. The function $f(z) = \sqrt[3]{z}$ is triple valued. That is, it assigns three distinct values to every value of z. Since the set of all values of z is $\aleph_1$, this would make the number of possible values of $f(z)$ equal to $3\aleph_1$—but that's pedantry; either of the first two sentences suffices.

b. The different values were shown in class to be found as

$$\sqrt[3]{z} = \sqrt[3]{r\,\mathrm{e}^{i(\phi+2\pi k)}} = \begin{cases} \sqrt[3]{r}\,\mathrm{e}^{i\phi/3}\,, & \text{for } k=0; \\ \sqrt[3]{r}\,\mathrm{e}^{i\phi/3+2\pi i/3}\,, & \text{for } k=1; \\ \sqrt[3]{r}\,\mathrm{e}^{i\phi/3+4\pi i/3}\,, & \text{for } k=2. \end{cases}$$

After $k{=}2$, the values of $\sqrt[3]{z}$ repeat cycling through the three values given above.

c. For determining where $f(z)$ is analytic, we need the $\Re e[z]$- and the $\Im m[z]$-derivatives of $\Re e[f(z)]$ and $\Im m[f(z)]$; $f(z)$ is not analytic where

$$\frac{\partial\,\Re e[f(z)]}{\partial x} = \frac{\partial\,\Im m[f(z)]}{\partial y}\,, \qquad \frac{\partial\,\Re e[f(z)]}{\partial y} = -\frac{\partial\,\Im m[f(z)]}{\partial x}$$

do not hold simultaneously. We'll need (taking either all upper signs, or all lower signs):

$$\frac{\partial(x\pm iy)^{\frac{1}{3}}}{\partial x} = \frac{(1)}{3(x\pm iy)^{\frac{2}{3}}}\,, \qquad \frac{\partial(x\pm iy)^{\frac{1}{3}}}{\partial y} = \frac{(\pm i)}{3(x\pm iy)^{\frac{2}{3}}}\,,$$

since $\Re e[f(z)] = (\sqrt[3]{x+iy} + \sqrt[3]{x-iy})/2$ and $\Im m[f(z)] = (\sqrt[3]{x+iy} - \sqrt[3]{x-iy})/2i$. Thus,

$$\begin{aligned} \frac{1}{2}\Big[\frac{1}{3(x{+}iy)^{\frac{2}{3}}} + \frac{1}{3(x{-}iy)^{\frac{2}{3}}}\Big] &\stackrel{?}{=} \frac{1}{2i}\Big[\frac{i}{3(x{+}iy)^{\frac{2}{3}}} - \frac{-i}{3(x{-}iy)^{\frac{2}{3}}}\Big]\,, \\ \frac{1}{2}\Big[\frac{i}{3(x{+}iy)^{\frac{2}{3}}} + \frac{-i}{3(x{-}iy)^{\frac{2}{3}}}\Big] &\stackrel{?}{=} -\frac{1}{2i}\Big[\frac{1}{3(x{+}iy)^{\frac{2}{3}}} - \frac{1}{3(x{-}iy)^{\frac{2}{3}}}\Big]\,. \end{aligned}$$

It is not hard to see that both of these equations hold—except at $x,y=0$, where neither expression is well defined. Therefore, $f(z)$ is analytic for all $z\neq 0$.

11. Determine where is the function $f(z) = \frac{z+i}{z-i}$ analytic. [=10pt]

As in the previous problem, we need to check

$$\frac{\partial\,\Re e[f(z)]}{\partial x} = \frac{\partial\,\Im m[f(z)]}{\partial y}\,, \qquad \frac{\partial\,\Re e[f(z)]}{\partial y} = -\frac{\partial\,\Im m[f(z)]}{\partial x}$$

and so need (taking either all upper signs, or all lower signs):

$$\frac{\partial}{\partial x}\Big(\frac{x\pm iy\pm i}{x\pm iy\mp i}\Big) = \frac{(1)(x\pm iy\mp i) - (x\pm iy\pm i)(1)}{(x\pm iy\mp i)^2} = \frac{\mp 2i}{(x\pm iy\mp i)^2}\,,$$

and

$$\frac{\partial}{\partial y}\Big(\frac{x\pm iy\pm i}{x\pm iy\mp i}\Big) = \frac{(\pm i)(x\pm iy\mp i) - (x\pm iy\pm i)(\pm i)}{(x\pm iy\mp i)^2} = \frac{+2}{(x\pm iy\mp i)^2}\,.$$

(Note that the sign of the numerator always comes out positive.) Thus:

$$\begin{aligned}
\frac{1}{2}\Big[\frac{-2i}{(x+iy-i)^2}+\frac{+2i}{(x-iy+i)^2}\Big] &\stackrel{?}{=} \frac{1}{2i}\Big[\frac{+2}{(x+iy-i)^2}-\frac{+2}{(x-iy+i)^2}\Big]\ ,\\
\frac{1}{2}\Big[\frac{+2}{(x+iy-i)^2}+\frac{+2}{(x-iy+i)^2}\Big] &\stackrel{?}{=} -\frac{1}{2i}\Big[\frac{-2i}{(x+iy-i)^2}-\frac{+2i}{(x-iy+i)^2}\Big]\ .
\end{aligned}$$

since $\frac{1}{2i}=-\frac{i}{2}$, both of these equations hold—except at $x,y=0$, where neither expression is well defined. Therefore, $f(z)=\frac{z+i}{z-i}$ is analytic for all $z\neq 0$.

Another (slightly longer) way to calculate is to first rewrite:

$$f(z)=\frac{z+i}{z-i}=\frac{z+i}{z-i}\frac{z^*+i}{z^*+i}=\frac{x^2+y^2-1}{x^2+y^2-2y+1}+i\frac{2x}{x^2+y^2-2y+1}\ ,$$

and then proceed with the calculation of the required partial derivatives.

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Compare now the outcomes of the two problems. In both cases, the function turned out to be analytic except at $z=0$. However, the first function is obviously 'better behaved' at $z=0$—it is finite (in fact, vanishes) there while the second one diverges (blows up).