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Mathematical Methods I — 21st Nov. '97.
2nd Midterm Exam — Solutions (T. Hübsch)

The solutions are presented here with much more detail than was expected of the students' answers in the exam. Hopefully, this will provide additional information and help understanding the material more fully.

**1.a.** $\sum_{n=0}^{\infty}\frac{(-1)^n n^3}{1+n^3}$ is a number series, so our concern is absolute or conditional convergence. It should be clear that this series does not converge absolutely. For example, testing with the integral test, we have that

$$\int_1^{\infty}\frac{\mathrm{d}x\; x^3}{1+x^3} > \int_1^{\infty}\mathrm{d}x = +\infty\ .$$

Furthermore, the series does not converge even conditionally, since it does not satisfy the Leibnitz criterion (for sufficiently large $n$):

$$\frac{(n{+}1)^3}{1+(n{+}1)^3} \not< \frac{n^2}{1+n^3}\ , \qquad \text{and} \qquad \lim_{n\to\infty}\frac{n^3}{1+n^3} = 1 \neq 0\ .$$

To see the inequality more clearly multiply both sides by $(1+(n{+}1)^3)(1+n^3)$, and expand:

$$(n{+}1)^3(1+n^3) \not< n^3(1+(n{+}1)^3)\ ,$$

$$n^6+3n^5{+}3n^4{+}2n^3{+}3n^2{+}3n{+}1 \not< n^6{+}3n^5{+}3n^4{+}2n^3\ ,$$

$$3n^2{+}3n{+}1 \not< 0\ , \qquad n \geq 0\ ,$$

where the last (and quite obvious) inequality was obtained by subtracting $n^6{+}3n^5{+}3n^4{+}2n^3$ from both sides.

**1.b.** $\sum_{n=0}^{\infty}\frac{n^x}{n!}$ is a function series. It is easy to calculate the radius of absolute convergence, using the ratio test ("$\stackrel{!}{>}$" means that the inequality is required):

$$1 \stackrel{!}{>} \lim_{n\to\infty}\left|\frac{(n{+}1)^x/(n+1)!}{n^x/n!}\right| = \lim_{n\to\infty}\frac{(n{+}1)^x}{(n+1)\,n!}\,\frac{n!}{n^x} = \lim_{n\to\infty}\frac{(\frac{n+1}{n})^x}{(n+1)} = \lim_{n\to\infty}\frac{(1{+}\frac{1}{n})^x}{(n+1)} = 0\ ,$$

as long as $|x| < \infty$. Whence the above series is absolutely convergent for $|x| < \infty$. By replacing $\frac{n^x}{n!}$ with $M_n \stackrel{\text{def}}{=} \frac{n^S}{n!}$ for some arbitrarily large but finite $S$, the above calculation stands for the Weierstrass $M$-test and proves absolute *and* uniform convergence as long as $|x| < S < \infty$.

**2.** Following the hint, we first determine that $\frac{1}{(k^2+1)} \sim \frac{1}{k^2}$ for large $k$, *i.e.*, the series converges as $1/k^2$. To improve convergence, we will need to add a multiple of $\alpha_1$ (which also converges as

$\sim 1/k^2$):

$$\sum_{k=0}^{\infty} \frac{1}{(k^2{+}1)} + a_1\alpha_1 \; = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{(k^2{+}1)} + \frac{a_1}{k(k{+}1)} \right] , \tag{1a}$$

$$\sum_{k=0}^{\infty} \frac{1}{(k^2{+}1)} + a_1 \; = 1 + \sum_{k=1}^{\infty} \frac{k(k{+}1) + a_1(k^2{+}1)}{(k^2{+}1)k(k{+}1)} , \tag{1b}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(1{+}a_1)k^2 + (k{+}a_1)}{(k^2{+}1)k(k{+}1)} . \tag{1c}$$

So, if we set $a_1 = -1$, the leading ($\sim k^2$) term in the numerator vanishes, leaving

$$\sum_{k=0}^{\infty} \frac{1}{(k^2{+}1)} + (-1) = 1 + \sum_{k=1}^{\infty} \frac{k{-}1}{(k^2{+}1)k(k{+}1)} ,$$

or

$$\sum_{k=0}^{\infty} \frac{1}{(k^2{+}1)} \quad = \quad 2 + \sum_{k=1}^{\infty} \frac{k{-}1}{(k^2{+}1)k(k{+}1)} .$$

**3.** The function $f(z) = \frac{1-e^{i\pi z}}{\sin(\pi z)}$ is a bit more complicated than it may first appear. One way to analyze this is to factor $\exp(\frac{1}{2}i\pi z)$ from the numerator and rewrite

$$f(z) \;=\; \frac{1 - \mathrm{e}^{i\pi z}}{\sin(\pi z)} \;=\; \mathrm{e}^{\frac{1}{2}i\pi z}\, \frac{\mathrm{e}^{-\frac{1}{2}i\pi z} - \mathrm{e}^{\frac{1}{2}i\pi z}}{\sin(\pi z)} \;=\; -2i\, \mathrm{e}^{\frac{1}{2}i\pi z}\, \frac{\sin(\frac{1}{2}\pi z)}{\sin(\pi z)} .$$

Now, $\mathrm{e}^{\frac{1}{2}i\pi z}$ neither vanishes nor blows up anywhere in the finite $z$-plane. When $z{=}n$ is an integer, the denominator vanishes and these are the potential locations of the poles. However, when $\frac{1}{2}z{=}k$ (and so $z{=}2k$ is an even integer), the numerator vanishes also, then $f(z)$ is of the form $\frac{0}{0}$, and can be shown by the use of L'Hospital's rule to be finite there. Thus, the poles are located where $z$ is an odd integer, *i.e.*, at $z_k{=}(2k{+}1)$.

**4.a.** The integrand of $\int_{-\infty}^{\infty} \frac{\mathrm{d}x\, \mathrm{e}^{+ix}}{x^3-8}$, regarded as a complex function, is $\frac{e^{iz}}{z^3-8}$ and has three simple poles where $z^3 = 8$, *i.e.*, at $z_k = 2\mathrm{e}^{2k\pi i/3}$, $k = 0, 1, 2$. The real integral may be thought of as a complex contour integral along the real axis, and can be closed by adding a semi-circle either in the upper half-plane or in the lower half-plane. The exponential factor in the integrand may be written $e^{iz} = e^{ix}e^{-y}$. For $\Im m(z) = y \to -\infty$, the integrand would diverge exponentially and the integral would be ill defined. Thus, we close the contour in the lower half-plane, where $\Im m(z) = y > 0$ and guarantees convergence of the integral. Moreover, along this arc at infinity with $\Im m(z) = y > 0$, the integral will vanish, by Jordan's lemma (see p.424–425). Note that the straight part of the contour goes right through the pole at $z_0 = 2$. Therefore, we need to make an $\epsilon$-arc detour. I choose the detour also to be in the upper half-plane, and note that the orientation of the so chosen detour is negative (see below). Therefore, we have (assuming the limits $R \to \infty$ and $\epsilon \to 0$):

$$\oint_C \frac{e^{iz}\mathrm{d}z}{z^3-8} = \int_{-R}^{2-\epsilon} \frac{e^{ix}\mathrm{d}x}{x^3-8} + \int_{C_\epsilon} \frac{e^{iz}\mathrm{d}z}{z^3-8} + \int_{2+\epsilon}^{R} \frac{e^{ix}\mathrm{d}x}{x^3-8} + \int_0^\pi \frac{e^{iRe^{i\theta}}\, Re^{i\theta} i\mathrm{d}\theta}{R^3e^{3i\theta}-8} , \tag{2a}$$

which becomes

$$2\pi i \operatorname*{Res}_{z=z_1}\left[\frac{e^{iz}}{z^3-8}\right] = \int_{-R}^{2-\epsilon} \frac{e^{ix}\mathrm{d}x}{x^3-8} - \pi i \operatorname*{Res}_{z=z_0}\left[\frac{e^{iz}}{z^3-8}\right] + \int_{2+\epsilon}^{R} \frac{e^{ix}\mathrm{d}x}{x^3-8} + 0 , \tag{2b}$$

$$2\pi i\left[\frac{e^{iz_1}}{3z_1^2}\right] = \mathscr{P}\int_{-\infty}^{\infty} \frac{e^{ix}\mathrm{d}x}{x^3-8} - \pi i\left[\frac{e^{iz_0}}{3z_0^2}\right] , \tag{2c}$$

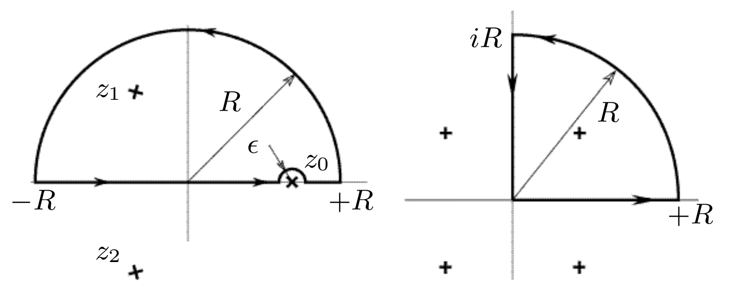
so

$$\mathscr{P}\int_{-\infty}^{\infty} \frac{e^{ix}\mathrm{d}x}{x^3-8} = 2\pi i\left[\frac{e^{iz_1}}{3z_1^2}\right] + \pi i\left[\frac{e^{iz_0}}{3z_0^2}\right] . \tag{3}$$

The two real integrals add up to the Cauchy principal part, and the residue of the simple poles was evaluated as

$$\operatorname*{Res}_{z=z_k}\left[\frac{e^{iz}}{z^3-8}\right] = \lim_{z\to z_k}\left[(z-z_k)\frac{e^{iz}}{z^3-8}\right] = \lim_{z\to z_k}\left[\frac{e^{iz}}{3z^2}\right] = \frac{e^{iz_k}}{3z_k^2} . \tag{4}$$

Since the integrand diverges at a point of the integration domain, the principal part had to be taken anyway.




The two contours: for problem # 4.a. on the left, and for # 4.b. on the right. The final evaluation of the expressions has been left out, as it does not simplify very much.

**4.b.** The integrand in $\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^4+1)}$ has simple poles at $z_k = \mathrm{e}^{(2k+1)\pi i/4}$, for $k = 0, 1, 2, 3$. Also, we note that the integral will look the same (up to an overall constant factor) if the variable $x$ is replaced by $z{=}x\,\mathrm{e}^{ik\pi/2}$, where $k{=}0, 1, 2, 3$. As for closing the contour, the integrand $\frac{z^2}{z^4+1}$ will vanish for $|z| \to \infty$, for all $\mathrm{Arg}(z)$. Thus, we have:

$$\oint_C \frac{\mathrm{d}z\ z^2}{z^4+1} = \int_0^R \frac{\mathrm{d}x\ x^2}{x^4+1} + \int_{C_R} \frac{\mathrm{d}z\ z^2}{z^4+1} + \int_{iR}^0 \frac{\mathrm{d}z\ z^2}{z^4+1} , \tag{5a}$$

which becomes ($z_1 = (1{+}i)/\sqrt{2}$)

$$2\pi i \operatorname*{Res}_{z=z_1}\left[\frac{z^2}{z^4+1}\right] = \int_0^\infty \frac{\mathrm{d}x\ x^2}{x^4+1} + 0 - \int_0^\infty \frac{\mathrm{d}(ix)\ (ix)^2}{(ix)^4+1} , \tag{5b}$$

$$2\pi i\,\frac{{z_1}^2}{4{z_1}^3} = (1+i)\int_0^\infty \frac{\mathrm{d}x\ x^2}{x^4+1} , \quad \Rightarrow \quad \int_0^\infty \frac{\mathrm{d}x\ x^2}{x^4+1} = \pi\sqrt{2} . \tag{5c}$$