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Mathematical Methods I — 10th Oct. '97.
1st Midterm Exam — Solution

The solutions are presented here with much more detail than was expected of the students' answers in the exam. Hopefully, this will provide additional information and help understanding the material more fully.

1.a: In spherical coordinates, the required integral becomes ($\mathrm{d}\vec{\sigma} = \hat{\mathrm{e}}_r\, r^2 \sin\theta \mathrm{d}\theta\, \mathrm{d}\phi$): $\vec{I} = \hat{\mathrm{e}}_r\, r^2 \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, r^{\frac{4}{3}}$ where we recognized that $x^2+y^2+z^2 = r^2$. It is terribly tempting to conclude that the θ, ϕ integrations produce simply a factor 4π, and the result is $4\pi\hat{\mathrm{e}}_r\, r^{\frac{10}{3}}$. This, however, is not true, since $\hat{\mathrm{e}}_r$ is *not* constant (as emphasized in class). Instead, we can express it in terms of constant vectors, say the Cartesian unit vectors:

$$\hat{\mathrm{e}}_r = \sin\theta\cos\phi\,\hat{\mathrm{e}}_x + \sin\theta\sin\phi\,\hat{\mathrm{e}}_y + \cos\theta\,\hat{\mathrm{e}}_z\ .$$

For the first two term, the ϕ-integrals are $\int_0^{2\pi} \sin\phi \mathrm{d}\phi = 0 = \int_0^{2\pi} \cos\phi \mathrm{d}\phi$, while for the thrid term, the θ-integral is (with $u = \cos\theta$) $\int_0^\pi \sin\theta\cos\theta \mathrm{d}\theta = -\int_1^{-1} \mathrm{d}u\, u = 0$. Thus, our integral vanishes.

1.b: For an indirect evaluation, think of the given integral as a surface-integral over the boundary surface, $S = \partial V$. We can then use a variant of Gauss's theorem, $\int_{S=\partial V} \mathrm{d}\vec{\sigma}\, f = \int_V \mathrm{d}^3\vec{r}\vec{\nabla} f$. Within the unit sphere centered at the origin, the volume integra becomes

$$\begin{aligned}\vec{I} = \int_0^1 r^2 \mathrm{d}r \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\ \vec{\nabla} r^{\frac{4}{3}} \ &=\ \int_0^1 r^2 \mathrm{d}r \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\ \hat{\mathrm{e}}_r \tfrac{4}{3} r^{\frac{1}{3}}\ ,\\ =\ \tfrac{4}{3} \int_0^1 r^{\frac{7}{3}} \mathrm{d}r \int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \hat{\mathrm{e}}_r\ .\end{aligned} \tag{1}$$

Again, $\hat{\mathrm{e}}_r$ is *not* constant; expressing it in terms of constant vectors as done in part a, will again yield the same, vanishing integrals.

2. To prove that the new system is *not* orthogonal, we only need a a single non-zero off-diagonal element in the metric. For this, we need the inverse relations, $x = \frac{1}{2}(\xi{+}\eta)$, $y = \frac{1}{2}(\xi{-}\eta)$, and after noting that $\vartheta = \frac{z^2}{(x+y)(x-y)} = \frac{z^2}{\xi\eta}$, we also have that $z = \sqrt{\xi\eta\vartheta}$. From here, it is obvious that partial derivatives of x, y with respect to ϑ vanish, whereupon $g_{\xi\vartheta}$ and $g_{\eta\vartheta}$ will have a single contribution, $\frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\vartheta}$ and $\frac{\partial z}{\partial\eta}\frac{\partial z}{\partial\vartheta}$ respectively, and hence no chance of cancellation of terms:

$$\begin{aligned} g_{\xi\theta} &\stackrel{\text{def}}{=} \frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\theta} + \frac{\partial y}{\partial\xi}\frac{\partial y}{\partial\theta} + \frac{\partial z}{\partial\xi}\frac{\partial z}{\partial\vartheta}\ ,\\ &= (\tfrac{1}{2})(0) + (\tfrac{1}{2})(0) + \Big(\frac{\eta\vartheta}{2\sqrt{\xi\eta\vartheta}}\Big)\Big(\frac{\xi\eta}{2\sqrt{\xi\eta\vartheta}}\Big)\ = \tfrac{1}{2}\eta\ . \end{aligned} \tag{2}$$

Similarly, $g_{\eta\vartheta} = \frac{1}{2}\xi$, and since these two off-diagonal elements of the metric are nonzero the coordinate system $\{\xi, \eta, \theta\}$ is not orthogonal.

3.a: The quantity $\sum_{i,j=1}^{3}(A_i g^{ij} B^j)$ transforms as a scalar, which is shown as follows:

$$\sum_{i,j=1}^{3}(A^i g_{ij} B^j) \;\to \sum_{i,j=1}^{3}(\widetilde{A}^i \widetilde{g}_{ij} \widetilde{B}^j) \;, \tag{3a}$$

$$= \sum_{i,j,k,l,m,n=1}^{3} \frac{\partial \widetilde{x}^i}{\partial x^k} A^k \frac{\partial x^l}{\partial \widetilde{x}^i} g_{lm} \frac{\partial x^m}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}^j}{\partial x^n} B^n \;, \tag{3b}$$

$$= \sum_{k,l,m,n=1}^{3} A^k \delta_k^l g_{lm} \delta_n^m B^n \;=\; \sum_{k,m=1} A^k g_{km} B^m \;, \tag{3a}$$

which, except for the relabeling $i, j \to k, m$ is the same as in the twiddled coordinate system—the hallmark of scalars.

3.b: The quantity $\sum_{i,j,k=1}^{3} A^k \frac{\partial}{\partial x^k}(A^i g_{ij} B^j)$ transforms as a scalar, which is seen as follows: In part a, we've shown that $\sum_{i,j=1}^{3}(A^i g_{ij} B^j)$ is a scalar. Then, $\frac{\partial}{\partial x^k}\big(\sum_{i,j=1}^{3} A^i g_{ij} B^j\big)$ must be a covariant vector; indeed:

$$\frac{\partial}{\partial x^k} \sum_{i,j=1}^{3}(A^i g_{ij} B^j) \;\to \frac{\partial}{\partial \widetilde{x}^k} \sum_{i,j=1}^{3}(\widetilde{A}^i \widetilde{g}_{ij} \widetilde{B}^j) \;= \tag{4a}$$

$$= \sum_{i,j,k,l,m,n,p,q=1}^{3} \frac{\partial x^p}{\partial \widetilde{x}^k} \frac{\partial}{\partial x^p} \Big(\frac{\partial \widetilde{x}^i}{\partial x^q} A^q \frac{\partial x^l}{\partial \widetilde{x}^i} g_{lm} \frac{\partial x^m}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}^j}{\partial x^n} B^n \Big) \;, \tag{4b}$$

$$= \sum_{k,l,m,n,p,q=1}^{3} \frac{\partial x^p}{\partial \widetilde{x}^k} \frac{\partial}{\partial x^p} \Big(A^q \delta_q^l g_{lm} \delta_n^m B^n \Big) \;=\; \sum_{p=1}^{3} \frac{\partial x^p}{\partial \widetilde{x}^k} \Big[\frac{\partial (AgB)}{\partial x^p} \Big] \;, \tag{4c}$$

where $(AgB) = \sum_{i,j=1}^{3}(A^i g_{ij} B^j)$. Finally, the contraction of a contravariant vector, with components A^k, and a covariant vector, with components $\frac{\partial(AgB)}{\partial x^k}$, must produce a scalar:

$$\sum_{k=1}^{3} A^k \frac{\partial (AgB)}{\partial x^k} \;\to \sum_{k=1}^{3} \widetilde{A}^k \frac{\partial \widetilde{(AgB)}}{\partial \widetilde{x}^k} \;=\; \sum_{i,j,k=1}^{3} \frac{\partial \widetilde{x}^k}{\partial x^i} A^i \frac{\partial x^j}{\partial \widetilde{x}^k} \frac{\partial (AgB)}{\partial x^j} \;, \tag{5a}$$

$$= \sum_{i,j=1}^{3} A^i \delta_i^j \frac{\partial (AgB)}{\partial x^j} \;= \sum_{i=1}^{3} A^i \frac{\partial (AgB)}{\partial x^i} \;. \tag{5b}$$

3.c: All products of the type (summation over repeated indices is implied)

$$(A^i g_{ij} A^j)^a \, (A^k g_{kl} B_l)^b \, (B_p g_{pq} B_q)^c$$

are all invariant, for arbitrary powers a, b, c. (Note the general fact here: once you have a scalar, any power of it is again a scalar.)

4.a: The eigenvalues of $M = \left(\begin{smallmatrix} 1 & \sqrt{3} \\ \sqrt{3} & a \end{smallmatrix}\right)$ must be real, since M is real and symmetric.

4.b: The determinant of a matrix must equal the product of eigenvalues (which are the only entries upon diagonalization). Hence, if one of the eigenvalues is to vanish, so must the determinant: $0 \stackrel{!}{=} \det[M] = a - 3$, whence $a = 3$.

4.c: The eigenvalues of M (with now a=3) are the solutions to the secular equation

$$0 \stackrel{!}{=} \det\left[M - \lambda \mathbb{1}\right] = (1-\lambda)(3-\lambda) - \sqrt{3}{\cdot}\sqrt{3} = \lambda^2 - 4\lambda \;, \tag{6}$$

whereupon the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 4$.

4.d: The eigenvectors are found by solving

$$\left[M - \lambda_i \mathbb{1}\right] |x_i\rangle = 0 \tag{7}$$

for each of the two eigenvalues.

$$\left[M - \lambda_1 \mathbb{1}\right] |x_1\rangle = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \;=\; 0 \;, \tag{8}$$

so $x_1 = -\sqrt{3} y_1$ and $|x_1\rangle = \frac{1}{2}\left(\begin{smallmatrix} \sqrt{3} \\ -1 \end{smallmatrix}\right)$.

$$\left[M - \lambda_2 \mathbb{1}\right] |x_2\rangle = \begin{bmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \;=\; 0 \;, \tag{9}$$

so $y_2 = \sqrt{3} x_2$ and $|x_2\rangle = \frac{1}{2}\left(\begin{smallmatrix} 1 \\ \sqrt{3} \end{smallmatrix}\right)$.

4.e: In general, $\sqrt{M}$ is to be understood in terms of a power expansion. However, owing to Cayley-Hamilton's theorem, we know that $\sqrt{M}$ is a 2×2 matrix expressible as $\alpha\mathbb{1} + \beta M$, since $M^2 - 8M - 9\mathbb{1} = 0$, and so all powers of M higher than the first are expressible as a linear combination of $\mathbb{1} = M^0$, and M. Then we have

$$M = \left(\sqrt{M}\right)^2 = \left(\alpha\mathbb{1} + \beta M\right)^2 = \begin{pmatrix} \alpha+\beta & \sqrt{3}\beta \\ \sqrt{3}\beta & \alpha+3\beta \end{pmatrix}^2 , \tag{10a}$$

$$= \begin{pmatrix} (\alpha+\beta)^2 + 3b^2 & \sqrt{3}\beta(2\alpha+4\beta) \\ \sqrt{3}\beta(2\alpha+8\beta) & (\alpha+3\beta)^2 + 3b^2 \end{pmatrix} , \tag{10b}$$

which yields three equations for the two variables α, β:

$$\begin{aligned} \alpha^2 + 2\alpha\beta + 4\beta^1 &= 1 , \\ 2\sqrt{3}\beta(\alpha + 2\beta) &= \sqrt{3} , \\ \alpha^2 + 6\alpha\beta + 12\beta^2 &= 3 . \end{aligned} \tag{11}$$

$\sqrt{3}$ times the first minus the second yields $\alpha = 0$, whereupon all three equations are solved by $\beta = \pm\frac{1}{2}$. The fact that the three equations for two variables did have solutions is a non-trivial verification of the Cayley-Hamilton theorem. The result is $\sqrt{M} = \frac{1}{2}M$.

This method of calculating transcendental functions of matrices using the Cayley-Hamilton theorem does not seem to be of startling significance in the present $2{\times}2$ case. However, consider calculating the square-root of an $n{\times}n$ matrix N. The conventional method would be to write $\sqrt{N}$ as another $n{\times}n$ matrix, R, with arbitrary n^2 variables and then solve the n^2 equations $R = N^2$. Contrast this with the method outlined above: there will only be n variables to determine in the expansion $\sqrt{N} = \sum_{k=0}^{n-1} c_k N^k$. For large matrices, the saving in computational time is then obvious—and so of special interest to computational mathematics.