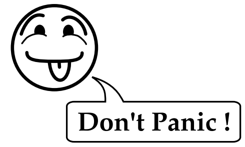
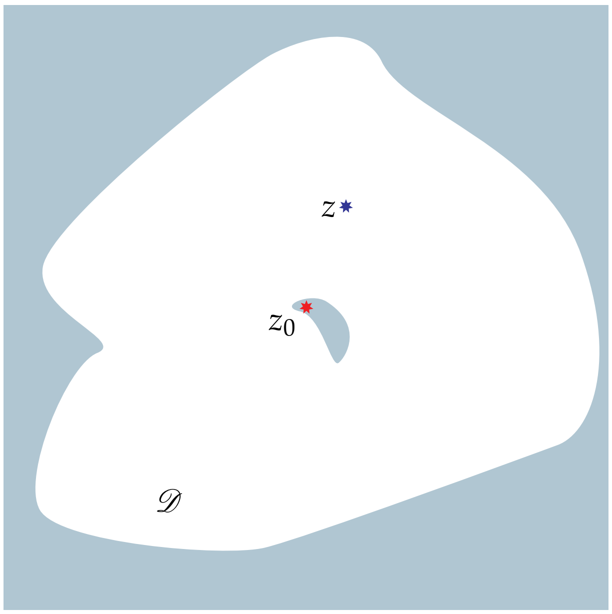
Laurent Series

(a few steps filled in)



T. Hübsch
Department of Physics & Astronomy,
Howard University, Washington, DC 20059
thubsch@howard.edu

Consider the situation presented in the display (1):



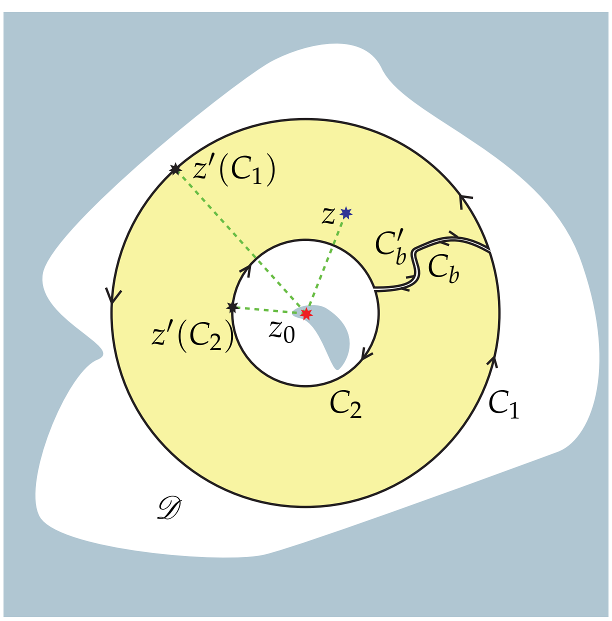
The function $f(z)$ is not known to be analytic—and may well not be analytic—in the grayish areas (outside the central region, and inside the little island in the middle), but is known to be analytic in the wiggly annular (white) area, denoted $\mathscr{D}$.

Suppose we know how the function $f(z)$ looks like at z_0 (red marker), where it is *not* analytic, and should like to determine its values in neighboring points, such as z (blue marker), where we *know* the function to be analytic, although we do not have a concrete expression. (1)

Taylor's series is out of question, since

$$f(z) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{k!} f^{(k)}(z_0) \tag{2}$$

is perfectly useless: As $f(z)$ is not analytic at $z = z_0$, we have no way of computing the derivatives $f^{(k)}(z_0)$, requisite for constructing Taylor's series. Consider then the arrangement



The combined contour $C := C_1 + C_b + C_2 + C'_b$ encloses the yellow region—wherein the function is perfectly analytic. We can therefore apply Cauchy's integral formula to evaluate $f(z)$ as an integral over this combined contour, and note that the integration over C_b will precisely cancel integration over C'_b. Also, the contour C_2 is initially oriented clockwise, as indicated in the figure, which evaluates to negative of the same contour integral done counter-clockwise. (3)

Performing then both *separate* circular contour integrals counter-clockwise, we have that $\oint_C = \oint_{C_1} - \oint_{C_2}$.

Cauchy's integral formula then gives:

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{\mathrm{d}z'\, f(z')}{(z'-z)} - \frac{1}{2\pi i} \oint_{C_2} \frac{\mathrm{d}z'\, f(z')}{(z'-z)}, \tag{4}$$

where we now attend to the two integrals in turn. As indicated in the illustration in (3), the distance $|z - z_0|$ is smaller than $|z'(C_1) - z_0|$, while z' is being swept along the contour C_1, *i.e.*, in the first integral. In turn

For the C_1-integral, then, we write

$$\frac{1}{(z'-z)} = \frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{(z'-z_0)} \frac{1}{\left[1 - (\frac{z-z_0}{z'-z_0})\right]} = \frac{1}{(z'-z_0)} \sum_{k=0}^{\infty} \Big(\frac{z-z_0}{z'-z_0}\Big)^k, \tag{5}$$

since

$$\left|\frac{z-z_0}{z'-z_0}\right| < 1 \qquad \text{while} \qquad z' \in C_1. \tag{6}$$

With this, the C_1-integral becomes

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{C_1} \frac{\mathrm{d}z'\, f(z')}{(z'-z)} &= \frac{1}{2\pi i} \oint_{C_1} \frac{\mathrm{d}z'\, f(z')}{(z'-z_0)-(z-z_0)} = \frac{1}{2\pi i} \oint_{C_1} \frac{\mathrm{d}z'\, f(z')}{(z'-z_0)} \frac{1}{1-(\frac{z-z_0}{z'-z_0})}, \\
&= \frac{1}{2\pi i} \oint_{C_1} \frac{\mathrm{d}z'\, f(z')}{(z'-z_0)} \sum_{k=0}^{\infty} \Big(\frac{z-z_0}{z'-z_0}\Big)^k = \sum_{n=0}^{\infty} (z-z_0)^n \left[\frac{1}{2\pi i} \oint_{C_1} \frac{\mathrm{d}z'\, f(z')}{(z'-z_0)^{n+1}}\right], \\
&= \sum_{n=0}^{\infty} a_n(z_0)\,(z-z_0)^n, \qquad\qquad (7)\\
a_n(z_0) &:= \frac{1}{2\pi i} \oint_{C_1} \frac{\mathrm{d}z'\, f(z')}{(z'-z_0)^{k+1}} = \frac{1}{n!} f^{(n)}(z_0), \qquad \text{for} \quad n = 0, 1, 2, 3 \ldots \qquad\qquad (8)
\end{aligned}$$

In turn, in the C_2-integral we use that now $|z - z_0| > |z'(C_1) - z_0|$, so that

$$\begin{aligned}
\frac{1}{(z'-z)} &= \frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{(z-z_0)} \frac{1}{\left[(\frac{z'-z_0}{z-z_0}) - 1\right]}, \\
&= \frac{-1}{(z-z_0)} \frac{1}{\left[1 - (\frac{z'-z_0}{z-z_0})\right]} = \frac{-1}{(z-z_0)} \sum_{k=0}^{\infty} \Big(\frac{z'-z_0}{z-z_0}\Big)^k = -\sum_{k=0}^{\infty} \frac{(z'-z_0)^k}{(z-z_0)^{k+1}}, \\
&\overset{\ell=k+1}{=} \; -\sum_{\ell=1}^{\infty} \frac{(z'-z_0)^{\ell-1}}{(z-z_0)^{\ell}} \overset{n=-\ell}{=} \; -\sum_{n=-1}^{-\infty} \frac{(z-z_0)^n}{(z'-z_0)^{n+1}}, \qquad\qquad (9)\\
\frac{1}{2\pi i} \oint_{C_2} \frac{\mathrm{d}z'\, f(z')}{(z'-z)} &= -\frac{1}{2\pi i} \oint_{C_2} \mathrm{d}z'\, f(z') \sum_{n=-\infty}^{-1} \frac{(z-z_0)^n}{(z'-z_0)^{n+1}}, \\
&= -\sum_{n=-\infty}^{-1} (z-z_0)^n \left[\frac{1}{2\pi i} \oint_{C_2} \frac{\mathrm{d}z'\, f(z')}{(z'-z_0)^{n+1}}\right]_{(n < \text{here!})}, \\
&= -\sum_{n=-\infty}^{-1} a_n(z_0)\,(z-z_0)^n, \qquad\qquad (10)\\
a_n(z_0) &:= \frac{1}{2\pi i} \oint_{C_2} \frac{\mathrm{d}z'\, f(z')}{(z'-z_0)^{n+1}}, \qquad n < 0,
\end{aligned}$$

$$= \frac{1}{2\pi i} \oint_{C_2} \mathrm{d}z' \, (z' - z_0)^{1-|n|} \, f(z'), \qquad \text{for} \quad n = -1, -2, -3 \ldots \tag{11}$$

Combining (7) and (10), we finally obtain:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0) \, (z - z_0)^n, \tag{12}$$

$$a_n(z_0) := \frac{1}{2\pi i} \oint_C \frac{\mathrm{d}z' \, f(z')}{(z' - z_0)^{n+1}}, \tag{13}$$

$$= \begin{cases} f^{(n)}(z_0), & n = 0, 1, 2, 3, \ldots \\ \frac{1}{2\pi i} \oint_C \mathrm{d}z' \, (z' - z_0)^{|n|-1} \, f(z'), & n = -1, -2, -3, \ldots \end{cases} \tag{14}$$

where the contour C is any contour within the domain of analyticity $\mathscr{D}$ that encircles the point z_0 once, counter-clockwise.

Thus, in complex analysis, it is possible to develop a (formal) power series of a function $f(z)$ around a point z_0—even if $f(z)$ is not analytic at z_0, as long as $f(z)$ is analytic in an annular domain around the point z_0, in the manner indicated in the pictures (1)–(3).