# The Gamma Function and its Cousins

## 1. The Gamma Function

Quantum mechanics (but also many other branches of Physics and Engineering) abounds with integrals where the integrand is a product of and exponential function and polynomial. Very often, these integrals can be brought to the form

$$\Gamma(z) \;\stackrel{\text{def}}{=}\; \int_0^\infty \mathrm{d}t\ t^{z-1}\, e^{-t}\ . \tag{1.1}$$

In doing so, the following maneuvers may be useful:

- ⋄ Reflecting the integration variable (replacing $t \to -t$ throughout in):

$$\int_{T_1}^{T_2} \mathrm{d}t\ f(t) \;=\; \int_{-T_1}^{-T_2} (-\mathrm{d}t)\ f(-t) \;=\; \int_{-T_2}^{-T_1} \mathrm{d}t\ f(-t)\ . \tag{1.2}$$

- ⋄ Dividing the symmetric integration range into two similar halves:

$$\begin{aligned}\int_{-T}^{+T} \mathrm{d}t\ f(t) &= \int_{-T}^{0} \mathrm{d}t\ f(t) + \int_{0}^{+T} \mathrm{d}t\ f(t) \stackrel{(1.2)}{=} \int_{0}^{+T} \mathrm{d}t\ f(-t) + \int_{0}^{+T} \mathrm{d}t\ f(t)\ ,\\ &= \int_0^{+T} \mathrm{d}t\ \big[f(-t) + f(t)\big]\ .\end{aligned} \tag{1.3}$$

  Since $f(-t)+f(t) = 2f(t)$ for even functions, while $f(-t)+f(t) = 0$ for odd functions,

$$\int_{-T}^{+T} \mathrm{d}t\ f_{\text{even}}(t) = 2\int_0^{+T} \mathrm{d}t\ f_{\text{even}}(t)\ , \qquad \int_{-T}^{+T} \mathrm{d}t\ f_{\text{odd}}(t) = 0\ . \tag{1.4}$$

- ⋄ General change of the integration variable(s)—should be a “no-brainer”:

$$\int_{x_0}^{x_1} \mathrm{d}x\, f(x) \;=\; \int_{t(x_0)}^{t(x_1)} \mathrm{d}t\ \Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big)\ f(x(t))\ . \tag{1.5}$$

- ⋄ Integration by parts—should be another “no-brainer”:

$$\int_{x_0}^{x_1} \mathrm{d}x\, f'(x)\, g(x) \;=\; \Big[f(x_1)\, g(x_1) - f(x_0)\, g(x_0)\Big] \;-\; \int_{x_0}^{x_1} \mathrm{d}x\, f(x)\, g'(x)\ . \tag{1.6}$$

  It should be quite clear that this is a straightforward consequence of the ‘product rule’: $\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$. (Hint: move the integral on the right over to the left.)

**Useful practice:** Derive the master formula:

$$\int_0^\infty \mathrm{d}t\ e^{-(\alpha t)^\beta}\ t^\gamma \;=\; \frac{\Gamma(\frac{\gamma+1}{\beta})}{\beta\,\alpha^{\gamma+1}}\ . \tag{1.7}$$

The formula will apply even for complex $\alpha, \beta, \gamma$, provided $\Re e(\alpha) > 0$. Integrals of the same type but the full range $-\infty < x < \infty$ are solved using (1.3) and this master formula. Integrals over the full range, but with a polynomial in the exponential instead of a simple power are solved by first completing the polynomial into a pure square, cube, etc., and then substituting so as to obtain a form of (1.7). So, for example,

$$\int_{-\infty}^{\infty} \mathrm{d}x \; e^{-a^2x^2+2abx} \, x^n = \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-(ax+b)^2} \, e^{b^2} \, x^n = \frac{e^{b^2}}{a^{n+1}} \int_{-\infty}^{\infty} \mathrm{d}t \; e^{-t^2} \, (t-b)^n \;, \tag{1.8}$$

whereupon you expand $(t-b)^n$ and solve each integral separately using (1.7). Note that any finite limit would have been shifted in the change $x \mapsto t = ax - b$.

### *1.1. Properties*

The benefit of changing an integral to the form (1.1) is seen upon noting that $\Gamma(z)$ satisfies a number of useful properties

$$\Gamma(1+z) = z\Gamma(z) \;, \tag{1.9a}$$

$$\Gamma(1-z) = \frac{\pi}{\Gamma(z)\sin(\pi z)} \;, \tag{1.9b}$$

$$\Gamma(k\,z) = (2\pi)^{\frac{1}{2}(1-k)} \, k^{kz-\frac{1}{2}} \prod_{r=0}^{k-1} \Gamma(z+\frac{r}{k}) \;, \qquad k \text{ an integer.} \tag{1.9c}$$

The first of these implies that $\Gamma(z) = \Gamma(1+z)/z$, which we can substitute in the second one and obtain the frequently useful *reflection formula*

$$\Gamma(1+z)\,\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)} \;. \tag{1.10}$$

Finally, for most physics applications it suffices to know that

$$\Gamma(1) = 1 \;, \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \;, \tag{1.11}$$

since most physics-related integrals (if they can be related to $\Gamma(z)$ at all) end up being expressed in terms of

$$\Gamma(n+1) = n! \;, \qquad \Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \;. \tag{1.12}$$

Here $n! \stackrel{\text{def}}{=} n{\cdot}(n-1)\cdots 2{\cdot}1$, while $n!! \stackrel{\text{def}}{=} n{\cdot}(n-2)\cdots 4{\cdot}2$ if $n$ is even, or $n!! \stackrel{\text{def}}{=} n{\cdot}(n-2)\cdots 3{\cdot}1$ if $n$ is odd. Also we have that

$$(2n)!! = 2^n \, n! \quad \text{and} \quad (2n+1)!! = \frac{(2n+1)!}{2^n \, n!} \tag{1.13}$$

The first result in (1.11) is elementary:

$$\Gamma(1) = \int_0^{\infty} \mathrm{d}t \; e^{-t} = \big(-e^{-t}\big)_{t=+\infty} - \big(-e^{-t}\big)_{t=0} = (0) - (-1) = 1 \;. \tag{1.14}$$

The second one requires a small maneuver:

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \mathrm{d}t\, t^{-\frac{1}{2}}\, e^{-t} = 2\int_0^\infty \mathrm{d}x\, e^{-x^2} = \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-x^2} \ , \tag{1.15}$$

where we changed the integration variable to $x = \sqrt{t}$, so $\frac{\mathrm{d}t}{\sqrt{t}} = 2\mathrm{d}x$ and used (1.3). Now comes a little trick (note the use of the distinct integration variables in the two factors of $\big[\Gamma(\frac{1}{2})\big]^2$):

$$\begin{aligned}
\big[\Gamma(\tfrac{1}{2})\big]^2 &= \Big[\int_{-\infty}^\infty \mathrm{d}x\, e^{-x^2}\Big]\Big[\int_{-\infty}^\infty \mathrm{d}y\, e^{-y^2}\Big] \;=\; \int_{xy-\mathrm{plane}} \mathrm{d}x\mathrm{d}y\, e^{-(x^2+y^2)} \ , \\
&= \int_0^\infty r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi\, e^{-r^2} \;=\; 2\pi\int_0^\infty r\mathrm{d}r\, e^{-r^2} \;=\; 2\pi \int_0^\infty (\tfrac{1}{2}\mathrm{d}u)\, e^{-u} \ , \\
&= \pi\Gamma(1) \;=\; \pi \ .
\end{aligned} \tag{1.16}$$

whence $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the second result in (1.11). (The second line began with the change of variables from Cartesian $(x, y)$ to polar $(r, \phi)$, where $x = r\cos\phi$ and $y = r\sin\phi$; the last integral in the second line follows upon the change of variables $u = r^2$.)

### *1.2. Related Integrals*

Other integrals that are related to the Gamma function include:

$$\Gamma(z) \;=\; \int_0^1 \mathrm{d}t\, \Big(\ln\big(\frac{1}{t}\big)\Big)^{z-1} \ , \tag{1.17a}$$

$$= \Big[\int_C \mathrm{d}\tau\ e^\tau\, \tau^{-z}\Big]^{-1} \ . \tag{1.17b}$$

The last integral is a contour integral in the complex $\tau$-plane, where the contour $C$ goes along the negative and just below the real-$\tau$ axis, encircles $\tau = 0$ counterclockwise and returns to $\tau = -\infty$ following the negative real-$\tau$ axis just above it.

Since

$$\int_0^1 \mathrm{d}t\, e^{-t}\, t^{z-1} = \sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{1}{z+k} \ , \tag{1.18}$$

(hint: expand the exponential and integrate term by term), we also have that

$$\Gamma(z) = \sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{1}{z+k} + \int_1^\infty \mathrm{d}t\ e^{-t}\, t^{z-1} \ . \tag{1.19}$$

It is not hard to prove that this integral converges for all finite $z$, so that the singularities $\Gamma(z)$ are those of the series: $z = 0, -1, -2\ldots$ Also, from this series+integral representation it is easy to calculate

$$\mathrm{Res}_{z=-n}\big(\Gamma(z)\big) = \frac{(-1)^n}{n!} \ . \tag{1.20}$$

## 2. The Psi Function

Defined as

$$\psi^{(m)}(x) \stackrel{\text{def}}{=} \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}} \ln\Gamma(z) , \qquad \psi(x) \stackrel{\text{def}}{=} \psi^{(0)}(x) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z} \ln\Gamma(z) , \tag{2.1}$$

the related function $\psi(z)$ satisfies

$$\psi(1+z) \;=\; \frac{1}{z} + \psi(z) , \tag{2.2$a$}$$

$$\psi(1-z) \;= \pi\cot(\pi z) + \psi(z) , \tag{2.2$b$}$$

$$k\psi(k\,z) \;= k\ln(k) + \sum_{r=0}^{k-1} \psi(z + \frac{r}{k}) , \qquad k \text{ an integer.} \tag{2.2$c$}$$

These are easily derived from the relations (1.9).

Integrals and sums relating to the psi function include:

$$\psi(z) \;= \int_0^\infty \frac{\mathrm{d}x}{x} \Big[e^{-x} - \frac{1}{(x+1)^2}\Big] \tag{2.3$a$}$$

$$= \int_0^\infty \frac{\mathrm{d}x}{x} \Big[e^{-x} - \frac{xe^{-xz}}{1-e^{-x}}\Big] \tag{2.3$b$}$$

$$= -\gamma + \int_0^\infty \mathrm{d}x \; \frac{e^{-x} - e^{-xz}}{1-e^{-x}} \tag{2.3$c$}$$

$$= -\gamma + \int_0^1 \mathrm{d}x \; \frac{1-x^{z-1}}{1-x} \tag{2.3$d$}$$

$$= -\gamma + \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+z}\Big) \tag{2.3$d$}$$

$$= -\gamma + \int_0^1 \mathrm{d}x \; \frac{1-x^{z-1}}{1-x} \tag{2.3$d$}$$

## 3. The Betta Function

Related is Euler's beta function (for $\Re e(x), \Re e(y) > 0$)

$$\begin{aligned} B(x,y) \stackrel{\text{def}}{=} \int_0^1 \mathrm{d}t\, t^{x-1}(1-t)^{y-1} \;&=\; \int_0^\infty \mathrm{d}u \, \frac{u^{x-1}}{(1+u)^{x+y}} , \\ = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \;&=\; \frac{x+y}{x\;y} \binom{x+y}{y}^{-1} . \end{aligned} \tag{3.1}$$

Here $\binom{x+y}{y}$ generalizes the (better be) well known binomial coefficient

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!\,(n-k)!} \;=\; \frac{n}{1}\frac{(n-1)}{2} \cdots \frac{(n-k+1)}{k} \tag{3.2}$$

from integral to complex values arguments (with positive real part). Note that the latter formula applies even if $n$ is not an integer, as long as $k$ is an integer. The Euler beta function (3.1), however, holds for even complex arguments, so we can *define* the binomial coefficient to be

$$\binom{x+y}{y} \stackrel{\text{def}}{=} \frac{(x+y)!}{x!\,y!} = \frac{x+y}{x\;y}\frac{\Gamma(x+y)}{\Gamma(x)\,\Gamma(y)} = \frac{x+y}{x\;y}\frac{1}{B(x,y)}\;, \tag{3.3}$$

which gives a well-defined result as long as $x{+}y$ is not a negative integer.

The (integral version of the) binomial coefficient appears in the *binomial expansion*:

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\;. \tag{3.4}$$

This is often used also as

$$(a+b)^n = a^n\Big[1+\Big(\frac{b}{a}\Big)\Big]^n = a^n\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{b}{a}\Big)^k\;, \tag{3.5}$$

and generalizes for cases when $n\to\nu$ is not an integer and/or $\nu<k$ into

$$(a+b)^\nu = a^\nu\Big[1+\Big(\frac{b}{a}\Big)\Big]^\nu = a^\nu\sum_{k=0}^{\infty}\frac{\nu}{k(\nu{-}k)}\frac{(b/a)^k}{B(\nu{-}k,k)}\;. \tag{3.6}$$

Since the $k$ in these expressions are always integers, the last expression in (3.2) always applies and is also the quickest way to calculate. Note, however, that once the series becomes infinite, there is the issue of convergence! The series (3.6) converges (absolutely) precisely if $b<a$. So, clearly, to apply (3.6), one factors out the larger of the two summands.

Integrals relating to the Betta function include

$$\int_0^\infty \mathrm{d}t\;\frac{\cosh(2xt)}{[\cosh(t)]^{2y}} = 4^{y-1}B(y{+}x,y{-}x)\;, \tag{3.7$a$}$$

$$\int_0^{\pi/2}\mathrm{d}\theta\;\cos^\mu(\theta)\;\sin^\nu(\theta) = \tfrac{1}{2}B\Big(\frac{\mu{+}1}{2},\frac{\nu{+}1}{2}\Big)\;, \tag{3.7$b$}$$

for $\Re e(y)>|\,\Re e(x)|$, and $\Re e(\mu),\Re e(\nu)>-1$.

## References

[1] G. Arfken: *Mathematical Methods for Physicists*, (Academic Press, New York, 1985).

[2] N.N. Lebedev: *Special Functions & Their Applications*, (Dover, New York, 1965).