The Jumping-Jack Integral

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Unh... re-using expressions on the board without carefully specifying the notation and any changes between the first writing and a later re-use leads to erroneous statements. So, here's what can be stated correctly.

1 A Curious Divergence

Consider the divergence

$$\vec{\nabla}\cdot\big(\vec{r}\,r^n\big) = \big(\vec{\nabla}\cdot\vec{r}\,\big)\,r^n + \vec{r}\cdot\big(\vec{\nabla}r^n\big) = \Big(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\Big)\,r^n + \vec{r}\cdot\Big(nr^{n-1}\big(\vec{\nabla}r\big)\Big),$$
$$= 3\,r^n + nr^{n-1}\,\vec{r}\cdot\Big(\frac{\vec{r}}{r}\Big) = 3\,r^n + nr^{n-1}\,\frac{r^2}{r} = (3+n)r^n, \tag{1}$$

where we used that

$$\vec{\nabla}\cdot\vec{r} = \Big[\hat{e}_x\frac{\partial}{\partial x}+\hat{e}_y\frac{\partial}{\partial y}+\hat{e}_z\frac{\partial}{\partial z}\Big]\cdot\big(\hat{e}_x x+\hat{e}_y y+\hat{e}_z z\big) = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 3; \tag{2}$$

$$\vec{\nabla}r = \Big[\hat{e}_x\frac{\partial}{\partial x}+\hat{e}_y\frac{\partial}{\partial y}+\hat{e}_z\frac{\partial}{\partial z}\Big]\sqrt{x^2+y^2+z^2} = \frac{\hat{e}_x 2x+\hat{e}_y 2y+\hat{e}_z 2z}{2\sqrt{x^2+y^2+z^2}},$$
$$= \frac{\hat{e}_x x+\hat{e}_y y+\hat{e}_z z}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r}. \tag{3}$$

So, Eq. (1) would seem to imply that

$$\vec{\nabla}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = (3-3)\,\frac{1}{r^3} = 0, \qquad \text{but this is not so everywhere.} \tag{4}$$

That something might be "fishy" with this computation is indicated by the fact that the function r^{-3} is ill-defined at $r=0$, and so must be its derivative. To be true, the function r^n is ill-defined for all $n<0$, but the result (1) that there's something special about $n=-3$ and $\vec{\nabla}\cdot\big(\frac{\vec{r}}{r^3}\big)$.

1.1 $n=-3$

Let's fix the value $n=-3$, and note that the computation

$$\vec{\nabla}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = (3-3)\,\frac{1}{r^3} = 0 \tag{5}$$

is safe for all values of $\vec{r}\neq\vec{0}$. Then, in particular, there is nothing wrong with integrating this function over any volume that does not include the origin. We pick such a volume constructed as follows:

1. To begin with, take V to be the inside of a (arbitrarily lumpy, large) surface S that encloses the origin.

2. Then construct $V \to V_*$, by:
   (a) excising a tiny ball (within a tiny sphere S') from around the origin,
   (b) excising a tiny wormhole/tunnel ("bridge", denoted B) connecting the inside of the excised inner sphere S' to the exterior of S.

See Fig. 1 for a sketch: both in mock-3D, and a cross-section through the "bridge" and the inner, tiny excised ball.

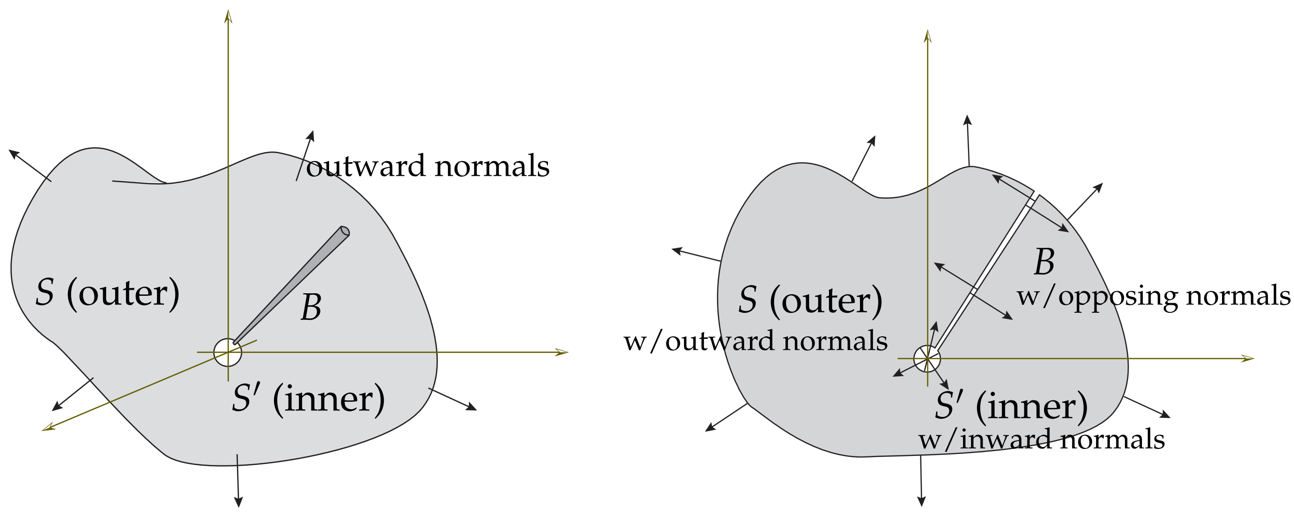

Figure 1: The surface combined as $S \cup B \cup S'$ and the volume V_* enclosed by it, which excludes the origin. The cross-section picture to the right indicates how the normals to the surface "propagate" from being outward on the outer surface S, opposing on the opposite but infinitesimally separated walls of the bridge/wormhole B, to being inwardly oriented on the inside sphere S'.

Since V_* explicitly excludes the origin, the computation (5) is correct everywhere within V_*, and also on its boundary $\partial V_* = S \cup B \cup S'$. That means we can employ Gauss's theorem

$$\int_{V_*} \mathrm{d}^3\vec{r}\, \vec{\nabla} \cdot \Big(\frac{\vec{r}}{r^3}\Big) = \int_{S\cup B\cup S'} \mathrm{d}^2\vec{\sigma} \cdot \Big(\frac{\vec{r}}{r^3}\Big), \tag{6a}$$

$$\| \qquad\qquad \|$$

$$\int_{V_*} \mathrm{d}^3\vec{r}\, 0 \qquad \Big(\int_S + \int_B + \int_{S'}\Big) \mathrm{d}^2\vec{\sigma} \cdot \Big(\frac{\vec{r}}{r^3}\Big), \tag{6b}$$

$$\| \qquad\qquad \|$$

$$0 \qquad \int_S \mathrm{d}^2\vec{\sigma} \cdot \Big(\frac{\vec{r}}{r^3}\Big) - \int_{-S'} \mathrm{d}^2\vec{\sigma} \cdot \Big(\frac{\vec{r}}{r^3}\Big). \tag{6c}$$

The tubular surface B is made infinitesimally narrow, so that contributions from the opposite points on the tube cancel each other out: the integrand will evaluate to the same magnitude but have the opposite signs, since the normals to the surface on opposite sides of a tupe are directed in opposite directions.

The inner spherical integral has inward (towards the origin) normals, so that the scalar product with $\vec{r}$ will be negative of what it would be if we had the usual, outward normals. Flipping the normals on S' $\to$ to become the more usual, outward ones reverses the orientation of this integral and changes $+\int_{S'} \to -\int_{-S'}$, and (6c) becomes

$$\int_S \mathrm{d}^2\vec{\sigma}\cdot\Big(\frac{\vec{r}}{r^3}\Big) \;=\; \int_{-S'} \mathrm{d}^2\vec{\sigma}\cdot\Big(\frac{\vec{r}}{r^3}\Big), \tag{7}$$

so that we can evaluate the integral over the large, arbitrarily lumpy surface S by means of evaluating the integral over the tiny surface S'.

Furthermore, we can choose S' to be a small, perfect sphere, of radius ϵ. Then, its (outward-oriented) surface element will be $\epsilon^2\,\sin\theta\mathrm{d}\theta,\mathrm{d}\phi$, so that

$$\int_{-S'} \mathrm{d}^2\vec{\sigma}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = \epsilon^2 \underbrace{\int_0^{\pi} \sin\theta\mathrm{d}\theta}_{=2} \underbrace{\int_0^{2\pi} \mathrm{d}\phi}_{=2\pi}\; \hat{r}\cdot\frac{\hat{r}}{\epsilon^2} = 4\pi, \qquad \text{since} \quad \hat{r}\cdot\hat{r} = 1. \tag{8}$$

Thus, we have that

$$\oint_S \mathrm{d}^2\vec{\sigma}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = 4\pi \tag{9}$$

for an arbitrarily large and lumpy surface that encloses the coordinate origin, where $\vec{r} = \vec{0}$. We now wish to extend this result to the left-hand side of Gauss's theorem

$$\int_V \mathrm{d}^3\vec{r}\;\vec{\nabla}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = \oint_S \mathrm{d}^2\vec{\sigma}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = 4\pi \tag{10}$$

were now the volume V (no longer V_*!!!) ***includes*** the origin, being the entire volume enclosed by the unmodified surface S.

The preceding computation relied on S enclosing the origin, and so it relies on V including the origin. Therefore, we have:

$$\int_V \mathrm{d}^3\vec{r}\;\vec{\nabla}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = \oint_S \mathrm{d}^2\vec{\sigma}\cdot\Big(\frac{\vec{r}}{r^3}\Big) = \begin{cases} 4\pi & \text{when} \quad V \text{ includes } \vec{0}, \\ 0 & \text{when} \quad V \text{ does not include } \ni \vec{0}. \end{cases} \tag{11}$$

This simple result indicates that the "function"

$$\delta^3(\vec{r}) \;:=\; \vec{\nabla}\cdot(\frac{\vec{r}}{r^3}) \tag{12}$$

is no ordinary function. For one thing, we can vary V as we like, as long as it includes the single point $\vec{r} = \vec{0}$, the integral is non-zero. And, as soon as this point is excluded, the integral is zero. For this to happen, $\delta^3(\vec{r})$ must in fact vanish everywere except at $\vec{r} = \vec{0}$, and there $\delta^3(\vec{r})$ must have such a "value" that its infinitesimal volume element multiple equals 4π. That is, the "value" $\delta^3(\vec{0})$ must be infinitely large, and precisely so that

$$\delta^3(\vec{0}) \;=\; \lim_{\triangle\mathrm{Vol}\to 0} \frac{4\pi}{\triangle\mathrm{Vol}}, \tag{13}$$

which is clearly undefined, limiting to division by zero. Nevertheless, as this *thing* occurs in all physics models, it is clearly a useful tool, and we refer to is as the Dirac "δ-function" in all physics literature

More properly called a *distribution*, the Dirac "δ-function" has a more rigorous definition as follows:

Definition 1.1 (Dirac's δ-Function) *For a well-defined class of (our purposes, everywhere once-differentiable) functions $f(x)$, the **Dirac δ-function** is defined to satisfy the following formula:*

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \delta(x-x_0)\, f(x) \;=\; f(x_0). \tag{14}$$

Remark 1.1: Owing to the assumption about the functions $f(x)$ for which the defining formula holds, we can do all the usual "tricks" such as differentiating the defining formula (14) by the parameter x_0, integrating both sides of (14) over x_0, integrate the integral on the right-hand side of (14) by parts and change variables in it.

Analogous "δ-functions" for (higher-dimensional) multiple integrals are obtained by multiplying such 1-dimensional "δ-functions":

$$\delta^3(\vec{r}-\vec{r}_0) \;=\; \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0) \tag{15}$$

in Cartesian coordinates. When changing to other types of coordinates, one will have to more carefully evaluate the consequences of changing variables, but this should get the Reader started.

1.2 $n > -3$

Now we have that

$$\vec{\nabla}\cdot\Big(\frac{\vec{r}}{r^n}\Big) = (3+n)r^n, \qquad \text{where} \quad 3{+}n > 0. \tag{16}$$

Consider, for example, $n = -2$. then

$$\int_V \mathrm{d}^3\vec{r}\,\frac{1}{r^2} = \int_{x_1}^{x_2}\mathrm{d}x \int_{y_1(x)}^{y_2(x)}\mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)}\mathrm{d}z\,\frac{1}{x^2+y^2+z^2}, \tag{17a}$$

$$= \int_{x_1}^{x_2}\mathrm{d}x \int_{y_1(x)}^{y_2(x)}\mathrm{d}y\,\frac{\Big[\arctan\big(\frac{z}{\sqrt{x^2+y^2}}\big)\Big]_{z_1(x,y)}^{z_2(x,y)}}{\sqrt{x^2+y^2}}, \tag{17b}$$

$$= \textit{etc.} \tag{17c}$$

where the next step in the integration can only be specified once we know the functions $z_i(x,y)$ in terms of which the volume domain of integration is specified. Amusingly, if we let V become all space, so that $x, y, z \in (-\infty, +\infty)$, we have:

$$\int_\infty \mathrm{d}^3\vec{r}\,\frac{1}{r^2} = \int_{-\infty}^{+\infty}\mathrm{d}x \int_{-\infty}^{+\infty}\mathrm{d}y \int_{\infty}^{\infty}\mathrm{d}z\,\frac{1}{x^2+y^2+z^2}, \tag{18a}$$

$$= \int_{-\infty}^{+\infty}\mathrm{d}x \int_{-\infty}^{+\infty}\mathrm{d}y\,\frac{\pi}{\sqrt{x^2+y^2}}, \tag{18b}$$

$$= \pi\int_{\infty}^{\infty}\mathrm{d}x\,\infty, \tag{18c}$$

where the y-integral diverges. Alternatively, we might attempt to solve (18b) by changing into polar coordinates: $(x,y) \to (\rho,\phi)$:

$$= \pi\int_0^\infty \rho\mathrm{d}\rho \underbrace{\int_0^{2\pi}\mathrm{d}\phi}_{=2\pi}\,\frac{1}{\rho} = 2\pi^2\int_0^\infty \mathrm{d}\rho = 2\pi^2\Big[\rho\Big]_0^\infty, \tag{18d}$$

which reveals that the integral diverges linearly, as $\lim_{\rho\to\infty}(\rho)$. It is not hard to determine that integrals for all $n > -3$ diverge if V does.

1.3 $n < -3$

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^n}\right) = (3+n) r^n, \qquad \text{where} \quad 3+n < 0. \tag{19}$$

Consider, for example, $n = -4$. then

$$\int_V \mathrm{d}^3\vec{r}\, \frac{1}{r^4} = \int_{x_1}^{x_2} \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} \mathrm{d}z\, \frac{1}{(x^2+y^2+z^2)^2}, \tag{20a}$$

$$= \int_{x_1}^{x_2} \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \left\{ \left[\frac{z}{2(x^2+y^2)(x^2+y^2+z^2)}\right]_{z_1(x,y)}^{z_2(x,y)} + \frac{\left[\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right]_{z_1(x,y)}^{z_2(x,y)}}{2(x^2+y^2)^{3/2}}, \right. \tag{20b}$$

$$= etc. \tag{20c}$$

where the next step in the integration can only be specified once we know the functions $z_i(x,y)$ in terms of which the volume domain of integration is specified. Amusingly, if we let V become all space, so that $x, y, z \in (-\infty, +\infty)$, we have:

$$\int_\infty \mathrm{d}^3\vec{r}\, \frac{1}{r^4} = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y \int_{\infty}^{\infty} \mathrm{d}z\, \frac{1}{(x^2+y^2+z^2)^2}, \tag{21a}$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y\, \frac{\pi}{2} \frac{1}{(x^2+y^2)^{3/2}}, \tag{21b}$$

where we immediately change to polar coordinates:

$$= \frac{\pi}{2} \int_0^\infty \rho \mathrm{d}\rho \underbrace{\int_0^{2\pi} \mathrm{d}\phi}_{=2\pi}\, \frac{1}{\rho^3} = \pi^2 \int_0^\infty \frac{\mathrm{d}\rho}{\rho^2} = 2\pi \left[\frac{1}{\rho}\right]_\infty^0, \tag{21c}$$

which reveals that the integral diverges linearly, as $\lim_{\rho\to 0}\left(\frac{1}{\rho}\right)$. It is not hard to determine that integrals for all $n < -3$ diverge if V includes the origin.

1.4 *So...*

There clearly exist choices of volume domains over which the integrals

$$\int_V \mathrm{d}^3\vec{r}\, \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^n}\right) \tag{22}$$

for any $n \neq -3$ diverge, and volume domains for which the integral for the same n is finite. The simple "3-cases" formula I wrote on the board was then misleading at best; it didn't even specify this drastic dependence on V. Nevertheless, the results for $n = -3$, for

$$\int_V \mathrm{d}^3\vec{r} \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^n}\right) \tag{23}$$

were the ones we were after. The rest was just to satisfy the curiosity of inquisitive Students.