Complex Integration

1. Singularities of Functions

First of all, you have to be able to locate and characterize the singularities of the integrand.

1.1. Location

To that end, note that if $g(z)$ and $h(z)$ are (simpler) functions of the complex variable z, then $f(z) = g(z)/h(z)$ diverges (tends to infinity) where $g(z)$ diverges and also where $h(z)$ vanishes. To discern the behavior of $f(z)$ *near* a singularity, it suffices to expand $g(z)$ and $h(z)$ in a Laurent series to lowest (most negative/least postitive) order. Thus:

$$z \sim 0 \;:\quad \cot(z) = \frac{\cos(z)}{\sin(z)} \approx \frac{1-\dots}{z-\dots} = \frac{1}{z}\ , \tag{1.1a}$$

$$z \sim \pi \;:\quad \cot(z) = \frac{\cos(z)}{\sin(z)} \approx \frac{(-1)-\dots}{(z-\pi)-\dots} = -\frac{1}{(z-\pi)}\ . \tag{1.1b}$$

Note that

$$\cos(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)\ , \tag{1.2}$$

so that $\cos(z)$ diverges when $y = \Im m(z)$ does, at $y \to \pm\infty$. So does, then, $\cot(z)$. To sum up, the singularities of $\cot(z)$ are at $z = 0, \pm\pi, \pm 2\pi, \dots$ and ∞.

1.2. Order (severity)

Next, we need to determine the severity of each singularity. Since

$$\cot z \sim \frac{1}{z} = z^{-1}\ , \qquad \text{for } z \sim 0\ , \tag{1.3}$$

we conclude that $z = 0$ is a pole of the 1st order of $\cot(z)$. That is, the calculation (1.1a) indicates (as easily verified by expanding beyond the leading power) that

$$\cot(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_0\, z^n\ , \tag{1.4}$$

where the (infinite) sum clearly defines the *analytic part* of the function $\cot(z)$. That is, in the corresponding Laurent expansion, $a_{-1} = 1$ and $a_n = 0$, $\forall n < -1$. This last fact implies, by definition, that $\cot(z)$ has a pole of the first order at $z = 0$. The same analysis will verify that $\cot(z)$ has poles of the first order at all of its singularitites, $z = 0, \pm 1, \pm 2, \dots, \infty$; e.g.:

$$\cot(z) = \frac{-1}{(z-\pi)} + \sum_{n=0}^{\infty} a_0'\,(z-\pi)^n\ , \qquad \textit{etc.} \tag{1.5}$$

By contrast, the function $\frac{1}{z}\cot(z)$ would have a pole of the *second* order at $z = 0$, but poles of the first order at $z = \pm 1, \pm 2, \dots, \infty$. Continuing in this vein, the function $\sec^2(z) = [\cos(z)]^{-2}$ has a (double) pole wherever $\cos(z) = 0$, *i.e.*, at $z = \pm\frac{1}{2}(2n+1)\pi$.

Finally, $e^{\frac{1}{z}}$ has an essential singularity at $z = 0$. To see this, note that the Taylor expansion $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ has an infinite radius of convergence, so that the expansion

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!} \tag{1.6}$$

is valid in the entire complex plan outside $z = 0$. This turns out to be the actual Laurant expansion of $e^{\frac{1}{z}}$, proving that there is a nonzero coefficient a_n for $n < N$, regardless how negative an N we chose: there is no limit to the "depth" of the singularity of $e^{\frac{1}{z}}$ at $z = 0$; it is *essential*. Stated differently, $\lim_{z\to 0} z^N e^{\frac{1}{z}} = \infty$ regardless of how large N is.

It is often useful to note that, if $f(z)$ has a pole of order m at z_0, so that

$$f(z) = \sum_{n\geq -m} a_n(z-z_0)^n \ , \tag{1.7}$$

the function $g(z) = [f(z)]^{-1}$ *vanishes* at z_0, and *at the rate* $(z-z_0)^m$, *i.e.*, we then know that

$$g(z) = \frac{1}{f(z)} = \sum_{n\geq +m} b_n(z - z_0)^n \ , \tag{1.8}$$

and where the Taylor coefficients b_n are related (but are not equal in general) to the Laurent coefficients a_{n-2m}. As a simple but nontrivial example, consider

$$f(z) = \frac{3}{4 - z^2} = \frac{3}{4}\left(\frac{1}{z+2} - \frac{1}{z-2}\right) = \frac{3}{4}\big(z - (-2)\big)^{-1} - \frac{3}{4}\big(z - (+2)\big)^{-1} \tag{1.9}$$

We easily read off $a_{-1} = \frac{3}{4}$ for $z_0 = -2$ and $a_{-1} = -\frac{3}{4}$ for $z_0 = +2$. On the other hand, the expansion of the reciprocal,

$$\begin{aligned}\frac{1}{f(z)} = g(z) = \tfrac{1}{3}(4-z^2) &= \tfrac{4}{3}(z-(-2)) - \tfrac{1}{3}(z-(-2))^2, \\ &= -\tfrac{4}{3}(z-(+2)) - \tfrac{1}{3}(z-(+2))^2,\end{aligned} \tag{1.10}$$

implies that $b_1 = +\frac{4}{3} = (a_{1-2} = a_{-1})^{-1}$ for $z_0 = -2$, and $b_1 = -\frac{4}{3} = (a_{1-2} = a_{-1})^{-1}$ for $z_0 = +2$. This $f(z)$ has a pole of order 1 at both $z_0 = \pm 2$, and $g(z)$ vanishes *linearly* at those same points (in all cases, it is the nonzero term of *lowest* order that determines the order of the pole and the rate of vanishing).

1.3. Residues

Finally, we will need the result

$$\oint_C \mathrm{d}z\ f(z) = 2\pi i \sum_{z_i \text{ within } C} \operatorname*{Res}_{z_i}[f(z)] \ , \tag{1.11}$$

and that

$$\operatorname*{Res}_{z_0}[f(z)] \stackrel{\text{def}}{=} a_{-1}(z_0) = \lim_{z\to z_0}\left[\frac{1}{m!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\Big((z-z_0)^m f(z)\Big)\right] \ , \tag{1.12}$$

where, when $f(z_0) = 0$, we have:

$$\lim_{z\to z_0}\left[\frac{(z-z_0)}{f(z)}\right] = \lim_{z\to z_0}\left[\frac{\frac{\mathrm{d}}{\mathrm{d}z}(z-z_0)}{\frac{\mathrm{d}}{\mathrm{d}z}f(z)}\right] = \lim_{z\to z_0}\left[\frac{1}{f'(z)}\right] \ . \tag{1.13}$$

2. Some Simple Samples

Consider, for starters, evaluating $I = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^{2n}}$, for $n = 1, 2, 3, \ldots$ We first extend the integrand into a function of the complex variable $z = x+iy = \rho e^{i\theta}$, and note that the (new, complexified) integrand satisfies

$$\lim_{\rho\to\infty} \frac{1}{1+z^{2n}} = \lim_{\rho\to\infty} \frac{1}{1+\rho^{2n}e^{2in\theta}} = 0 \;, \qquad \forall\theta \;. \tag{2.1}$$

We thus complete the contour $z \in (-\infty, +\infty)$ by adding to it a semi-circle at infinity, choosing the semicircle in the upper half-plain. Then

$$\begin{aligned}
\oint_C \frac{\mathrm{d}z}{1+z^{2n}} &= I + \int\limits_{\frac{1}{2}C@\infty,\ \Im m(z)\geq 0} \frac{\mathrm{d}z}{1+z^{2n}} = I + \lim_{\rho\to\infty} \int_0^\pi \frac{i\mathrm{d}\theta\rho e^{i\theta}}{1+\rho^{2n}e^{2in\theta}} \;, \\
&= I + i \lim_{\rho\to\infty} \rho^{1-2n} \int_0^\pi \frac{\mathrm{d}\theta e^{i\theta}}{\rho^{-2n}+e^{2in\theta}} \;, \qquad (\text{note: } \rho^{-2n}\to 0) \\
&= I + i \underbrace{\lim_{\rho\to\infty} \rho^{1-2n}}_{\to 0} \int_0^\pi \mathrm{d}\theta e^{i\theta(1-2n)} \;.
\end{aligned} \tag{2.2}$$

That is, the integral along the semi-circle at infinity vanishes, and the whole contour integral receives contributions only from the original I. Thus,

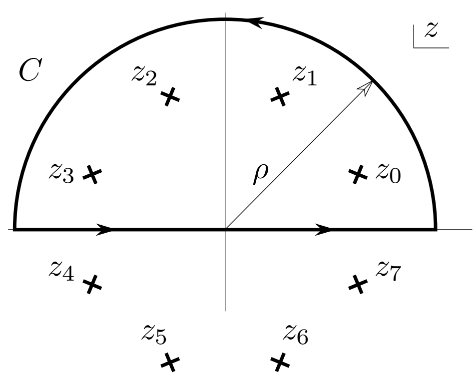
$$I = \oint_C \frac{\mathrm{d}z}{1+z^{2n}} = 2\pi i \sum_{\Im m(z_k)>0} \operatorname*{Res}_{z_k} \left[\frac{1}{1+z^{2n}}\right] \;, \tag{2.3}$$

where z_k are the poles of $\frac{1}{1+z^{2n}}$, *i.e.*, the zeros of $1+z^{2n}$. Since $z^{2n} = -1 = e^{i\pi} = e^{i\pi+2ki\pi}$, $\forall k \in \mathbb{Z}$, we have that $z_k = e^{i\pi\frac{2k+1}{2n}}$. Since

$$z_{k+2n} = e^{i\pi\frac{2(k+2n)+1}{2n}} = e^{i\pi\frac{2k+1}{2n}+2i\pi} = e^{i\pi\frac{2k+1}{2n}+2i\pi} = e^{i\pi\frac{2k+1}{2n}} = z_k \;, \tag{2.4}$$

there are only a finite $(2n)$ number of inequivalent choices of k, and for convenience we can let $k = 0, 1, \ldots, (2n-1)$ rather than $k = 1, 2, \ldots, 2n$. Note that only those z_k are enclosed in the semicircular contour for which $\Im m(z_k) > 0$, as stated in the condition for the summation in Eq. (2.3); these turn out to be z_k, $k = 0, 1, \ldots, (n-1)$; see the figure below.

$\rho \to \infty$



the $n = 4$ case

Thus,

$$I = \oint_C \frac{\mathrm{d}z}{1+z^{2n}} = 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z_k}\left[\frac{1}{1+z^{2n}}\right] , \qquad z_k = e^{i\pi\frac{2k+1}{2n}} . \tag{2.5}$$

I'll let you compute this using Eq. (1.12). Also, try closing the contour in the lower half-plain (with $\Im m(z) \leq 0$), and verify that the result remains the same.

3. A Nifty Complex Integral

The integral

$$\int_0^\infty \frac{\mathrm{d}x\ \mathrm{e}^{2ix}}{1-x^5}$$

can be evaluated as a complex contour integral, by interpreting the integral along the real axis as open contour (with a detour around the pole at $x = 1$. However, note that the integrand is neither (anti)symmetric nor does it become a simple multiple of itself under $x \to \mathrm{e}^{2\pi i/5}x$. Therefore, closing the contour along a spoke $\lim_{x\to\infty}[0, \mathrm{e}^{i\phi}x)$, for any $\phi \neq 0$ will end up involving another unknown integral, and so will be of no help in evaluating the above integral itself.

Recall however that the $\ln(z)$ function customarily has a branch-cut along the positive axis, so that $\ln(z)$ becomes $\ln(z)-2\pi i$ just below the axis if one arrives there by going from just above the real axis counter-clockwise around the origin, or clockwise around infinity. We then consider the closed-contour integral

$$I = \oint_C \frac{\mathrm{d}z\ \mathrm{e}^{2iz}}{1-z^5} \ln(z) . \tag{3.1}$$

The contour C goes along the real axis (with a clockwise, upper semicircular detour around z=1), encircles z=∞ in the clockwise fashion, runs just below the real axis (with a clockwise, lower semicircular detour around z=1), and encircles z=0 in a counterclockwise fashion. Thus:

$$I = I^+_{\epsilon,1-\delta} + I^+_{\frac{1}{2}C^{(CW)}_\delta} + I^+_{1+\delta,\infty} + I_\infty + I^-_{\infty,1+\delta} + I^-_{\frac{1}{2}C^{(CW)}_\delta} + I^-_{1-\delta,0} + I_0 , \tag{3.2}$$

where

$$I^+_{\epsilon,1-\delta} = \int_\epsilon^{1-\delta} \frac{\mathrm{d}x\ \mathrm{e}^{2ix}}{1-x^5} \ln(x) , \tag{3.3a}$$

$$I^+_{\frac{1}{2}C^{(CW)}_\delta} = -i\pi \operatorname*{Res}_{z=1}\left[\frac{\mathrm{e}^{2iz}}{1-z^5}\ln(z)\right] , \tag{3.3b}$$

$$I^+_{1+\delta,\infty} = \int_{1+\delta}^\infty \frac{\mathrm{d}x\ \mathrm{e}^{2ix}}{1-x^5} \ln(x) , \tag{3.3c}$$

$$I_\infty = \int_{C_\infty} \frac{\mathrm{d}z\ \mathrm{e}^{2iz}}{1-z^5} \ln(z) , \tag{3.3d}$$

$$I^-_{\infty,1+\delta} = \int_\infty^{1+\delta} \frac{\mathrm{d}x\ \mathrm{e}^{2ix}}{1-x^5} [\ln(x) - 2\pi i] , \tag{3.3e}$$

$$I^-_{\frac{1}{2}C^{(CW)}_\delta} = -i\pi \operatorname*{Res}_{z=1}\Big[\frac{e^{2iz}}{1-z^5}[\ln(z)-2\pi i]\Big] \ , \tag{3.3f}$$

$$I^-_{1-\delta,\epsilon} = \int_{1-\delta}^{\epsilon} \frac{dx\ e^{2ix}}{1-x^5}[\ln(x)-2\pi i] \ , \tag{3.3g}$$

$$I_\epsilon = \int_{C_\epsilon} \frac{dz\ e^{2iz}}{1-z^5} \ln(z) \ . \tag{3.3h}$$

Now notice that the limits on (3.3e, g) are opposite of those in (3.3a, c); we then flip the limits on the former two and find that

$$\lim_{\epsilon,\delta\to 0}\Big[I^+_{\epsilon,1-\delta} + I^+_{1+\delta,\infty} + I^-_{\infty,1+\delta} + I^-_{1-\delta,\epsilon}\Big] = +2\pi i\, \mathcal{P}\int_0^\infty \frac{dx\ e^{2ix}}{1-x^5} \ , \tag{3.4}$$

since the terms involving the logarithm cancel between the I^+'s and the I^-'s. Similarly,

$$\begin{aligned} I^+_{\frac{1}{2}C^{(CW)}_\delta} + I^-_{\frac{1}{2}C^{(CW)}_\delta} &= -2\pi i \operatorname*{Res}_{z=1}\Big[\frac{e^{2iz}}{1-z^5}\ln(z)\Big] \ + \ -i\pi \operatorname*{Res}_{z=1}\Big[\frac{e^{2iz}}{1-z^5}(\ln(z)-2\pi i)\Big] \ , \\ &= -2\pi i\big[0\big] \ + \ -i\pi\big[\frac{2\pi i e^{2i}}{5}\big] \ = \ \frac{2\pi^2 e^{2i}}{5} \ . \end{aligned} \tag{3.5}$$

Next (recall that although $\lim_{\epsilon\to 0}\ln(\epsilon) = -\infty$, $\lim_{\epsilon\to 0}(\epsilon\ln(\epsilon)) = 0$),

$$I_\epsilon = \lim_{\epsilon\to 0}\int_{2\pi}^0 \frac{\epsilon e^{i\phi} i d\phi\ e^{2i\epsilon e^{i\phi}}}{1-\epsilon^5 e^{5i\phi}}(\ln\epsilon + i\phi) = \lim_{\epsilon\to 0} i\epsilon\int_{2\pi}^0 e^{i\phi}d\phi(\ln\epsilon + i\phi) = 0 \ . \tag{3.6}$$

Finally,

$$I_\infty = \lim_{\substack{\lambda\to\infty\\ \epsilon\to 0}}\int_{2\pi}^0 \frac{(\lambda+\epsilon e^{i\phi}) i d\phi\ e^{2i(\lambda+\epsilon e^{i\phi})}}{1-(\lambda+\epsilon e^{i\phi})^5}\big(\ln(\lambda+\epsilon e^{i\phi})\big) \ , \tag{3.7a}$$

$$= i \lim_{\lambda\to\infty}\frac{e^{2i\lambda}\ln(\lambda)}{\lambda^4}\int_{2\pi}^0 d\phi = 2\pi i \lim_{\lambda\to\infty}\frac{e^{2i\lambda}\ln(\lambda)}{\lambda^4} = 0 \ . \tag{3.7b}$$

So,

$$I = 2\pi i\,\mathcal{P}\int_0^\infty \frac{dx\ e^{2ix}}{1-x^5} + \frac{2\pi^2 e^{2i}}{5} \ . \tag{3.8}$$

On the other hand, as a contour integral I may be evaluated as the sum of residues of the poles encircled — which is only the z=1, and it is encircled clockwise:

$$I = -2\pi i \operatorname*{Res}_{z=1}\Big[\frac{e^{2iz}\ln(z)}{1-z^5}\Big] = -2\pi i \lim_{z\to 1}\Big[\frac{e^{2iz}\ln(z)}{-5z^4}\Big] = 0 \ . \tag{3.9}$$

Note that while it is true that $\ln(z)$ diverges at $z=0,\infty$, these are not poles! If they were, than $\lim_{z\to 0} z^n\ln(z)$ would be non-zero for some integral n. In fact, this limit vanishes for any, even fractional $n > 0$. (As for the $z=\infty$ singularity, the simple substitution $\zeta = 1/z$ lets us repeat the argument with $\zeta \to 0$.) In other words, the singularities of $\ln(z)$ are milder than any pole, even of fractional order. This makes the above trick especially useful.

The final result then is:

$$0 = I = 2\pi i\,\mathcal{P}\int_0^\infty \frac{dx\ e^{2ix}}{1-x^5} + \frac{2\pi^2 e^{2i}}{5} \ , \tag{3.10}$$

or

$$\mathcal{P} \int_0^\infty \frac{\mathrm{d}x\ \mathrm{e}^{2ix}}{1-x^5} \;=\; \frac{i\pi \mathrm{e}^{2i}}{5} \;=\; -\tfrac{\pi}{5}\sin(2) + i\tfrac{\pi}{5}\cos(2)\ . \tag{3.11}$$