BTW Notes

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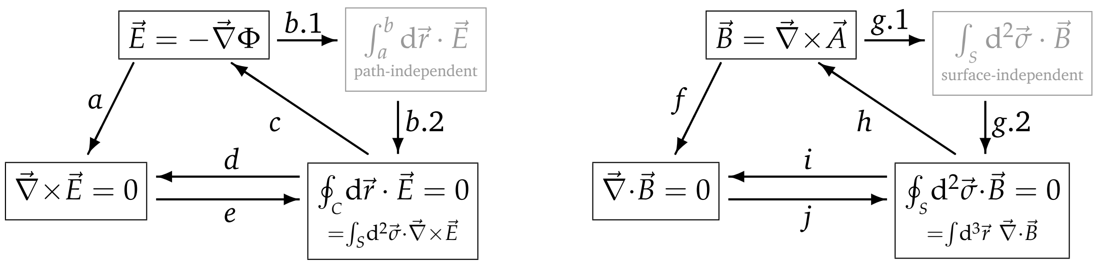
1 Potential Theory

In physics applications, a "potential" is a function from which the more imminently needed one can be derived. This motivates the notation and the sign conventions used. The discussion revolves about showing the equivalence of two groups of three statements, as indicated in the twin diagrams below:



(1.1)

We now proceed explaining how to derive these results one from another, as indicated by the arrows.

1.1 Electrostatic Field

We proceed with the left-hand side diagram:

a: We simply apply $\vec{\nabla}\times$ to both sides of $\vec{E} = -\vec{\nabla}\Phi$ and obtain

$$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\Phi) \equiv 0, \qquad \Rightarrow \qquad \vec{\nabla}\times\vec{E} = 0, \tag{1.2}$$

since the curl of the gradient of any (well-defined, twice differentiable) scalar function is zero.

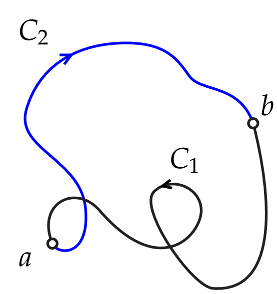
b.1: We integrate both sides of $\vec{E} = -\vec{\nabla}\Phi$ along a contour C_1, from the point a to b, and use the fundamental theorem of calculus to obtain:

$$\int_{C_1} \mathrm{d}\vec{r}\cdot\vec{E} = \int_a^b \mathrm{d}\vec{r}\cdot\vec{E} = -\int_a^b \mathrm{d}\vec{r}\cdot\vec{\nabla}\Phi = -\Phi\big|_a^b = \Phi(a) - \Phi(b). \tag{1.3}$$

Since the result only depends on the end-points[1], it is path-independent, and we could have integrated along a different contour, C_2, that begins and ends at the same points:

$$\Phi(a) - \Phi(b) = -\int_a^b \mathrm{d}\vec{r}\cdot\vec{\nabla}\Phi = \int_{C_2} \mathrm{d}\vec{r}\cdot\vec{E}. \tag{1.4}$$

b.2: Subtracting (1.4) from (1.3), we obtain



$$0 = \int_{a\ (C_1)}^{b} \mathrm{d}\vec{r}\cdot\vec{E} - \int_{a\ (C_2)}^{b} \mathrm{d}\vec{r}\cdot\vec{E} = \int_{a\ (C_1)}^{b} \mathrm{d}\vec{r}\cdot\vec{E} + \int_{b\ (-C_2)}^{a} \mathrm{d}\vec{r}\cdot\vec{E} = \oint_C \mathrm{d}\vec{r}\cdot\vec{E}, \tag{1.5}$$

where $C = C_1 + C_2$ is the closed contour following C_1 from a to b, and then C_2 from b back to a; $-C_2$ implies following the contour against its indicated orientation.

[1] Caveat: see Section 1.2 below.

c: If we know that $\oint_C \mathrm{d}\vec{r}\cdot\vec{E} = 0$, mark off two points on the contour C and call them a and b; label one portion of the closed contour C_1, the other C_2. The closed-contour integration is then the sum of two integrals:

$$0 = \oint_C \mathrm{d}\vec{r}\cdot\vec{E} = \int_{a\,(C_1)}^{b} \mathrm{d}\vec{r}\cdot\vec{E} + \int_{b\,(-C_2)}^{a} \mathrm{d}\vec{r}\cdot\vec{E} = \int_{a\,(C_1)}^{b} \mathrm{d}\vec{r}\cdot\vec{E} - \int_{b\,(C_2)}^{a} \mathrm{d}\vec{r}\cdot\vec{E} \quad\Rightarrow\quad \int_{a\,(C_1)}^{b} \mathrm{d}\vec{r}\cdot\vec{E} = \int_{b\,(C_2)}^{a} \mathrm{d}\vec{r}\cdot\vec{E}, \tag{1.6}$$

and the integral is path-independent, and depends only on the end-points. Introducing the symbol $\Phi := -\int \mathrm{d}\vec{r}\cdot\vec{E}$ for the *anti-derivative* (indefinite integral), we then have:

$$\int_a^b \mathrm{d}\vec{r}\cdot\vec{E} = \Phi(a) - \Phi(b) \qquad\Rightarrow\qquad \vec{E} = -\vec{\nabla}\Phi. \tag{1.7}$$

d: Once we have that $\oint_C \mathrm{d}\vec{r}\cdot\vec{E} = 0$, we use Stokes' theorem "backwards," and write:

$$0 = \oint_C \mathrm{d}\vec{r}\cdot\vec{E} = \int_S \mathrm{d}^2\vec{r}\cdot(\vec{\nabla}\times\vec{E}), \tag{1.8}$$

where S is *any* surface that is bounded by the closed contour C. Since this integral vanishes for *every* surface bounded by the closed contour C, it must be the integrand that vanishes, and we conclude that $\vec{\nabla}\times\vec{E} = 0$.

e: This one is easy: knowing that $\vec{\nabla}\times\vec{E} = 0$, integrate over any surface S, taking the scalar product between the area element, $\mathrm{d}^2\vec{\sigma}$ and the vector $\vec{\nabla}\times\vec{E}$: $\int_S \mathrm{d}^2\vec{\sigma}\cdot(\vec{\nabla}\times\vec{E}) = 0$, since the integrand is zero.

It should be noted that the above computations are all done *assuming* that the various functions and their various derivatives are well-defined (bounded) throughout the region where these computations are expected to hold.

f: Much as in part ***a***, we simply apply $\vec{\nabla}\cdot$ on both sides of $\vec{B} = \vec{\nabla}\times\vec{A}$, and obtain:

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0, \tag{1.9}$$

since the divergence of the curl of any (well-defined, twice differentiable) vector function is zero.

g.1: We now integrate both sides of $\vec{B} = \vec{\nabla}\times\vec{A}$ over a surface S_1 bounded by a contour $C = \partial S_1$, and use the Stokes' theorem to obtain:

$$\int_{S_1} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = \int_{S_1} \mathrm{d}^2\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A}) = \oint_C \mathrm{d}\vec{r}\cdot\vec{A}. \tag{1.10}$$

Since the result only depends on the computation of the integral along the boundary C, we could have just as well integrated over any other surface, S_2, as long as it is bounded by the same contour:

$$\oint_C \mathrm{d}\vec{r}\cdot\vec{A} = \int_{S_2} \mathrm{d}^2\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A}) = \int_{S_2} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} \tag{1.11}$$

g.2: Subtracting (1.11) from (1.10), we obtain

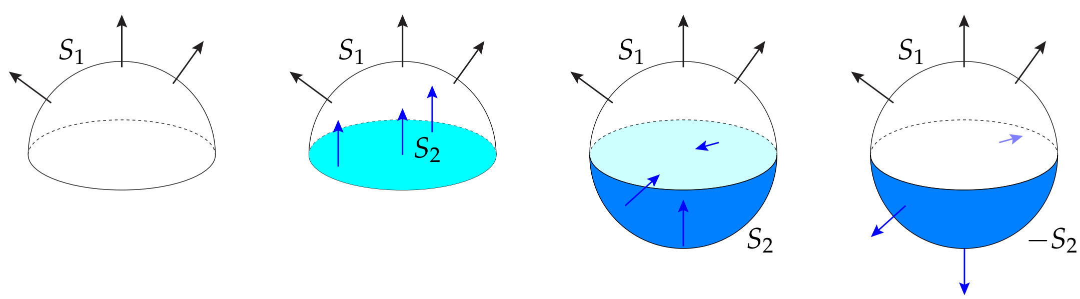
$$0 = \int_{S_1} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} - \int_{S_2} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = \int_{S_1} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} + \int_{-S_2} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = \oint_S \mathrm{d}^2\vec{\sigma}\cdot\vec{B}, \tag{1.12}$$

where $S = S_1 + S_2$ is the closed surface put together from S_1 and S_2 by virtue of them both being bounded by the same contour. (Think of the Northern and the Southern hemisphere both bounded by the equator.) Integrating over $-S_2$ implies integrating over the same surface, S_2, but with the opposite normal. Think of it this way: let's suppose first that S_1 represents the Northern hemisphere and C the equator. Now imagine constructing S_2 by deforming the Northern hemisphere, by "lowering" the surface of S_2 below the Earth surface, until it becomes the plane containing the equator. The normals to this version of S_2 are all parallel to each other and point directly to the North pole from the center of this version of S_2, which happens also to be the center of the Earth.

Now continue deforming S_2 further in the general direction of the South pole, so that it eventually conforms to the Southern hemisphere. Keeping the normals to the surface uniform through this continuous process, we now have S_2 with its normals all pointing towards the center of the Earth. Notice that this configuration was reached by continuously deforming Northern hemisphere S_1 into the Southern hemisphere S_2, and that:

(a) the normals to S_1 are *outward,* and point away from the center of Earth,
(b) the normals to S_2 are *inward,* and point towards the center of Earth.

Thus, as one would follow the normals from the Northern hemisphere across the equator to the Southern hemisphere, their direction changed *discontinuously* on the equator. Switching the second integral from S_2 to $-S_2$ then equips also the Southern hemisphere with *outward* (directed away from the center of the Earth) normals, and now the whole closed surface has uniformly defined normals—this then is S.



(1.13)

h: In turn, if we know that $\oint_S \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = 0$, divide this surface into two regions by a closed contour $C \subset S$. Label one of those regions S_1 and the other $-S_2$. They are both bounded by C and jointly form S, so the integral over S is the sum of the integrals over S_1 and $-S_2$:

$$0 = \oint_S \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = \int_{S_1} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} + \int_{-S_2} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = \int_{S_1} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} - \int_{S_2} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} \quad \Rightarrow \quad \int_{S_1} \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = \int_{S_2} \mathrm{d}^2\vec{\sigma}\cdot\vec{B}, \tag{1.14}$$

and the integral is independent of the actual surface along which it is computed—provided the surface is bounded by the fixed contour C. It follows that the integral is *completely* determined by data along this contour. It follows that the integrals (1.14) must equal a contour integral $\oint_C \mathrm{d}\vec{r}\cdot\vec{A}$, so that

$$\oint_C \mathrm{d}\vec{r}\cdot\vec{A} = \int_S \mathrm{d}^2\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A}) = \int_S \mathrm{d}^2\vec{\sigma}\cdot\vec{B} \qquad \Rightarrow \qquad \vec{B} = \vec{\nabla}\times\vec{A}, \tag{1.15}$$

since the second equality holds for every surface that is bounded by C, and the first equality is Stokes' theorem.

i: Once we have that $\oint_S \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = 0$, we use Gauss' theorem "backwards," and write:

$$0 = \oint_S \mathrm{d}^2\vec{\sigma}\cdot\vec{B} = \int_V \mathrm{d}^3\vec{r}\,(\vec{\nabla}\cdot\vec{B}), \tag{1.16}$$

where V is the volume that is bounded by the closed surface S. Since this integral vanishes for *every* surface bounded by the closed contour C, it must be the integrand that vanishes, and we conclude that $\vec{\nabla} \times \vec{E} = 0$.

j: This one is easy: knowing that $\vec{\nabla}\cdot\vec{B} = 0$, integrate over any volume V: $\int_V \mathrm{d}^3\vec{r}(\vec{\nabla}\cdot\vec{B}) = 0$, since the integrand is zero.

1.2 Sources and Sinks

The above analysis within the Section 1 implicitly assumed that in all cases the integrands and the integrals were well-defined.