Divergent Series in Calculus

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This is a commentary on the YouTube video by Michael Penn, entitled "Alternating sum of natural numbers using.... differential equations???"
I recommend you watch the video itself and to its homework.

Consider the simple differential equation

$$y(x) = ?: \qquad y'(x) + y(x) = f(x), \quad f(x) \in C^\infty(\mathbb{R}). \tag{1}$$

The right-hand side function $f(x)$ is assumed to take real ("$\mathbb{R}$") arguments and have real values, and be "infinitely differentiable," i.e., its derivative of *any* desired order *can* be computed... The calculation of derivatives proceeds iteratively: $f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(n-1)}(x)$, so *of course* the ∞'th derivative is not computable in any finite amount of time, except in special cases. This *given* function $f(x)$ is often called the "source" or the "driving force" or the "forcing function." By contrast, $y(x)$ is the "unknown function" or the "sought-for function."

1 The General Solution

The equation is "inhomogeneous" in the standard sense (in the context of differential equations): if we rescale the sought-for $y(x) \to \lambda\, y(x)$ and nothing else, the various terms in (1) scale with different powers of λ.

1.1 The Unforced Solution

Omitting $f(x)$ reduces (1) to its *sourceless* (or *unforced*) version, which then happens to be homogeneous: $y'(x) + y(x) = 0$, which is easily solved in the form $y(x) = N\, e^{kx}$, leading to:

$$0 = y'(x) + y(x) = N(k\, e^{kx}) + N\, e^{kx} = N(k{+}1)\, e^{kx}, \quad N,\, e^{kx} \neq 0 \quad \Rightarrow \quad k = -1, \tag{2}$$

so that $y_h(x) = N\, e^{-x}$ is the *sourceless,* and here *homogeneous* solution, up to the undetermined (normalization) constant, N. The general solution is then sought for in the form $y(x) = y_p(x) + y_h(x)$ such that $y'_p(x){+}y_p(x) = f(x)$, and we now focus on the *particular* part, $y_p(x)$. Had the sourceless reduction of (1) not been homogeneous, the simple sum of the *unforced* and *forced* solutions would not be a solution!

1.2 The Forced Solution

The particular (forced) solution can be found using only the assumptions given in (1), which includes that not only is $f(x)$ *given,* but any derivative of it is also (implicitly) given, and so we can endeavor to fashion a *particular solution* — the one that correctly reproduces the right-hand side of (1) — with the entire a priori infinite sequence: $f(x)$, $f'(x)$, $f''(x)$,... $f^{(n)}(x)$,... To that end, let us assume

$$y_p(x) = \sum_{n=0}^{\infty} c_n\, f^{(n)}(x), \qquad f^{(0)}(x) \equiv f(x), \tag{3}$$

where c_n are constants to be determined. Plugging into (1), we obtain

$$0 = y_p'(x) + y_p(x) = \sum_{n=0}^{\infty} c_n\, f^{(n+1)}(x) + \sum_{n=0}^{\infty} c_n\, f^{(n)}(x) = \underbrace{\sum_{n=0}^{\infty} c_n\, f^{(n+1)}(x)}_{n\mapsto n-1} + c_0\, f(x) + \sum_{n=1}^{\infty} c_n\, f^{(n)}(x), \tag{4a}$$

$$= c_0\, f(x) + \sum_{n-1=0}^{\infty} c_{n-1}\, f^{(n-1+1)}(x) + \sum_{n=0}^{\infty} c_n\, f^{(n)}(x) = c_0\, f(x) + \sum_{n=1}^{\infty} c_{n-1}\, f^{(n)}(x) + \sum_{n=0}^{\infty} c_n\, f^{(n)}(x), \tag{4b}$$

$$= c_0\, f(x) + \sum_{n=1}^{\infty} (c_{n-1} + c_n)\, f^{(n)}(x). \tag{4c}$$

Without assuming *something additional* about the forcing function $f(x)$ in (1), the only way the ultimate result in (4) can vanish is if we set

$$c_0 \stackrel{!}{=} 1 \quad \& \quad c_n \stackrel{!}{=} -c_{n-1}, \quad n = 1, 2, 3, \dots \tag{5}$$

That is, we've just derived that

$$c_0 = 1, \quad c_1 = -c_0 = -1, \quad c_2 = -c_1 = -(-1) = +1, \dots \quad c_n = (-1)^n. \tag{6}$$

That is, $y_p(x) = \sum_{n=0}^{\infty} (-1)^n f^{(n)}(x)$, so that

$$y(x) = N\, e^{-x} + \sum_{n=0}^{\infty} (-1)^n f^{(n)}(x) \tag{7}$$

is the complete solution.

2 Special Cases

2.1 Termination

If $f(x)$ is a polynomial of a finite degree d, then $f^{(n)}(x) \equiv 0$ for all $n > d$, and the series in (7) is said to *terminate*: it reduces to a finite sum. This is Michael's first example; try it out yourself, with *any* finite-degree polynomial.

2.2 Exponential Forcing Function

This is Michael's second example, and connects to the *divergent series* in the title of these notes. He chooses $f(x) = e^x$, in which case the series on the right-hand side of (7) is easy to simplify, since $\frac{d^n}{dx^n} e^x = e^x$, so

$$y(x) = N\, e^{-x} + \sum_{n=0}^{\infty} (-1)^n\, e^x = N\, e^{-x} + S\, e^x, \quad S := \sum_{n=0}^{\infty} (-1)^n, \tag{8}$$

which gives us a motivation to *evaluate* the infinite series, $S := \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$ *Everything else* in the solution (8) has a definite value at any finite value of x, so it standard to reason that we should expect the series S to also have a *definite value* that can be *assigned* unambiguously. It is clear that the series S *does not converge* to anything definite, since its partial sums are

$$\mathcal{s}_N := \sum_{n=0}^{N} (-1)^n = \begin{cases} 0, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}, \end{cases} \qquad \text{and} \quad \Big(\lim_{N\to\infty} \mathcal{s}_N\Big) \ \underline{\underline{\text{does not exist}}}. \tag{9}$$

Nevertheless, we can easily determine the (unambiguously assignable value of) S by plugging into (1):

$$e^x = y_p'(x) + y_p(x) = \Big(\frac{\mathrm{d}}{\mathrm{d}x}\, S\, e^x\Big) + \Big(S\, e^x\Big) = 2S\, e^x, \qquad \Rightarrow \qquad S = \tfrac{1}{2}. \tag{10}$$

Comparing (8) with (10), we have computed the *assignment*

$$\sum_{n=0}^{\infty} (-1)^n \;\mapsto\; \frac{1}{2}. \tag{11}$$

Indeed, this is not only the result obtained for this (non-convergent, a.k.a. "formally divergent") Grandi's series by many other means and by many mathematicians, but is also the unique value computable only by using the expected *stability* and *linearity* assumptions about series, and so is *summable* (not convergent!) by this ("Pickwickian" [1]) method:

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \dots \tag{12a}$$
$$= 1 + \big(-1 + 1 - 1 + 1 - 1 + \dots\big), \qquad \text{(stability)} \tag{12b}$$
$$= 1 - \big(+1 - 1 + 1 - 1 + 1 - \dots\big), \qquad \text{(linearity)} \tag{12c}$$
$$= 1 - S, \qquad \Rightarrow \quad 2S = 1, \quad \Rightarrow \quad S = \tfrac{1}{2}. \tag{12d}$$

In fact, since this value can be obtained solely by using the expected *stability* and *linearity* assumptions about series, it follows that any method of evaluation which would give S a different value would have to violate at least one of these using the expected assumptions about series.

Counter-Example? In turn, choosing $f(x) = e^{-x}$ results in

$$y(x) = N\, e^{-x} + \sum_{n=0}^{\infty} (-1)^n\, e^{-x} = (N{+}A)\, e^{-x}, \quad A := \sum_{n=0}^{\infty} 1, \tag{13}$$

which should immediately look suspicious: it purports that the particular solution (for which $y_p'{+}y_p{=}f(x)$) is proportional to the homogeneous solution, for which $y_h'{+}y_h = 0$. That is,

$$e^{-x} = f(x) = y_p'(x) + y_p(x) = \Big(\frac{\mathrm{d}}{\mathrm{d}x}\, A\, e^{-x}\Big) + \Big(A\, e^{-x}\Big) = \Big(-A\, e^{-x}\Big) + \Big(A\, e^{-x}\Big) = A(1{-}1)e^{-x}, \tag{14}$$
$$\Rightarrow \quad A{\cdot}0 = 1, \tag{15}$$

so that $A := \sum_{n=0}^{\infty} 1$ cannot be assigned any definite value. Consistently, the series $\sum_{n=0}^{\infty} 1$ cannot be assigned a value by Hardy's "Pickwickian" method either.

So, no this is not a counter-example — it is the exceptional case to which the method does not apply. The reason for this is that the forcing function was selected to equal the homogeneous solution. This fact will turn up again in § 10, on the Green's function.

2.3 Building Further

In Michael's third example, he chooses $f(x) = x\, e^x$, for which

$$f'(x) = e^x + x\, e^x, \quad f''(x) = 2e^x + x\, e^x, \dots \quad f^{(n)}(x) = n\, e^x + x\, e^x. \tag{16}$$

Plugging into (7), this produces

$$y(x) = N\, e^{-x} + \sum_{n=0}^{\infty} (-1)^n f^{(n)}(x) = N\, e^{-x} + \sum_{n=0}^{\infty} (-1)^n \big(n\, e^x + x\, e^x\big), \tag{17a}$$

$$= N\,e^{-x} + e^x \sum_{n=0}^{\infty}(-1)^n\,n + x\,e^x \sum_{n=0}^{\infty}(-1)^n = N\,e^{-x} + \tfrac{1}{2}xe^x + P\,e^x, \quad P := \sum_{n=0}^{\infty}(-1)^n\,n. \tag{17b}$$

Plugging this into (1), we find

$$f(x) = xe^x = y'(x) + y(x) = \big(-N\,e^{-x} + \tfrac{1}{2}(e^x + xe^x) + Pe^x\big) + \big(N\,e^{-x} + \tfrac{1}{2}xe^x + P\,e^x\big), \tag{18a}$$

$$= (N-N)e^{-x} + (\tfrac{1}{2}+\tfrac{1}{2})xe^x + (2P+\tfrac{1}{2})\,e^x, \qquad P = -\tfrac{1}{4}, \tag{18b}$$

$$y(x) = N\,e^{-x} + \tfrac{1}{2}xe^x - \tfrac{1}{4}\,e^x. \tag{18c}$$

Comparing (17) with (18), we have computed the *assignment*

$$\sum_{n=0}^{\infty}(-1)^n\,n \;\mapsto\; -\frac{1}{4}. \tag{19}$$

2.4 Some Conclusions

Recipe: In retrospect, the method worked since the infinite sequence of derivatives of the forcing function $f(x)$ in (7):

1. includes a collection of functions, each distinct from the homogeneous solution;
2. each multiplied by a numerical infinite series;
3. each of which is assumed to have an assignable definite value;
4. which values are then computed by requiring that the so-parametrized purported solution satisfy the original differential equation (1).

Afterthought: In fact, step 1 may be used simply to obtain that collection of functions to write an Ansatz (solution template) for step 4; the *evaluation* of the infinite series identified in step 2 is a by-product of this method. It does however provide a motivation for seeking to assign a value to such infinite series.

For example, for the case $f(x) = \sin(x)$ and knowing that $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) \propto \cos(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) \propto \sin(x)$, we seek a solution in the form $y(x) = Ne^{-x} + A\sin(x) + B\cos(x)$ and compute:

$$\sin(x) = y'(x) + y(x) = \big(-Ne^{-x} + A\cos(x) - B\sin(x)\big) + \big(Ne^{-x} + A\sin(x) + B\cos(x)\big), \tag{20a}$$

$$= (A-B)\sin(x) + (A+B)\cos(x); \qquad \Rightarrow \quad B = -A, \quad \Rightarrow \quad A = \tfrac{1}{2}, \;\; B = -\tfrac{1}{2}; \tag{20b}$$

$$y(x) = Ne^{-x} + \tfrac{1}{2}\sin(x) - \tfrac{1}{2}\cos(x) = Ne^{-x} + \tfrac{1}{\sqrt{2}}\sin\big(x - \tfrac{\pi}{2}\big) \tag{20c}$$

References

[1] G. H. Hardy, *Divergent Series*. American Mathematical Society, 2nd ed., 1992. Original publ. 1929. 3