All references are to *Elementary Differential Equations*, 5th ed., by C.H. Edwards and D.E. Penney.

Instructor's Warning: This solution set presents considerably more detail than was required of Students in their assignments; hopefully y'all will learn from this.

§ 1.3

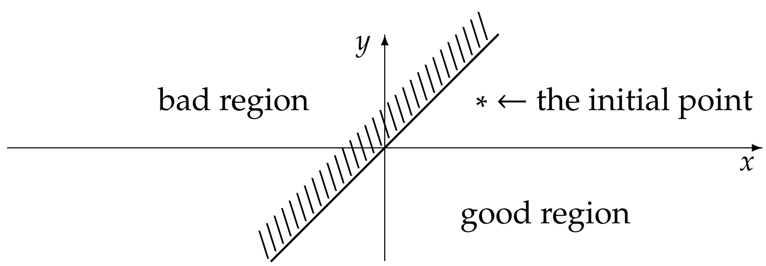
 # 16: We need to discuss the applicability of Theorem 1 (p.23) to $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \sqrt{x-y}$, and thus whether or not a unique solution satisfying $y(2) = 1$ exists. To that end, we need to check the assumptions of Theorem 1: are

1. $\sqrt{x-y}$ and
2. $\frac{\partial}{\partial y}\sqrt{x-y} = \frac{1}{2\sqrt{x-y}}\frac{\partial x-y}{\partial y} = \frac{-1}{2\sqrt{x-y}}$

continuous in a rectangle containing $(x,y) = (2,1)$? Indeed, they are:

1. $\sqrt{x-y}$ becomes imaginary when $y > x$: in the standard (x,y)-plane, this is to the left of the straight line $y = x$.
2. $\frac{-1}{2\sqrt{x-y}}$ becomes "$\frac{1}{0}$" nonsense on this diagonal line, $y = x$, so that we are left with the whole open half-plane being to the right of the $y = x$ 45° diagonal.

The point $(x,y) = (2,1)$ is well within this half-plane, and we can easily find a rectangle around this point and within the "kosher" half-plane:



The conditions of Theorem 1 satisfied in any such rectangle, the conclusion of Theorem 1 holds: there is a unique solution to this differential equation, satisfying the condition that $y(2) = 1$.

 # 30: Several students have returned blanks on this one. None of whom then even tried sketching the solution—as suggested in the problem itself!? And, not "mentally"; try it with paper and pen(cil). In particular, see if the piece-wise defined function is continuous and smooth at the junction points. What happens to the graph when you vary c? And, finally, calculate the derivative, $y'(x)$, as well as the expression $-\sqrt{1-y^2(x)}$ in each of the three regions, and see if the statement of the differential equation,

$$y'(x) \stackrel{?}{=} -\sqrt{1-y^2(x)}\,, \tag{1.3:1}$$

is satisfied in all three regions and at the junctions.

§ 1.4

 # 42: The moon rock. They talk about the number of potassium and argon atoms, as a function of time; let's call these functions $K(t)$ and $A(t)$, respectively. The next sentence is packed with info, so let's disjoin it:

1. it says that potassium decays radioactively. Then, Eq. (14) applies, and we have, without much ado:

$$\frac{\mathrm{d}\,K}{\mathrm{d}\,t} = -k\,K(t)\;, \qquad \Rightarrow \qquad K(t) = K_0\,e^{-kt} = K_0\,2^{-t/\tau}\;, \tag{1.4:2}$$

where in the last equality we've used the relation (18), that $k = \frac{\ln(2)}{\tau}$, so that

$$e^{-kt} = e^{-\ln(2)t/\tau} = \left(e^{\ln(2)}\right)^{-t/\tau} = 2^{-t/\tau}\;. \tag{1.4:3}$$

OK so far? Now we know how the number of potassium atoms changes over time, $K(t) = K_0\,2^{-t/\tau}$.

So, how about the argon atoms? Well, there's:

2. That second sentence also says that only one in nine potassium atom decays produces an argon atom. So, how many potassium atoms have decayed by time t? Easy: To begin with, there were K_0, and at time t, there were only $K_0\,2^{-t/\tau}$; thus, $K_0 - K_0\,2^{-t/\tau}$ have decayed. *And*, a ninth of those produced argon atoms. Therefore, the number of argon atoms has to be:

$$A(t) = \tfrac{1}{9}\left(K_0 - K_0\,2^{-t/\tau}\right) = \tfrac{1}{9}K_0(1 - 2^{-t/\tau})\;. \tag{1.4:4}$$

OK. Now back up to that *first* sentence: "A certain moon rock was found [at time t_* since the time $t = 0$, when it contained only potassium] to contain equal numbers of potassium and argon atoms." Ha! That's it! This says that:

$$A(t_*) \overset{!}{=} K(t_*)\;, \qquad i.e. \qquad \tfrac{1}{9}\,K_0\,(1 - 2^{-t_*/\tau}) \overset{!}{=} K_0\,2^{-t_*/\tau}\;. \tag{1.4:5}$$

Canceling K_0, multiplying through by 9, and rearranging terms left-right, we get

$$1 \overset{!}{=} 10\,2^{-t_*/\tau}\;, \qquad \Rightarrow \qquad \tfrac{1}{10} \overset{!}{=} 2^{-t_*/\tau}\;, \qquad \Rightarrow \qquad \log_2(\tfrac{1}{10}) \overset{!}{=} -t_*/\tau\;, \tag{1.4:6}$$

or

$$t_* = \tau\,\log_2(10) \approx (1.28{\times}10^9\text{ years})(3.3219) \approx 4.2521{\times}10^9\text{ years}\;. \tag{1.4:7}$$

The result, $1.25{\times}10^9$ years, quoted in the text is a typo.

§ 1.5

 # 14: The differential equation $xy' - 3y = x^3$—or, equivalently, $y' - \frac{3}{x}y = x^2$ is of the form of Eq. (3) on p.45, with $P(x) = -\frac{3}{x}$ and $Q(x) = x^2$. Therefore, the solution in Eq. (6) on p.45 applies. In there, the C-independent term is the *particluar* solution. 'nuff said:

$$\begin{aligned} y_p(x) &= e^{-\int \mathrm{d}x\;P(x)} \int \mathrm{d}x\;Q(x)\,e^{+\int \mathrm{d}x\;P(x)} \;=\; e^{+3\int\frac{\mathrm{d}x}{x}} \int \mathrm{d}x\;x^2\,e^{-3\int\frac{\mathrm{d}x}{x}}\;,\\ &= e^{+3\ln(x)} \int \mathrm{d}x\;x^2\,e^{-3\ln(x)} \;=\; x^3 \int \mathrm{d}x\;x^2\,x^{-3} \;=\; x^3 \int \frac{\mathrm{d}x}{x} \;=\; x^3\ln(x)\;. \end{aligned} \tag{1.5:8}$$

§ 1.6

Errata: The last display on p.68 and the firs on p.69, should have, respectively

$$x(y) = \int \frac{\mathrm{d}\,x}{\mathrm{d}\,y}\,\mathrm{d}y = \int \frac{1}{\mathrm{d}y/\mathrm{d}x}\,\mathrm{d}y = \cdots$$

and

$$y\,p\,\frac{\mathrm{d}\,p}{\mathrm{d}\,y} = p^2\,.$$

Do you now see their typos? Doesn't it make much more sense?

 # 25: The differential equation $y^2(xy' + y)(1 + x^4)^{1/2} = x$ contains the pattern $xy' + y$ which *should* be obvious to simplify as

$$x\,y' + y \;=\; x\,\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + 1{\cdot}y \;=\; x\,\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + \frac{\mathrm{d}\,x}{\mathrm{d}\,x}\,y \;=\; \frac{\mathrm{d}}{\mathrm{d}\,x}(x\,y)\,. \tag{1.6:9}$$

This suggests *trying* the substitution $z = x\,y$, which is indeed easy to invert:

$$z = x\,y\,, \qquad \Rightarrow \qquad y = \frac{z}{x}\,. \tag{1.6:10}$$

Since y' occurs nowhere else, this is enough to complete the substitution $y \to z = x\,y$, and obtain

$$y^2\,(x\,y' + y)\,\sqrt{1 + x^4} \;=\; x\,, \qquad \Rightarrow \qquad \Big(\frac{z}{x}\Big)^2\,\frac{\mathrm{d}\,z}{\mathrm{d}\,x}\,\sqrt{1 + x^4} \;=\; x\,, \tag{1.6:11}$$

which separates:

$$z^2\,\mathrm{d}z \;=\; \frac{x^3\,\mathrm{d}x}{\sqrt{1 + x^4}} \;\overset{(1+x^4)=u}{=}\; \tfrac{1}{4}\,\frac{\mathrm{d}u}{\sqrt{u}}\,, \tag{1.6:12}$$

and can be integrated into:

$$\tfrac{1}{3}\,z^3 \;=\; \tfrac{1}{2}\,\sqrt{u} + \tfrac{1}{3}C\,, \qquad \Rightarrow \qquad 2\,x^3\,y^3 \;=\; 3\,\sqrt{1 + x^4} + C\,. \tag{1.6:13}$$

where I judiciously included a $\frac{1}{3}$ in the initial appearance of the integration constant, C.

This *implicit* solution can be solved for y:

$$y(x) = \frac{1}{x}\,\sqrt[3]{\frac{3\,\sqrt{1 + x^4}}{2} + \frac{C}{2}}\,. \tag{1.6:14}$$

 # 54: The differential equation $y\,y'' = 3(y')^2$ is *precisely* of one of the two reduction types, and closely follows Example 11—where, however, there was a typo. But, never fear, we proceed as instructed: the *independent* variable, x, does not occur in this differential equation, so we substitute:

$$p := y'\,, \qquad \text{and} \qquad y'' = p\,\frac{\mathrm{d}\,p}{\mathrm{d}\,y}\,. \tag{1.6:15}$$

The differential equation then becomes

$$y\,y'' = 3(y')^2\,, \qquad \Rightarrow \qquad y\Big(p\,\frac{\mathrm{d}\,p}{\mathrm{d}\,y}\Big) = 3\,p^2\,, \tag{1.6:16}$$

which separates into:

$$\frac{\mathrm{d}p}{p} = 3\frac{\mathrm{d}y}{y}\,, \qquad \Rightarrow \qquad \ln(p) = 3\ln y + \ln(C) = \ln(y^3) + \ln(C) = \ln(C\,y^3)\,, \tag{1.6:17}$$

which produces, upon exponentiating both sides:

$$p = C\,y^3\,, \qquad \Rightarrow \qquad \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = C\,y^3\,. \tag{1.6:18}$$

This last equation separates and can be integrated:

$$\frac{\mathrm{d}y}{y^3} = C\,\mathrm{d}x\,, \qquad \Rightarrow \qquad -\tfrac{1}{2}\,y^{-2} = C\,(x - x_0)\,, \qquad \Rightarrow \qquad y^{-2} = 2\,C\,(x_0 - x)\,, \tag{1.6:19}$$

and finally,

$$1 = (2C)\,y^2\,(x_0 - x)\,, \tag{1.6:20}$$

which agrees with the text's solution, upon identifying $A = 2C$ and $B = x_0$. A bit more user-friendly format of the same solution is

$$y(x) = \frac{1}{\sqrt{2C(x_0 - x)}}\,. \tag{1.6:21}$$