**Delaware State University** **Mid-Term Exam: 10/10/07**
**Department of Mathematics**
**Math-351: Differential Equations** **Solution**

This solution set is more complete than what was expected of students to obtain in the exam. I hope this helps understanding it all better.

**1.** Consider the given differential equation:

$$\frac{\mathrm{d}\,(\ln(y))}{\mathrm{d}\,x} - \frac{x}{y} - \frac{3}{x} = 0\,. \tag{1}$$

**a.** Re-write (1) as a (more standard) differential equation for $y(x)$. [10%]

The unexpected occurrence of $y$ in $\mathrm{d}\ln(y)/\mathrm{d}x$ suggests we expand that; indeed:

$$\frac{1}{y}\frac{\mathrm{d}\,y}{\mathrm{d}\,x} - \frac{x}{y} - \frac{3}{x} = 0\,, \qquad \Rightarrow \qquad \frac{\mathrm{d}\,y}{\mathrm{d}\,x} - x - \frac{3}{x}y = 0\,, \qquad \Rightarrow \qquad \boxed{\frac{\mathrm{d}\,y}{\mathrm{d}\,x} - \frac{3}{x}y = x}\,. \tag{2}$$

**b.** Compute the general solution of (1) using your result in part a. [20%]

From (2), the associated equation is $y' - 3x^{-1}y = 0$, which separates into

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{3}{x}\,y\,, \quad \Rightarrow \quad \frac{\mathrm{d}\,y}{y} = \frac{3\,\mathrm{d}\,x}{x}\,, \quad \Rightarrow \quad \int\frac{\mathrm{d}\,y}{y} = 3\int\frac{\mathrm{d}\,x}{x}\,, \quad \Rightarrow \quad \ln(y) - \ln(C) = 3\ln(x)\,, \tag{3}$$

from which $\boxed{y_c(x) = C\,x^3}$. Check by substituting into $y' - 3x^{-1}y = 0$.

Next, we need the particular part, to which end we clear the denominator in (2):

$$x\,y' - 3y = x^2\,. \tag{4}$$

The right-hand side function being a polynomial of order 2, we try $y_p(x) = a + bx + cx^2$ and compute:

$$\left.\begin{aligned} y_p &= a + bx + cx^2\,,\\ y_p' &= b + 2cx\,, \end{aligned}\right\} \quad \text{Eq. (4) becomes} \quad \begin{aligned} x(b + 2cx) - 3(a + bx + cx^2) &= x^2\,,\\ -3a - 2bx - cx^2 &= x^2\,, \end{aligned} \tag{5}$$

which implies that $a = 0 = b$ and $c = -1$. Thus, $\boxed{y_p(x) = -x^2}$.

Thus, $\boxed{y(x) = C\,x^3 - x^2}$ is the general solution.

**2.** For the differential equation $2y'' - y' - y = x$,

**a.** Compute the complementary solution. [20%]

This being a linear differential equation with constant coefficients, we try $y_c(x) = e^{rx}$, and obtain the characteristic equation and its solutions:

$$0 = (2\,r^2 - r - 1) = (2r + 1)(r - 1) \quad \Rightarrow \quad r_1 = -\tfrac{1}{2}\,,\ r_2 = +1\,. \tag{6}$$

This implies that the complementary solution is $\boxed{y_c(x) = C_1\,e^{-x/2} + C_2\,e^{x}}$.

**b.** Compute the particular solution. [10%]

Since the source function is a polynomial of order 1, we try $y_p(x) = a + bx$. Then $y_p'(x) = b$ and $y_p''(x) = 0$. Substituting this into $2y'' - y' - y = x$ yields:

$$2(0) - (b) - (a + b\,x) = -b\,x - (a{+}b) \overset{!}{=} x\,, \quad \Rightarrow \quad \left\{ \begin{array}{rcl} -b &=& 1 \\ a+b &=& 0 \end{array} \right\} \quad \Rightarrow \quad a = 1\,. \tag{7}$$

Thus, the particular solution is $\boxed{y_p(x) = 1 - x}$.

**c.** Determine all constants in the general solution so that $y(0) = 0 = y'(0)$. [10%]

So, the general solution is $y(x) = 1 - x + C_1\,e^{-x/2} + C_2\,e^x$, and $y'(x) = -1 - \frac{1}{2}C_1\,e^{-x/2} + C_2\,e^x$. Evaluated at $x = 0$, imposing the "boundary" conditions, we have:

$$y(0) = \quad 1 + C_1 + C_2 = 0\,, \tag{8}$$

$$y'(0) = -1 - \tfrac{1}{2}\,C_1 + C_2 = 0\,, \tag{9}$$

from which

$$Eq.\,(8) - Eq.\,(9): \quad 2 + \tfrac{3}{2}\,C_1 = 0\,, \quad \Rightarrow \quad C_1 = -\tfrac{4}{3}\,, \tag{10}$$

$$Eq.\,(8) + 2Eq.\,(9): \quad -1 + 3\,C_2 = 0\,, \quad \Rightarrow \quad C_2 = +\tfrac{1}{3}\,. \tag{11}$$

Thus,

$$y(0) = 0 = y'(0) \qquad \Rightarrow \qquad \boxed{y(x) \;=\; 1 - x - \tfrac{4}{3}\,e^{-x/2} + \tfrac{1}{3}\,e^x}\,.$$

**3.** For the differential equation $x^2\,y'' - x\,y' + y = \ln(x)$,

**a.** Compute the complementary solution. [20%]

This differential equation exhibits the re-scaling symmetry with respect to $x \to ł\,x$, and so we use the substitution

$$\ln(x) = \xi\,, \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\,x} = e^{-\xi}\frac{\mathrm{d}}{\mathrm{d}\,\xi}\,, \quad \Rightarrow \quad \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} \equiv \frac{\mathrm{d}}{\mathrm{d}\,x}\frac{\mathrm{d}}{\mathrm{d}\,x} = e^{-\xi}\frac{\mathrm{d}}{\mathrm{d}\,\xi}e^{-\xi}\frac{\mathrm{d}}{\mathrm{d}\,\xi} = e^{-2\xi}\frac{\mathrm{d}^2}{\mathrm{d}\,\xi^2} - e^{-2\xi}\frac{\mathrm{d}}{\mathrm{d}\,\xi}\,.$$

With this, $x^2\,y'' - x\,y' + y = \ln(x)$ becomes

$$e^{2\xi}\Big(e^{-2\xi}\frac{\mathrm{d}^2\,y}{\mathrm{d}\,\xi^2} - e^{-2\xi}\frac{\mathrm{d}\,y}{\mathrm{d}\,\xi}\Big) - e^{\xi}\Big(e^{-\xi}\frac{\mathrm{d}\,y}{\mathrm{d}\,\xi}\Big) + y = \xi\,,$$

$$\frac{\mathrm{d}^2\,y}{\mathrm{d}\,\xi^2} - 2\frac{\mathrm{d}\,y}{\mathrm{d}\,\xi} + y = \xi\,, \quad \text{or} \quad y''(\xi) - 2y'(\xi) + y(\xi) = \xi\,,$$

a linear, 2nd order differential equation with constant coefficients.

The associated equation is $y''(\xi) - 2y'(\xi) + y(\xi) = 0$, and has solutions of the form $e^{r\xi}$. Inserting produces the characteristic equation

$$0 \;=\; r^2 - 2r + 1 \;=\; (r-1)^2\,, \qquad \Rightarrow \qquad r_1 = 1 = r_2\,. \tag{12}$$

Since this is a double (repeated) solution, we have $\boxed{y_c(\xi) = (C_1 + C_2\,\xi)\,e^{\xi}}$. Substituting $\xi = \ln(x)$ back, we have $\boxed{y_c(x) = (C_1 + C_2\ln(x))\,x}$. Either version is correct; I'll work with both, for giggles.

**b.** Compute the particular solution by varying the parameters in the coplementary solution. [20%]

**b.1.** First of all, finding the particular solution by guessing is really easy in the $\xi$-version: The differential equation is $y'' - 2y' + y = \xi$, suggesting that we try $y_p(\xi) = (a + b\xi)$. Then, $y_p'(\xi) = b$ and $y_p''(\xi) = 0$. Substituting, we get

$$(0) - 2(b) + (a + b\xi) = b\,\xi + (a - 2b) \stackrel{!}{=} \xi\,, \quad \Rightarrow \quad \begin{cases} b &= 1\,, \\ a &= 2\,. \end{cases} \tag{13}$$

Thus, $\boxed{y_p(\xi) = 2 + \xi \text{ and } y_p(x) = 2 + \ln(x)}$.

**b.2.** However, I wanted you to solve this by varying the coefficients in $y_c$—for which there is a "ready-to-use" formula in the text:

$$y_p(\xi) = -y_1(\xi) \int d\xi\, \frac{y_2(\xi)\, f(\xi)}{W(\xi)} + y_2(\xi) \int d\xi\, \frac{y_1(\xi)\, f(\xi)}{W(\xi)}\,, \tag{14}$$

where $f(\xi) = \xi$, $y_1(\xi) = e^{\xi}$, $y_2(\xi) = \xi\, e^{\xi}$, and

$$W(\xi) = W_0\, e^{-\int d\xi\, p(\xi)} = W_0\, e^{-\int d\xi\,(-2)} = W_0\, e^{2\xi}\,. \tag{15}$$

Substituting, we get

$$\begin{aligned} y_p(\xi) &= -\frac{e^{\xi}}{W_0} \int d\xi\, (\xi\, e^{\xi})(\xi)(e^{-2\xi}) + \frac{\xi\, e^{\xi}}{W_0} \int d\xi\, (e^{\xi})(\xi)(e^{-2\xi})\,, \\ &= -\frac{e^{\xi}}{W_0} \int d\xi\, e^{-\xi}\, \xi^2 + \frac{\xi\, e^{\xi}}{W_0} \int d\xi\, e^{-\xi}\, \xi\,. \end{aligned}$$

Now the integrals of this form lend themselves to integration by parts:

$$\begin{aligned} \int dt\, e^{-t}\, t^n &= \Big[-e^{-t}\, t^n\Big] - n \int dt\, (-e^{-t})\, t^{n-1}\,, \\ &= \Big[-e^{-t}\, t^n\Big] + n\Big[-e^{-t}\, t^{n-1}\Big] + n(n{-}1) \int dt\, (-e^{-t})\, t^{n-2}\,, \\ &= -\Big[e^{-t}\, t^n\Big] - n\Big[e^{-t}\, t^{n-1}\Big] - \cdots - n!\, e^{-t}\,, \\ &= -e^{-t}\Big(t^n + n\, t^{n-1} + n(n{-}1)\, t^{n-2} + \cdots + n!\, t + n!\Big) \;=\; -e^{-t} \sum_{k=0}^{n} \frac{n!}{k!} t^k\,. \end{aligned} \tag{16}$$

So,

$$\int d\xi\, e^{-\xi}\, \xi^2 = -e^{-\xi}(\xi^2 + 2\xi + 2)\,, \quad \text{and} \quad \int d\xi\, e^{-\xi}\, \xi = -e^{-\xi}(\xi + 1)\,, \tag{17}$$

and

$$\begin{aligned} y_p(\xi) &= +\frac{e^{\xi}}{W_0}\Big(e^{-\xi}(\xi^2 + 2\xi + 2)\Big) - \frac{\xi\, e^{\xi}}{W_0}\Big(e^{-\xi}(\xi + 1)\Big)\,, \\ &= \frac{1}{W_0}\Big(\xi^2 + 2\xi + 2 - \xi^2 - \xi\Big) = \frac{1}{W_0}\,(\xi + 2)\,. \end{aligned}$$

Finally, to determine $W_0$, compute $y_p'(\xi) = \frac{1}{W_0}$ and $y_p''(\xi) = 0$, so that inserting in the differential equation $y'' - 2y' + y = \xi$ fixes $W_0 = 1$, and $\boxed{y_p(\xi) = 2 + \xi \text{ and } y_p(x) = 2 + \ln(x)}$.

**b.3.** Let's see what happens in the $x$-version. The differential equation now is $x^2 y'' - xy' + y = \ln(x)$, which my be rewritten as

$$y''(x) - \frac{1}{x} y'(x) + \frac{1}{x^2} y(x) = \frac{\ln(x)}{x^2}\,, \quad \text{so now} \quad p(x) = -\frac{1}{x}\,,\ q(x) = +\frac{1}{x^2}\,,\ f(x) = \frac{\ln(x)}{x^2}\,. \tag{18}$$

The Wronskian is now

$$W(x) = W_0\, e^{-\int \mathrm{d}x\, p(x)} = W_0\, e^{+\int \frac{\mathrm{d}x}{x}} = W_0\, e^{\ln(x)} = W_0\, x\,. \tag{19}$$

The "ready-to-use" solution in the $x$-version becomes

$$\begin{aligned} y_p(x) &= -y_1(x) \int \mathrm{d}x\, \frac{y_2(x)\, f(x)}{W(x)} + y_2(x) \int \mathrm{d}x\, \frac{y_1(x)\, f(x)}{W(x)}\,,\\ &= -x \int \mathrm{d}x\, \frac{x \ln(x)\ \ln(x) x^{-2}}{W_0\, x} + x\ln(x) \int \mathrm{d}x\, \frac{x\ \ln(x)\, x^{-2}}{W_0\, x}\,,\\ &= -\frac{x}{W_0} \int \mathrm{d}x \frac{\ln^2(x)}{x^2} + x\ln(x) \int \mathrm{d}x \frac{\ln(x)}{x^2}\,. \end{aligned}$$

These are integrals that can be solved by integration by parts[1], obtaining

$$\int \mathrm{d}x \frac{\ln^2(x)}{x^2} = -\frac{\ln^2(x)}{x} - \frac{2\ln(x)}{x} - \frac{2}{x}\,, \qquad \int \mathrm{d}x \frac{\ln(x)}{x^2} = -\frac{\ln(x)}{x} - \frac{1}{x}\,, \tag{20}$$

producing

$$\begin{aligned} y_p(x) &= -\frac{x}{W_0}\Big( -\frac{\ln^2(x)}{x} - \frac{2\ln(x)}{x} - \frac{2}{x}\Big) + x\ln(x)\Big( -\frac{\ln(x)}{x} - \frac{1}{x}\Big)\,,\\ &= \frac{1}{W_0}\Big( \ln^2(x) + 2\ln(x) + 2\ -\ \ln^2(x) - \ln(x)\Big) \ =\ \frac{1}{W_0}\Big(\ln(x) + 2\Big)\,. \end{aligned}$$

Again, inserting this in the differential equation $x^2 y'' - xy' + y = \ln(x)$ fixes $W_0 = 1$, and $\boxed{y_p(\xi) = 2 + \xi \text{ and } y_p(x) = 2 + \ln(x)}$.

— ★ —

Thus, when possible, do use the method illustrated here as **b.1.**; but if that doesn't pan out, the method of varying constants in the complementary solution always works and straightforwardly, albeit not infrequently at quite some expense in labor and tedium.

[1] Or guessing the leading term and then correcting with the (sequence of) sub-leading one(s), as I've shown at the beginning of the class.