1 Some Substitution Samples

Here are some additional examples of solving differential equations by means of *judicious*[1] substitutions.

A lot will depend on your familiarity with "usual" algebra: if you can recognize that $a^3b^3 + 3a^2cb^2 + 3ac^2b + c^3 = (ab + c)^3$, whatever a, b and c are, substituting the left-hand side by the right-hand side does simplify things. Similarly, knowing that $\frac{\mathrm{d}}{\mathrm{d}x}\big(y(x)\,z(x)\big) = y'(x)\,z(x) + y(x)\,z'(x)$ will help simplifying:

Example 1:

Consider:

$$\sin(3x)\,y'(x) + 3\cos(3x)\,y(x) = x^4\ . \tag{1.1}$$

Seeing that the left-hand side is in fact $\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(3x)\,y(x)\big)$, so Eq. (1.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\sin(3x)\,y(x)\Big) = x^4\ , \tag{1.2}$$

helps. Just multiply through by $\mathrm{d}x$ and integrate:

$$\int \mathrm{d}\Big(\sin(3x)\,y(x)\Big) = \int \mathrm{d}x\ x^4$$

$$\sin(3x)\,y(x) + C = \tfrac{1}{5}\,x^5 \qquad \Rightarrow \qquad y(x) = (\tfrac{1}{5}x^5 - C)\csc(3x)\ , \tag{1.3}$$

or

$$y(x) = \tfrac{1}{5}\,x^5\csc(3x) - C\csc(3x)\ , \tag{1.4}$$

where it is evident that the first, C'-independent term is the *particular* solution, and the second, C'-dependent term is the *complementary* solution. Of course, C is the integration constant.

Note that Eq. (1.1) could have been recast into another suggestive form if one divided through by $\sin(3x)$:

$$y'(x) + 3\cot(3x)\,y(x) = x^4\ \csc(3x)\ . \tag{1.1'}$$

This now is of the form $y'(x) + p(x)\,y(x) = q(x)$, with $p(x) = 3\cot(3x)$ and $q(x) = x^4\ \csc(3x)$; to this, the solution (6) on p. 45 applies. But, don't trust me; try it out yourself.

Example 2:

Consider now the differential equation:

$$y'(x) = \frac{\sqrt{3 + \big(x - y(x)\big)^2}}{x - y(x)}\ . \tag{1.5}$$

The admittedly ugly expression on the right-hand side persistently has $x - y(x)$ and—*this is crucial*—$y(x)$ occurs in no other way in this differential equation. This last fact ensures that we may trade in $y(x)$ for

$$z(x)\ :=\ x - y(x)\ , \qquad \text{or} \qquad y(x) = x - z(x)\ . \tag{1.6}$$

[1] This is the key word: judicious! And, it's *you* who has to get to a point of starting to *see* what is and what isn't judicious. Unfortunately, there is no recipe for this, except practice, practice, and then practice. . .

From this,

$$y'(x) = \frac{\mathrm{d}}{\mathrm{d}\,x}\Big(x - z(x)\Big) = 1 - z'(x)\,, \tag{1.7}$$

which we substitute in the left-hand side of (1.5), whereas (1.6) is substituted in the right-hand side of same. This produces:

$$1 - z'(x) = \frac{\sqrt{3+z^2(x)}}{z(x)}\,. \tag{1.8}$$

where I used the convention that $z^2(x)$ denotes the square of the function $z(x)$. The differential equation (1.8) now in fact separates:

$$\begin{aligned} z'(x) &= 1 - \frac{\sqrt{3+z^2(x)}}{z(x)}\,, \\ \int \frac{\mathrm{d}z}{1-\frac{\sqrt{3+z^2}}{z}} &= \int \mathrm{d}x \qquad\qquad \Rightarrow \quad -\tfrac{1}{9}\Big(z^3 + (3+z^2)^{3/2}\Big) = x + C\,, \end{aligned} \tag{1.9}$$

where C is the constant of integration. Remembering the substitution (1.6), we now have:

$$x + C + \tfrac{1}{9}\Big((x-y)^3 + (3+(x-y)^2)^{3/2}\Big) = 0\,. \tag{1.10}$$

This actually *can* be solved, either as y in terms of x or the other way around, but it is quite hilariously complicated. For giggles, we get:

$$x_{\pm,\pm}(y) = \frac{y-C}{2} \;\pm\; \frac{\sqrt{A+B}}{2} \;\pm\; \frac{1}{2}\sqrt{D - B - \frac{E}{\sqrt{A+B}}}\,, \tag{1.11}$$

where

$$\begin{aligned} A &= (C{+}y{-}2)(C{+}y{+}2)\,, \qquad & B &= \Big(6(C+y)^2+8\Big)^{\frac{2}{3}}\,, \\ D &= 2(C-y)^2 + 8(Cy-1)\,, \qquad & E &= 2(C+y)\Big((C+y)^2+12\Big)\,. \end{aligned}$$

The explicit, closed-form solutions for y in terms of x are much, much more complicated expressions. *That*'s why, not infrequently, we content ourselves with the *implicit* solution (1.10).

— ★ —

Between these two (admittedly simple) examples provided to complement Example 1 on pages 57–58, this idea of seeking out *patterns that you know how to simplify* and then doing so should start sinking in. It is of course possible to iterate this, and keep simplifying until, after several steps, the differential equation becomes something you know how to solve. . . . or you get stuck with a differential equation that cannot be solved by substitutions (alone).

Practice *will* help.

2 Partial Derivatives

For the benefit of the students who have not seen "partial derivatives" before, here's a little introduction[2]: ***WORK through it!***

Consider a functional expression, $f(x, y)$, involving two distinct variables, x and y, and where the latter is regarded as a function of x. This is not at all outlandish; Let $f(x, y(x)) = x^3 \big(y(x)\big)^2$. Then,

$$\frac{\mathrm{d}\, f}{\mathrm{d}\, x} \;=\; \frac{\mathrm{d}\, x^3}{\mathrm{d}\, x}\,(y(x))^2 + x^3 \frac{\mathrm{d}\big(y(x)\big)^2}{\mathrm{d}\, x} \;=\; 3\,x^2\,(y(x))^2 + x^3\Big(2\,y(x)\frac{\mathrm{d}\,y(x)}{\mathrm{d}\,x}\Big)\,, \tag{2.1}$$

where the first equality follows on applying the product rule, and the second uses the chain rule.

Note that in the two parts of the calculation, we have *temporarily* treated x and y as if they were independent variables. The first term in (2.1) was obtained by pretending that $y(x)$ was in fact independent of x; the second, by ignoring the x^3 factor, and considering the derivative of only the y^2 factor.

This temporary independence is precisely the feature that is formalized by the notion of *partial* derivatives. Let's then define a *partial derivative of $f(x, y)$ by x* to mean the derivative of f by x *while holding y constant*—and the other way around, with $x \leftrightarrow y$. To distinguish this notion of a derivative from the "ordinary" one, we'll write this as:

$$f(x, y(x)) = x^3\big(y(x)\big)^2\,, \qquad \frac{\partial f}{\partial x} = 3\,x^2\big(y(x)\big)^2, \qquad \frac{\partial f}{\partial y} = x^3\,2\,y(x)\,, \tag{2.2}$$

so that

$$\frac{\mathrm{d}\, f}{\mathrm{d}\, x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = \Big(3\,x^2\,(y(x))^2\Big) + \Big(x^3\,2\,y(x)\Big)\Big(\frac{\mathrm{d}\,y}{\mathrm{d}\,x}\Big)\,. \tag{2.3}$$

A more general use of partial derivatives (one that you have actually already used, but have not called it so) is the use of the chain rule in a calculation such as:

$$\frac{\mathrm{d} f(a(x), b(x))}{\mathrm{d}\, x} = \frac{\partial f}{\partial a}\,\frac{\mathrm{d}\, a}{\mathrm{d}\, x} + \frac{\partial f}{\partial b}\,\frac{\mathrm{d}\, b}{\mathrm{d}\, x}\,,$$

such as in the example:

$$\frac{\mathrm{d}\,\sin^2(x)\cos^3(x)}{\mathrm{d}\, x} = \Big(2\,\sin(x)\,\frac{\mathrm{d}\,\sin(x)}{\mathrm{d}\, x}\Big)\Big(\cos^3(x)\Big) + \Big(\sin^2(x)\Big)\Big(3\,\cos^2(x)\,\frac{\mathrm{d}\,\cos(x)}{\mathrm{d}\, x}\Big) \tag{2.4}$$

also follows from (2.3), by identifying $a(x) = \sin(x)$ and $b(x) = \cos(x)$, so that we abbreviate $f(a, b) = a^2\,b^3$:

$$= \Big(2\,a\,b^3\Big)\Big(\frac{\mathrm{d}\,a}{\mathrm{d}\,x}\Big) + \Big(3\,a^2\,b^2\Big)\Big(\frac{\mathrm{d}\,b}{\mathrm{d}\,x}\Big)\,, \quad \text{which fully agrees with (2.4),} \tag{2.5}$$

and where we used that

$$\frac{\partial a^2\,b^3}{\partial a} = 2\,a\,b^3\,, \qquad \text{and} \qquad \frac{\partial a^2\,b^3}{\partial b} = 2\,a^2\,b^2\,. \tag{2.6}$$

[2]This is *by no means* a substitute for the relevant parts of the calculus I, II, III sequence, which will have *properly* introduced the subject and with all the necessary rigor and precision!

For a scientist, perhaps, a reasonable way of thinking of partial derivatives is in trying to evaluate the original question: how does a compound function vary:

$$\mathrm{d}\, f(x, y(x)) \;=\; \frac{\partial f}{\partial x}\,\mathrm{d}x \;+\; \frac{\partial f}{\partial y}\mathrm{d}y \tag{2.7}$$

saying simply that the function $f(x, y)$ changes (**a**) through its direct dependence on x—and ignoring the dependence on $y(x)$, and (**b**) through its dependence on $y(x)$—while ignoring the direct dependence on x. Of course, we then note that $\mathrm{d}y = y'(x)\,\mathrm{d}x$, and obtain the standard expression:

$$\mathrm{d}\, f(x, y(x)) \;=\; \frac{\partial f}{\partial x}\,\mathrm{d}x \;+\; \frac{\partial f}{\partial y}\, y'(x)\,\mathrm{d}x \tag{2.8}$$

While you may not heave *written* this out in this particular way, you must have encountered this notion before.

A few more examples should clarify the matter; the key point is that while calculating the *partial derivative of* f *by* x, you ignore all other, implicit dependences of f on x through some other function(s). Thus:

$$\begin{aligned}
\frac{\partial}{\partial x}\Big(\sqrt{2x-3y(x)}\Big) &= \frac{1}{2\sqrt{2x-3y}}\frac{\partial(2x-3y)}{\partial x} = \frac{1}{2\sqrt{2x-3y}}\,2 = \frac{1}{\sqrt{2x-3y(x)}}\,,\\
\frac{\partial}{\partial y}\Big(\sqrt{2x-3y(x)}\Big) &= \frac{1}{2\sqrt{2x-3y}}\frac{\partial(2x-3y)}{\partial y} = \frac{1}{2\sqrt{2x-3y}}\,(-3) = -\frac{3}{2\sqrt{2x-3y(x)}}\,,\\
\frac{\partial}{\partial x}\left(x\sqrt{\frac{1-y(x)}{z(x)+3}}\right) &= \frac{\partial}{\partial x}\left(x\sqrt{\frac{1-y}{z+3}}\right) = \frac{\partial x}{\partial x}\,\sqrt{\frac{1-y}{z+3}} = \sqrt{\frac{1-y(x)}{z(x)+3}}\,,\\
\frac{\partial}{\partial y}\left(x\sqrt{\frac{1-y(x)}{z(x)+3}}\right) &= \frac{\partial}{\partial y}\left(x\sqrt{\frac{1-y}{z+3}}\right) = \frac{x}{\sqrt{z+3}}\,\frac{\partial\sqrt{1-y}}{\partial y} = \frac{x}{\sqrt{z+3}}\,\frac{1}{2\sqrt{1-y}}\,\frac{\partial(1-y)}{\partial y}\\
&= \frac{x}{\sqrt{z+3}}\,\frac{1}{2\sqrt{1-y}}\,(-y) = \frac{-x\,y(x)}{\sqrt{(z(x)+3)(1-y(x))}}\,,\\
\frac{\partial}{\partial x}\left(\frac{\sin(y(x))}{\cos(x)}\right) &= \frac{\partial}{\partial x}\left(\frac{\sin(y)}{\cos(x)}\right) = \sin(y)\,\frac{\partial}{\partial x}\left(\frac{1}{\cos(x)}\right) = \sin(y)\left(-\frac{1}{\cos^2(x)}\,\frac{\partial\cos(x)}{\partial x}\right)\\
&= \sin(y)\left(-\frac{1}{\cos^2(x)}\Big(-\sin(x)\Big)\right) = \frac{\sin(y(x))\sin(x)}{\cos^2(x)}\,,\\
\frac{\partial}{\partial y}\left(\frac{\sin(y(x))}{\cos(x)}\right) &= \frac{\partial}{\partial y}\left(\frac{\sin(y)}{\cos(x)}\right) = \frac{1}{\cos(x)}\,\frac{\partial\sin(y)}{\partial y} = \frac{\cos(y(x))}{\cos(x)}\,,\\
\vdots\quad &= \quad\vdots
\end{aligned}$$

Got it?

3 Matrices, Determinants and Linear Systems

For the benefit of the students who have not seen "matrices" and "determinants" before, here's a little introduction[3]: ***WORK through it!***

Matrices are rectangular lists or tables of numbers or functions, such as

$$\mathbb{A} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \qquad \mathbb{B} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \\ 4 & 6 \end{bmatrix}, \qquad \mathbb{C} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad \mathbb{D} = \begin{bmatrix} -1 & 3 & 8 \\ 5 & 2 & -7 \\ 4 & -6 & 9 \end{bmatrix}. \tag{3.1}$$

Two matrices, $\mathbb{A}$ and $\mathbb{B}$, can be multiplied if the number of columns of the left factor equals to number of rows of the right factor. In that case, the multiplication goes as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + \cdots) & (a_{11}b_{12} + a_{12}b_{22} + \cdots) & \cdots \\ (a_{21}b_{11} + a_{22}b_{21} + \cdots) & (a_{21}b_{12} + a_{22}b_{22} + \cdots) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{3.2}$$

So,

$$\mathbb{A}\mathbb{B} = \begin{bmatrix} (1{\cdot}1 + 3{\cdot}5 + 5{\cdot}4) & (1{\cdot}3 + 3{\cdot}2 + 5{\cdot}6) \\ (2{\cdot}1 + 4{\cdot}5 + 6{\cdot}4) & (2{\cdot}3 + 4{\cdot}2 + 6{\cdot}6) \end{bmatrix} = \begin{bmatrix} 36 & 39 \\ 46 & 50 \end{bmatrix}, \tag{3.3}$$

$$\mathbb{A}\mathbb{D} = \begin{bmatrix} (1{\cdot}(-1) + 3{\cdot}5 + 5{\cdot}4) & (1{\cdot}3 + 3{\cdot}2 + 5{\cdot}(-6)) & (1{\cdot}8 + 3{\cdot}(-7) + 5{\cdot}9) \\ (2{\cdot}(-1) + 4{\cdot}5 + 6{\cdot}4) & (2{\cdot}3 + 4{\cdot}2 + 6{\cdot}(-6)) & (2{\cdot}8 + 4{\cdot}(-7) + 6{\cdot}9) \end{bmatrix} = \begin{bmatrix} 34 & -21 & 32 \\ 42 & -22 & 42 \end{bmatrix}, \tag{3.4}$$

$$\mathbb{B}\mathbb{A} = \begin{bmatrix} (1{\cdot}1 + 3{\cdot}2) & (1{\cdot}3 + 3{\cdot}4) & (1{\cdot}5 + 3{\cdot}6) \\ (5{\cdot}1 + 2{\cdot}2) & (5{\cdot}3 + 2{\cdot}4) & (5{\cdot}5 + 2{\cdot}6) \\ (4{\cdot}1 + 6{\cdot}2) & (4{\cdot}3 + 6{\cdot}4) & (4{\cdot}5 + 6{\cdot}6) \end{bmatrix} = \begin{bmatrix} 7 & 15 & 23 \\ 9 & 23 & 37 \\ 16 & 36 & 56 \end{bmatrix}, \tag{3.5}$$

$$\mathbb{B}\mathbb{C} = \begin{bmatrix} (1{\cdot}1 + 3{\cdot}2) & (1{\cdot}3 + 3{\cdot}4) \\ (5{\cdot}1 + 2{\cdot}2) & (5{\cdot}3 + 2{\cdot}4) \\ (4{\cdot}1 + 6{\cdot}2) & (4{\cdot}3 + 6{\cdot}4) \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 9 & 23 \\ 16 & 36 \end{bmatrix}, \tag{3.6}$$

$$\mathbb{C}\mathbb{A} = \begin{bmatrix} (1{\cdot}1 + 3{\cdot}2) & (1{\cdot}3 + 3{\cdot}4) & (1{\cdot}5 + 3{\cdot}6) \\ (2{\cdot}1 + 4{\cdot}2) & (2{\cdot}3 + 4{\cdot}4) & (2{\cdot}5 + 4{\cdot}6) \end{bmatrix} = \begin{bmatrix} 7 & 15 & 23 \\ 10 & 22 & 34 \end{bmatrix}, \tag{3.7}$$

$$\mathbb{C}\mathbb{C} = \begin{bmatrix} (1{\cdot}1 + 3{\cdot}2) & (1{\cdot}3 + 3{\cdot}4) \\ (2{\cdot}1 + 4{\cdot}2) & (2{\cdot}3 + 4{\cdot}4) \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix} \stackrel{\text{def}}{=} \mathbb{C}^2, \tag{3.8}$$

$$\mathbb{D}\mathbb{B} = \begin{bmatrix} ((-1){\cdot}1 + 3{\cdot}5 + 8{\cdot}4) & ((-1){\cdot}3 + 3{\cdot}2 + 8{\cdot}6) \\ (5{\cdot}1 + 2{\cdot}5 + (-7){\cdot}4) & (5{\cdot}3 + 2{\cdot}2 + (-7){\cdot}6) \\ (4{\cdot}1 + (-6){\cdot}5 + 9{\cdot}4) & (4{\cdot}3 + (-6){\cdot}2 + 9{\cdot}6) \end{bmatrix} = \begin{bmatrix} 46 & 51 \\ -13 & -23 \\ 10 & 54 \end{bmatrix}, \tag{3.9}$$

$$\mathbb{D}\mathbb{D} = \begin{bmatrix} ((-1){\cdot}1 + 3{\cdot}5 + 8{\cdot}4) & ((-1){\cdot}3 + 3{\cdot}2 + 8{\cdot}6) \\ (5{\cdot}1 + 2{\cdot}5 + (-7){\cdot}4) & (5{\cdot}3 + 2{\cdot}2 + (-7){\cdot}6) \\ (4{\cdot}1 + (-6){\cdot}5 + 9{\cdot}4) & (4{\cdot}3 + (-6){\cdot}2 + 9{\cdot}6) \end{bmatrix} = \begin{bmatrix} 48 & -45 & 43 \\ -23 & 61 & -37 \\ 2 & -54 & 155 \end{bmatrix}. \tag{3.10}$$

[3]This is *by no means* a substitute for the relevant parts of linear algebra, which will have *properly* introduced the subject and with all the necessary rigor and precision!

Furthermore, the products not displayed above, $\mathbb{A}^2$, $\mathbb{AC}$, $\mathbb{B}^2$, $\mathbb{BD}$, $\mathbb{CB}$, $\mathbb{CD}$, $\mathbb{DA}$ and $\mathbb{DC}$, do not in fact exist.

It is useful to note the special matrices:

$$\mathbb{O} = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \qquad \text{and} \qquad \mathbb{1} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.11}$$

which satisfy the usual properties

$$\mathbb{1}\,\mathbb{A} = \mathbb{A} = \mathbb{A}\,\mathbb{1}\,, \qquad \text{and} \qquad \mathbb{O}\,\mathbb{A} = \mathbb{O} = \mathbb{A}\,\mathbb{O}\,, \qquad \forall\,\mathbb{A}\,. \tag{3.12}$$

Caution: the matrix-multiplication is not commutative: $\mathbb{AB} \neq \mathbb{BA}$, above. Also:

$$\mathbb{E} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}, \qquad \text{then} \qquad \mathbb{CE} = \begin{bmatrix} -8 & 10 \\ -10 & 12 \end{bmatrix} \neq \mathbb{EC} = \begin{bmatrix} -3 & -5 \\ 5 & 7 \end{bmatrix}! \tag{3.13}$$

Matrices also can square to $\mathbb{O}$, without being equal to $\mathbb{O}$:

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}^2 = \mathbb{O}\,. \tag{3.14}$$

That means, you can't (necessarily) *divide* by matrices. There is, however, such a concept as an *inverse matrix*. That is, some matrices have them, others don't. Also, for non-square matrices, the *left-inverse* and the *right-inverse* may be very different. For example, $\mathbb{A}$ above does not have a left inverse:

$$\text{There exists no} \quad \mathbb{F}\,, \quad \text{such that} \quad \mathbb{FA} = \mathbb{1}\,. \tag{3.15}$$

But,

$$\mathbb{A}\begin{bmatrix} -2+\alpha & \frac{3}{2}+\beta \\ 1-2\alpha & -\frac{1}{2}-2\beta \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.16}$$

for arbitrary α, β! We'd say that $\mathbb{A}$ has no left-inverse, but has two parameters worth of right-inverses.

The *determinant* of a *square* matrix is defined by the following recursive pair of definitions:

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}\,, \tag{3.17a}$$

and expanding by the first row:

$$\begin{aligned} \det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} &= a_{11}\cdot\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - a_{12}\cdot\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} + a_{13}\cdot\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \cdots \\ &= a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{23} & \cdots \\ a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} - a_{12}\cdot\det\begin{bmatrix} a_{21} & a_{23} & \cdots \\ a_{31} & a_{33} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} + a_{13}\cdot\det\begin{bmatrix} a_{21} & a_{22} & \cdots \\ a_{31} & a_{32} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} - \cdots \end{aligned} \tag{3.17b}$$

Note the alternating signs in this "expansion by the first row" (so it ends when you run out of elements in the first row). Notice also that the resulting determinants are all smaller. So, after iterating this, you'll end up with determinants of 2×2 matrices, for which you use (3.17a), and you're done.

For example

$$\begin{aligned}\det\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} &= 1\cdot\det\begin{bmatrix}5&6\\8&9\end{bmatrix} - 2\cdot\det\begin{bmatrix}4&6\\7&9\end{bmatrix} + 3\cdot\det\begin{bmatrix}4&5\\7&8\end{bmatrix}\\ &= 1(5{\cdot}9-6{\cdot}8)-2(4{\cdot}9-6{\cdot}7)+3(4{\cdot}8-5{\cdot}7)\\ &= 1(-3)-2(-6)+3(-3) = -3+12-9=0\,,\end{aligned} \tag{3.18}$$

but

$$\begin{aligned}\det\begin{bmatrix}1&2&3\\4&5&6\\7&8&-9\end{bmatrix} &= 1\cdot\det\begin{bmatrix}5&6\\8&-9\end{bmatrix} - 2\cdot\det\begin{bmatrix}4&6\\7&-9\end{bmatrix} + 3\cdot\det\begin{bmatrix}4&5\\7&8\end{bmatrix}\\ &= 1(5{\cdot}(-9)-6{\cdot}8)-2(4{\cdot}(-9)-6{\cdot}7)+3(4{\cdot}8-5{\cdot}7)\\ &= 1(-93)-2(-78)+3(-3) = -3+12-9=54\,.\end{aligned} \tag{3.19}$$

In fact, for square matrices, the condition for an inverse matrix to exist is that its determinant should be nonzero. Therefore the matrix $\left[\begin{smallmatrix}1&2&3\\4&5&6\\7&8&9\end{smallmatrix}\right]$ has no inverse, whereas $\left[\begin{smallmatrix}1&2&3\\4&5&6\\7&8&-9\end{smallmatrix}\right]$ has:

$$\mathbb{G}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&-9\end{bmatrix}\,, \qquad \text{then} \qquad \mathbb{G}^{-1}=\begin{bmatrix}-\frac{31}{18}&\frac{7}{9}&-\frac{1}{18}\\ \frac{13}{9}&-\frac{5}{9}&\frac{1}{9}\\ -\frac{1}{18}&\frac{1}{9}&-\frac{1}{18}\end{bmatrix}\,. \tag{3.20}$$

Indeed, check by explicit matrix multiplication that $\mathbb{G}^{-1}\,\mathbb{G} = \mathbb{1} = \mathbb{G}\,\mathbb{G}^{-1}$.

OK, so what does this have to do with the price of beans in China?

I don't know. But I know that it helps solving systems of algebraic equations. Suppose we have the system of algebraic equations:

$$\begin{cases} ax + by + cz = l\,,\\ dx + ey + fz = m\,,\\ gx + hy + kz = n\,.\end{cases} \tag{3.21}$$

That's three linear equations for three unknown variables, x, y, z, and depending on the twelve coefficients: $a, b, c, d, e, f, g, h, k$ and l, m, n. Using the above matrix notation, we can also rewrite this as a matrix equation:

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}l\\m\\n\end{bmatrix}\,. \tag{3.22}$$

Moreover, if we can find the inverse matrix to the square matrix $\left[\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & k \end{smallmatrix}\right]$, called the matrix of the system (3.21), we can even solve the system (3.21):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}^{-1} \begin{bmatrix} l \\ m \\ n \end{bmatrix} . \tag{3.23}$$

For example,

$$\left\{ \begin{aligned} x + 2y + 3z &= 18 , \\ 4x + 5y + 6z &= 18 , \\ 7x + 8y - 9z &= 18 . \end{aligned} \right. \qquad \textit{i.e.} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \\ 18 \end{bmatrix} \tag{3.24}$$

is solved by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{31}{18} & \frac{7}{9} & -\frac{1}{18} \\ \frac{13}{9} & -\frac{5}{9} & \frac{1}{9} \\ -\frac{1}{18} & \frac{1}{9} & -\frac{1}{18} \end{bmatrix} \begin{bmatrix} 18 \\ 18 \\ 18 \end{bmatrix} = \begin{bmatrix} -31 + 14 - 1 \\ 13 - 10 + 2 \\ -1 + 2 - 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 18 \\ 0 \end{bmatrix} , \qquad \Rightarrow \qquad \left\{ \begin{aligned} x &= -18 , \\ y &= 18 , \\ z &= 0 . \end{aligned} \right. \tag{3.25}$$

Now, look at the special case of (3.21), when $l = m = n = 0$:

$$\left\{ \begin{aligned} ax + by + cz &= 0 , \\ dx + ey + fz &= 0 , \\ gx + hy + kz &= 0 . \end{aligned} \right. \qquad \textit{i.e.} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbb{O} . \tag{3.26}$$

If the square matrix $\left[\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & k \end{smallmatrix}\right]$ has an inverse, then we obtain that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \left\{ \begin{aligned} x &= 0 , \\ y &= 0 , \\ z &= 0 . \end{aligned} \right. \tag{3.27}$$

So, the only hope for non-trivial x, y, z to solve the system (3.26) is for the matrix $\left[\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & k \end{smallmatrix}\right]$ to have no inverse, which can only happen if its determinant is zero.

A homogeneous system of n linear equations in n variables has a non-trivial solution only if the determinant of the system vanishes.

Indeed:

$$\left\{ \begin{aligned} x + 2y + 3z &= 0 , \\ 4x + 5y + 6z &= 0 , \\ 7x + 8y + 9z &= 0 . \end{aligned} \right. \qquad \textit{i.e.} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.28}$$

is solved by $y = -2x$ and $z = x$. That is, instead of a single solution such as (3.25), we now have a 1-parameter family of solutions: $(x, y, z) = (x, -2x, x)$, with x *arbitrary* and parametrizing the *continuum* of solutions.