

(Fundamentalna) Fizika Elementarnih Čestica

Dan 08a: Kalibracioni princip gravitacije
i geometrizacija fizike

Tristan Hübsch

Department of Physics and Astronomy, Howard University, Washington DC

Department of Mathematics, University of Maryland, College Park, MD

Department of Physics, Faculty of Natural Sciences, Novi Sad, Serbia

<https://tristan.nfshost.com/>

Fundamentalna Fizika Elementarnih čestica

Program

- Kovariantni izvod i Christoffel-ov simbol
 - Koordinatni bazis
 - Kovariantni izvod
 - Metričnost Christoffel-ovog simbola
- Zakrivljenost prostor-vremena
 - Tenzor zakrivljenosti
 - Uslovi i kontrakcije
 - Einstein-Hilbert-ovo dejstvo
- Sprega supstancije i gravitacije
 - “Kovariantizovanje” Lagranžijana
 - Einstein-ove jednačine
 - Dve obavezne paralele
- Specialna rešenja

Kovarijantni izvod i Christoffel-ov simbol

Koordinatni Bazis

...potsetnik/obnova

- Lorentz-invarijantni interval u proizvoljnim koordinatama:
princip ekvivalencije

$$ds^2 := dx^\mu (-\eta_{\mu\nu}) dx^\nu = dy^\rho \underbrace{\left(\frac{\partial x^\mu}{\partial y^\rho} \right) (-\eta_{\mu\nu}) \left(\frac{\partial x^\nu}{\partial y^\sigma} \right)}_{\text{metrički tensorni faktor}} dy^\sigma = dy^\rho g_{\rho\sigma}(y) dy^\sigma,$$

- što definiše

- **metrički tensor**, $g_{\rho\sigma}(y) := \left(\frac{\partial x^\mu}{\partial y^\rho} \right) (-\eta_{\mu\nu}) \left(\frac{\partial x^\nu}{\partial y^\sigma} \right)$

- Prelaskom u neki drugi koordinatni sistem, metrički tensor je:

$$g_{\mu\nu}(z) = \frac{\partial y^\rho}{\partial z^\mu} \frac{\partial y^\sigma}{\partial z^\nu} g_{\rho\sigma}(y)$$

- Determinanta je

$$\sqrt{-g(x)} = \det \left[\frac{\partial z}{\partial x} \right] \sqrt{-g(z)}$$

- Pošto je

$$d^4x = \det \left[\frac{\partial x}{\partial z} \right] d^4z$$

onda je $\sqrt{-g(x)} d^4x = \sqrt{-g(z)} d^4z$ **invarijanta.**

Kovarijantni izvod i Christoffel-ov simbol

Koordinatni Bazis

...potsetnik/obnova

- Bazisni vektori:

$$\vec{x}_\mu := (\partial_\mu \vec{r}) \quad \text{i} \quad \vec{x}^\mu := g^{\mu\nu}(\mathbf{x}) \vec{x}_\nu,$$

- tako

$$A_\mu := \vec{x}_\mu \cdot \vec{A}, \quad A^\mu := \vec{x}^\mu \cdot \vec{A}, \quad \text{i} \quad \vec{A} = A_\mu \vec{x}^\mu = A^\mu \vec{x}_\mu,$$

- i

$$\vec{x}_\mu \cdot \vec{x}_\nu = g_{\mu\nu}(\mathbf{x}) \quad \text{i} \quad \vec{x}^\mu \cdot \vec{x}^\nu = g^{\mu\nu}(\mathbf{x}).$$

- Onda

$$\Gamma_{\mu\nu}^\rho : \quad (\partial_\nu \vec{x}_\mu) = \Gamma_{\mu\nu}^\rho \vec{x}_\rho \quad \text{zbog kompletnosti bazisa}$$

- Stoga,

$$\Gamma_{\mu\nu}^\rho \vec{x}_\rho := (\partial_\mu \vec{x}_\nu) = (\partial_\mu \partial_\nu \vec{r}) = (\partial_\nu \partial_\mu \vec{r}) = (\partial_\nu \vec{x}_\mu) = \Gamma_{\nu\mu}^\rho \vec{x}_\rho.$$

- i

$$(\partial_\mu \vec{x}^\rho) = -\Gamma_{\mu\nu}^\rho \vec{x}^\nu \quad \text{jer} \quad \partial_\mu (\vec{x}_\mu \cdot \vec{x}^\nu = \delta_\mu^\nu) = 0.$$

Kovarijantni izvod i Christoffel-ov simbol

Kovarijantni izvod

• Sledi:

$$\vec{A} := A^\rho \vec{x}_\rho \quad \& \quad (\partial_\mu \vec{x}_\nu) =: \Gamma_{\mu\nu}^\rho \vec{x}_\rho \quad \Rightarrow \quad (\partial_\mu \vec{A}) = [(\partial_\mu A^\rho) + \Gamma_{\mu\nu}^\rho A^\nu] \vec{x}_\rho;$$

$$\vec{B} := B_\rho \vec{x}^\rho \quad \& \quad (\partial_\mu \vec{x}^\rho) =: -\Gamma_{\mu\nu}^\rho \vec{x}^\nu \quad \Rightarrow \quad (\partial_\mu \vec{B}) = [(\partial_\mu B_\nu) - \Gamma_{\mu\nu}^\rho B_\rho] \vec{x}^\nu.$$

• Definišimo:

$$\mathcal{D}_\mu A^\rho := (\partial_\mu A^\rho) + \Gamma_{\mu\nu}^\rho A^\nu \quad \text{i} \quad \mathcal{D}_\mu B_\nu := (\partial_\mu B_\nu) - \Gamma_{\mu\nu}^\rho B_\rho$$

• Zahvaljujući Weyl-ovoj konstrukciji,

$$T(p, q; w) := C^w \otimes \mathcal{YS} \left[\underbrace{A \otimes \cdots \otimes A}_p \otimes \underbrace{B \otimes \cdots \otimes B}_q \right]$$

• onda *sledi* (po pravilu izvoda proizvoda) da:

$$(\mathcal{D}_\mu \mathbb{T})_{\mu_1 \cdots \mu_q}^{\nu_1 \cdots \nu_p} = (\partial_\mu \mathbb{T})_{\mu_1 \cdots \mu_q}^{\nu_1 \cdots \nu_p} + \sum_{i=1}^p \Gamma_{\mu \sigma_i}^{\nu_i} T_{\mu_1 \cdots \cdots \cdots \mu_q}^{\nu_1 \cdots \sigma_i \cdots \nu_p} - \sum_{i=1}^q \Gamma_{\mu \rho_i}^{\sigma_i} T_{\mu_1 \cdots \sigma_i \cdots \mu_q}^{\nu_1 \cdots \cdots \cdots \nu_p}$$

Kovarijantni izvod i Christoffel-ov simbol

Kovarijantni izvod

Čak,

$$X_{\rho_1 \dots \rho_q; \mu}^{\nu_1 \dots \nu_p} := (\mathcal{D}_\mu \mathbb{X})_{\rho_1 \dots \rho_q}^{\nu_1 \dots \nu_p}$$

- se transformiše kao tensorska gustina tipa $(p, q+1; w)$.
- Pošto se parcijalni izvod tensora ranga ≥ 0 ne transformiše kao tenzor (proveriti), ne može ni Γ_μ -simbol—radi kompenzacije:

$$\Gamma_{\mu\nu}^\rho(\mathbf{x}) = \underbrace{\frac{\partial \mathbf{x}^\rho}{\partial y^\sigma} \frac{\partial y^\kappa}{\partial \mathbf{x}^\mu} \frac{\partial y^\lambda}{\partial \mathbf{x}^\nu} \Gamma_{\kappa\lambda}^\sigma(y)}_{\text{tenzorski deo}} + \underbrace{\frac{\partial \mathbf{x}^\rho}{\partial y^\sigma} \frac{\partial^2 y^\sigma}{\partial \mathbf{x}^\mu \partial \mathbf{x}^\nu}}_{\text{nehomogeni deo}},$$

je tenzor ako i samo ako je transformacija $x \rightarrow y$ linearna

u kom slučaju, Γ_μ ionako nije potrebno. 😎

Važi za rotacije i translacije dekartovskih koordinata.

Kovarijantni izvod i Christoffel-ov simbol

Kovarijantni izvod

- Stoga se Γ_μ ponaša kao kalibracioni 4-vektorski potencijal, osim što ima jednu dodatnu transformacionu matricu:

$$\Gamma'_\mu = [\mathbf{U}]_\mu^\nu \mathbf{U} \Gamma_\nu \mathbf{U}^{-1} + \mathbf{U} \partial_\mu \mathbf{U}^{-1}$$

- I, još jedna stvar:

$$[\mathbb{A}_\mu \cdot \Psi]^\alpha = [A_\mu]_\beta^\alpha \cdot \Psi^\beta \quad \leftrightarrow \quad [\Gamma_\mu \cdot V]^\rho = [\Gamma_\mu]_\nu^\rho V^\nu$$

bez relacije \longleftrightarrow simetrično

- Tu se ogleda konceptualna nelinearnost:
$$\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$$
- Transformacije faza zavise od prostor-vremena
- Transformacije prostor-vremenskih koordinata zavisi of prostor-vremena
- Yang-Mills \mathbb{A}_μ je prostor-vremenski 4-vektor matrica u *prostoru "boje"*.
- Christoffel Γ_μ je prostor-vremenski 4-vektor matrica u *prostor-vremenu*.

Kovarijantni izvod i Christoffel-ov simbol

Metričnost Christoffel-ovog Simbola

- Zadavanjem relacija

$$(\partial_\nu \vec{x}_\mu) = \Gamma_{\mu\nu}^\rho \vec{x}_\rho \quad \text{i} \quad \vec{x}_\mu \cdot \vec{x}_\nu = g_{\mu\nu}(x)$$

- sledi da mora da postoji relacija izmedju Γ_μ i metrike.

Stvarno,

$$(\partial_\mu g_{\nu\rho}) = (\partial_\mu (\vec{x}_\nu \cdot \vec{x}_\rho)) = \Gamma_{\mu\nu}^\sigma \vec{x}_\sigma \cdot \vec{x}_\rho + \vec{x}_\nu \cdot \Gamma_{\mu\rho}^\sigma \vec{x}_\sigma = g_{\sigma\rho} \Gamma_{\mu\nu}^\sigma + g_{\sigma\nu} \Gamma_{\mu\rho}^\sigma$$

- daje

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} [(\partial_\mu g_{\nu\sigma}) + (\partial_\nu g_{\mu\sigma}) - (\partial_\sigma g_{\mu\nu})] \quad (\heartsuit)$$

- koje zadovoljava

$$\mathcal{D}_\mu g_{\nu\rho} = 0 = \mathcal{D}_\mu g^{\nu\rho}$$

metrika je
kovarijantno konstantna

- i obratno: $\mathcal{D}_\mu g_{\nu\rho} = 0$ sa $\mathcal{D}_\mu = \partial_\mu + \Gamma_\mu$ implicira relaciju (\heartsuit) .

- Ovaj (Christoffel-ov) Γ_μ -simbol je metričan.

Zakrivljenost prostor-vremena

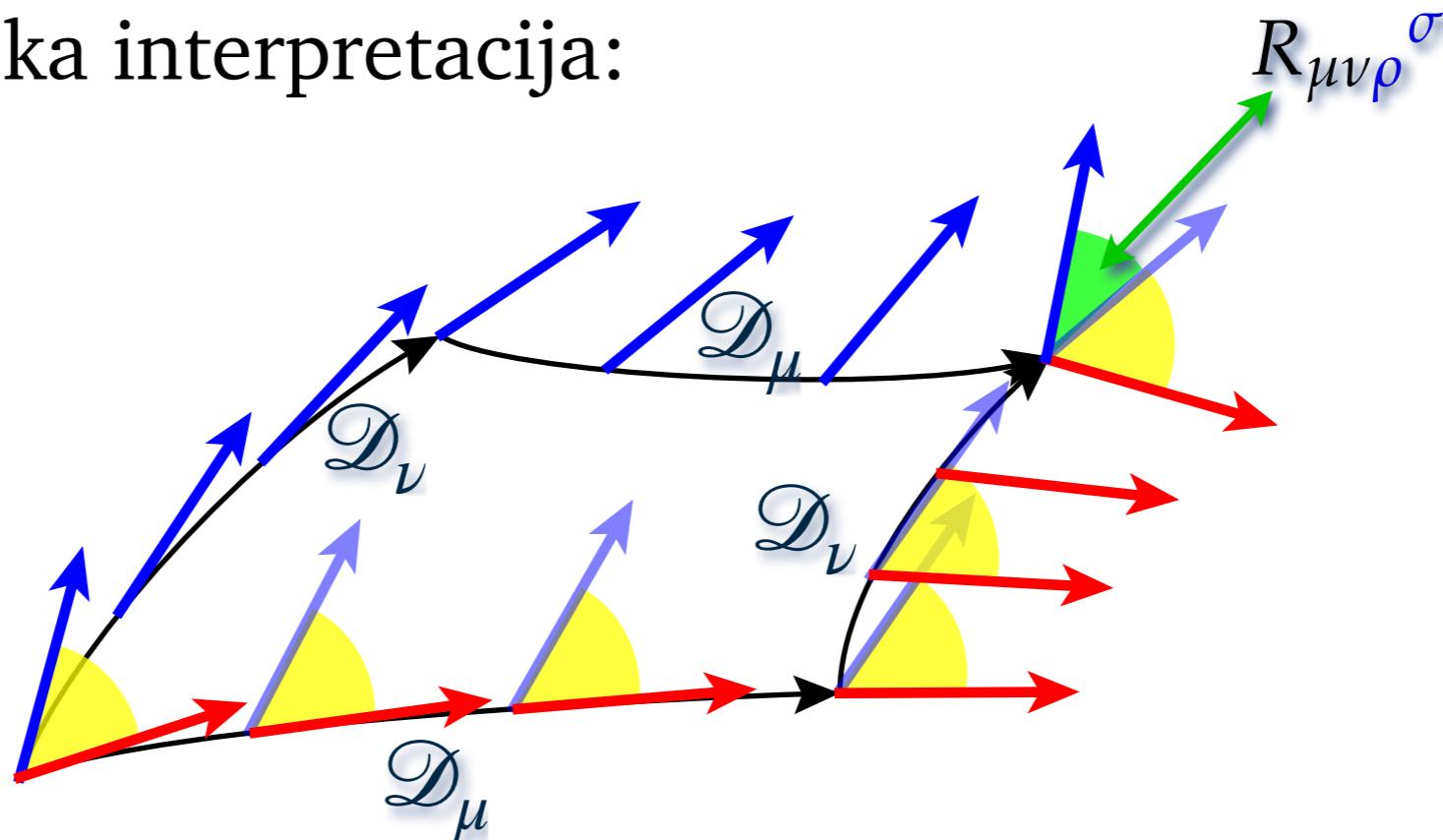
Tenzor zakrivljenosti

• Baš kao $[\mathbb{F}_{\mu\nu}]_{\alpha}^{\beta} := \frac{\hbar c}{ig_c} [\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]_{\alpha}^{\beta}$

• definišemo

$$\begin{aligned} R_{\mu\nu\rho}^{\sigma} &:= [\mathcal{D}_{\mu}, \mathcal{D}_{\nu}]_{\rho}^{\sigma} = [(\delta_{\lambda}^{\sigma} \partial_{\nu} + \Gamma_{\nu\lambda}^{\sigma}) \Gamma_{\mu\rho}^{\lambda}] - [(\delta_{\lambda}^{\sigma} \partial_{\mu} + \Gamma_{\mu\lambda}^{\sigma}) \Gamma_{\nu\rho}^{\lambda}] \\ &= \partial_{\nu} \Gamma_{\mu\rho}^{\sigma} - \partial_{\mu} \Gamma_{\nu\rho}^{\sigma} + \Gamma_{\nu\lambda}^{\sigma} \Gamma_{\mu\rho}^{\lambda} - \Gamma_{\mu\lambda}^{\sigma} \Gamma_{\nu\rho}^{\lambda} \end{aligned}$$

• Geometrijska interpretacija:



Zakrivljenost prostor-vremena

Uslovi i kontrakcije

- Definišemo $R_{\mu\nu\rho\sigma} := R_{\mu\nu\rho}^{\lambda} g_{\lambda\sigma}$ (za $\mathbb{F}_{\mu\nu}$ to ne postoji)
- Riemann-ov tenzor zadovoljava sledeće identitete:

$$R_{\mu\nu\rho}^{\rho} = 0$$

$$\text{(neabelovski)} \quad \text{Tr}[\mathbb{F}_{\mu\nu}] = 0$$

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$$

$$\mathbb{F}_{\mu\nu} = -\mathbb{F}_{\nu\mu}$$

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$$

—

$$R_{\mu\nu\rho\sigma} = +R_{\rho\sigma\mu\nu}$$

—

$$\varepsilon^{\lambda\mu\nu\rho} R_{\mu\nu\rho}^{\sigma} = 0$$

1. Bianchi-ev identitet

—

$$\varepsilon^{\kappa\lambda\mu\nu} \mathcal{D}_{\lambda} R_{\mu\nu\rho}^{\sigma} = 0$$

2. Bianchi-ev identitet

$$\varepsilon^{\kappa\lambda\mu\nu} D_{\lambda} \mathbb{F}_{\mu\nu} = 0$$

- Riemann-ov tenzor je delom prvi izvod Γ_{μ} , delom “kvadrat” $[\Gamma_{\mu}, \Gamma_{\nu}]$

- ...kao što je $\mathbb{F}_{\mu\nu}$ delom prvi izvod \mathbb{A}_{μ} , delom “kvadrat” $[\mathbb{A}_{\mu}, \mathbb{A}_{\nu}]$

- ...a drugog reda po izvodima metrike, $g_{\mu\nu}$, i homogen!

Takodje zavisi of $g^{\mu\nu}$, što je veoma nelinearno po $g_{\mu\nu}$!

Zakrivljenost prostor-vremena

Uslovi i kontrakcije

- Za tenzor polja Yang-Mills tipa,

$$g^{\mu\nu} F_{\mu\nu} \equiv 0, \quad \begin{cases} \text{Tr}[F_{\mu\nu}] = [F_{\mu\nu}]_{\alpha}^{\alpha} = 0, & \text{za poluproste Lie-jeve grupe,} \\ \text{Tr}[F_{\mu\nu}] = F_{\mu\nu}, & \text{za } U(1) \text{ faktore.} \end{cases}$$

- Pošto su sva 4 indeksa $R_{\mu\nu\rho\sigma}$ istog tipa, možemo da definišemo:

Ricci-jev tenzor:

$$R_{\mu\rho} := R_{\mu\nu\rho}{}^{\nu},$$

skalarna zakrivljenost:

$$R := g^{\mu\rho} R_{\mu\rho} = g^{\mu\rho} R_{\mu\nu\rho}{}^{\nu}.$$

- Stoga je moguće definisati:

invarijanta

$S_{\mu\nu\rho\sigma}$, deo "čisti trag", $= \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$.

$E_{\mu\nu\rho\sigma}$, deo "polu-bestraga", $= (g_{\mu[\rho}S_{\nu]\sigma} - g_{\nu[\rho}S_{\sigma]\mu})$; $S_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

$C_{\mu\nu\rho\sigma}$, deo bez traga, tzv. Weyl-ov tenzor (conformne zakrivljenosti).

- Takodje: invarijanta

invarijanta

$$\|R_{\mu\nu}\|^2 := R_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} R_{\rho\sigma}$$

$$\|R_{\mu\nu\rho}{}^{\sigma}\|^2 := R_{\mu\nu\rho}{}^{\sigma} g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g_{\sigma\delta} R_{\alpha\beta\gamma}{}^{\delta}$$

Zakrivljenost prostor-vremena

Einstein-Hilbert-ovo dejstvo

- Za slučaj Yang-Mills polja, jedini član pogodan za Lagranžijansku gustinu a kvadratan po $F_{\mu\nu}$ je $\propto \text{Tr}[F_{\mu\nu} F^{\mu\nu}]$.

- Po istom rezonu, pogledajmo: kvadratno po R

$$\int \sqrt{-g} d^4x R_{\mu\nu\rho}{}^\sigma g^{\mu\kappa} g^{\nu\lambda} R_{\kappa\lambda\sigma}{}^\rho.$$

- Variranjem po komponentama Γ_μ daje PDJ po Γ_μ 2. reda

- Variranjem po komponentama $g_{\mu\nu}$ daje PDJ po $g_{\mu\nu}$ 4. reda

- Za razliku od Yang-Mills $F_{\mu\nu}$, sada imamo i linearnu invarijantu R , pa:

$$\frac{c^3}{16\pi G_N} \int \sqrt{-g} d^4x R,$$

- je Einstein-Hilbert-ovo dejstvo.

- Tako da su jedinice ML^2/T , gde su E -jedinice $[d^4x] = 4$ i $[g_{\mu\nu}] = 0$

- Variranjem po komponentama $g_{\mu\nu}$ daje PDJ po $g_{\mu\nu}$ 2. reda.

Sprega supstancije i gravitacije

“Kovarijantizovanje” Lagranžijana

- Variranje Einstein-Hilbert-ovog dejstva daje

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0.$$

Prazno prostor-vreme!

- Ovo je PDJ kretanja za $g_{\mu\nu}$ 2. reda.

- Kako su i $R_{\mu\nu\rho}^{\sigma}$ i $R_{\mu\nu}$ i R su svi (vrlo) nelinearni po $g_{\mu\nu}$, ovo je vrlo netrivijalan, nelinearan sistem PDJ.

- Sprezanje svega ostalog sa ovom kalibracionom teorjom OKT:

$$\begin{aligned} S[\phi_i(\mathbf{x})] &= \int d^4x \mathcal{L}(\phi_i, (\partial_\mu \phi_i), \dots; \mathbf{x}; C_a) \\ &\rightarrow \int \sqrt{|g|} d^4x \left[\frac{c^3}{16\pi G_N} R - \mathcal{L}(\phi_i, (D_\mu \phi_i), \dots; \mathbf{x}; C_a) \right] \end{aligned}$$

Opšte
Koordinatne
Transformacije

svako ne-metričko/ne-Christoffel-ovo polje

Hvala na pažnji!

Tristan Hübsch

Department of Physics and Astronomy, Howard University, Washington DC

Department of Mathematics, University of Maryland, College Park, MD

Department of Physics, Faculty of Natural Sciences, Novi Sad, Serbia

<https://tristan.nfshost.com/>